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SOME STABILITY RESULTS FOR REACTIVE NAVIER-STOKES-POISSON SYSTEMS

B. DUCOMET

Département de Physique Théorique et Appliquée
CEA Bruyères-le-Chatel, BP 12
F-91680, Bruyères-le-Chatel, France
E-mail: ducomet@bruyeres.cea.fr

Abstract. We review the main results concerning the global existence and the stability of solutions for some models of viscous compressible self-gravitating fluids used in classical astrophysics.

1. Introduction. Hydrodynamics is a very convenient tool to investigate the stability properties of stellar structures [1] [7]. In fact, in various circumstances, a star can be considered as a continuous medium, with some additional particularities, due to gravitation, radiation and thermonuclear reactions.

The physical framework we consider here is that of a self-gravitating perfect fluid, compressible, viscous and heat-conducting, in local equilibrium with the radiation, which is considered by the astrophysicists as a reasonable model for protostars [1] [7].

To take into account thermonuclear processes, we consider a self-consistent production of energy inside the fluid, modelling the "burning" of the constitutive elements of the star.

In fact, these thermonuclear reactions are described by a lot of coupled equations of reaction-diffusion type (hundred of reactions are currently taken into account in realistic numerical simulations).

In order to get a tractable problem, we introduce a simple reacting process with a first order kinetic [8].

As the boundary of the structure is not a priori known, we consider the free-boundary case: the star is situated in a connected region $\Omega_t \subset \mathbf{R}^3$ and its (unknown) boundary $S_t \equiv \partial\Omega_t$ is allowed to fluctuate.

So the equations describing the model are those of self-gravitating hydrodynamics, with radiation and first-order Arrhenius kinetics.

So, for each $y \in \Omega_t$, and each $t \geq 0$, the problem to be solved is the following

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Navier-Stokes-Poisson system:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ \rho \dfrac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{s} - \rho \nabla \Phi, \\ \rho \dfrac{De}{Dt} = \mathbf{s} : \mathbf{d} - \nabla \cdot \mathbf{Q}_{th} + \lambda \phi(\theta, Z) \\ \rho \dfrac{DZ}{Dt} = -\nabla \cdot \mathbf{Q}_{ch} - \phi(\theta, Z), \\ \Delta \Phi = 4\pi G \rho, \end{cases} \tag{1}$$

where $\mathbf{d}$ is the linearized strain tensor, with entries: $\mathbf{d}_{ij} = 1/2(\partial_j v_i + \partial_i v_j)$. denote also by We denote by ":" the contraction for two tensors $\mathbf{a}$ and $\mathbf{b}$, so that $\mathbf{a} : \mathbf{b} = \sum_{ij} \mathbf{a}_{ij} \mathbf{b}_{ij}$.

We denote also by $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$, the "material derivative".

If one solves the Poisson equation giving the gravitational potential Φ, we get the formula:

$$\Phi(y, t) = -G \int_{\Omega_t} \frac{\rho(z, t)}{|y - z|} dz, \tag{2}$$

where G is the Newton constant. So, the unknown quantities are finally the density $\rho(y, t)$, the velocity $\mathbf{v}(y, t) = (v_1, v_2, v_3)$, the temperature $\theta(y, t)$, and the fraction of reactant $Z(y, t)$.

The thermodynamical and mechanical properties of the fluid are determined by the expressions of:

1. **The stress tensor**:

$$\mathbf{s} = -p\mathbf{I} + 2\nu \mathbf{d} + \mu \; Tr(\mathbf{d}) \; \mathbf{I},$$

where:

$$p(\rho, \theta) = R\rho\theta + \frac{a}{3}\theta^4,$$

is the pressure of the gas (gaseous and radiative), R is the perfect gas constant, a is the Stefan-Boltzmann constant, $\mathbf{I}$ is the unit tensor, $Tr(\mathbf{d})$ denotes the trace of $\mathbf{d}$: $Tr(\mathbf{d}) = \sum_i \mathbf{d}_{ii}$, , and ν and μ are two (positive) viscosity coefficients.

2. **The energy density**:

$$e(\rho, \theta) = C_v \theta + a \frac{\theta^4}{\rho},$$

where C_v is the gaseous conductivity.

3. **The thermal flux**:

$$\mathbf{Q}_{th} = -\chi(\rho, \theta) \nabla \theta,$$

where:

$$\chi(\rho, \theta) = \kappa_1 + \kappa_2 \frac{\theta^q}{\rho},$$

is the thermal conductivity (gaseous and radiative), with κ_1, κ_2, q some positive constant.

4. **The chemical flux**:

$$\mathbf{Q}_{ch} = -d\rho \nabla Z,$$

where d is a diffusion constant.

5. **The rate function** $\phi(\theta, Z)$ is determined by the Arrhenius law:

$$\phi(\theta, Z) = KZ\theta^{\beta} e^{-\frac{E}{\theta}},$$

where E is the activation energy, β a non-negative number, and K the coefficient of rate of reactant. The coefficient λ is the difference in the heat of formation of the reactants.

We consider, for a given initial configuration Ω_0, and for each y in Ω_0, the initial conditions:

$$(\rho, v, \theta, Z)(y, 0) = (\rho_0, v_0, \theta_0, Z_0)(y) \tag{3}$$

We take, for each $t \geq 0$, the following dynamical boundary conditions:

$$\begin{cases} (\mathbf{s} + P\mathbf{I}) \cdot n = 0 \\ \dfrac{D\psi}{Dt} = 0, \end{cases} \tag{4}$$

where P is a pressure, modelling the external medium ($P = 0$ corresponds to the vacuum), and $\psi(x, t) = 0$ is the equation of the boundary S_t. The second equation in (4) tells us that S_t is a material boundary: at each time t it follows the motion of the same particles.

We consider also Neumann thermal and chemical boundary conditions:

$$\begin{cases} \mathbf{Q}_{th} = 0, \\ \mathbf{Q}_{ch} = 0. \end{cases} \tag{5}$$

We suppose also that the data $(\rho_0, v_0, \theta_0, Z_0)(y)$ have sufficient regularity (see below), and that ρ_0, θ_0 and Z_0, are non-negative everywhere.

The natural question is then to show that the problem (1)-(5) has a unique global solution, and study its behaviour at large times, under various conditions on the physical parameters.

The plan of the paper is as follows. In section 2, we give some ideas concerning the "state of the art" and difficulties of the 3d problem. Then (section 3), we consider the spherical case with a hard core, for the non-radiative case. Then we briefly analyze, in the monodimensional case, the competition between the radiative parts of p and e, and the conductivity χ.

2. The 3d problem

2.1. Local and global existence in the general case. The local-in-time existence of a solution can be proved [12] by using the Schauder fixed point method after Secchi [9], the only differences being the different dynamical boundary condition, and the coupling of the chemical process by the diffusion equation for $\bar{Z}$, and uniqueness is proved by using a simple "Gronwall" argument.

Concerning the global existence, the problem is largely open.

In the non-gravitational case and non-reactive case, when radiation is absent, W. M. Zajączkowski and E. Zadrzyńska [5] have recently proved the global existence and stability for small perturbations of a spherical equilibrium.

Even more recently, G. Ströhmer and W. M. Zajączkowski [6] have extended this result to the barotropic gravitational case.

It is easy to realize that, in the gravitational case, a serious difficulty comes from the non-definite-positiveness of the energy.

In fact, by using the equations of motion, a simple computation gives the conservation of the total energy:

$$\forall t \geq 0, \quad E(t) = E_c(t) + E_{th}(t) + E_{ch}(t) + E_P(t) - E_g(t) = E(0),$$

where: $E_c = \int_{\Omega_t} \frac{1}{2}\rho v^2 \ dx$ is the kinetic contribution, $E_{th} = \int_{\Omega_t} \rho e \ dx$ is the thermal and radiating contribution, $E_{ch} = \int_{\Omega_t} \lambda\rho Z \ dx$ is the chemical contribution, $E_P = P\ |\Omega_t|$, is the contribution of the external pressure (positive if $P > 0$), and $E_g = \frac{G}{2}\int_{\Omega_t}\int_{\Omega_t} \frac{\rho(x,t)\rho(y,t)}{|x-y|} \ dx \ dy$, is the gravitational energy.

It appears that the gravitational contribution has the wrong sign, so, a bound on $E(t)$ is not helpful to get individual ones: in particular, singularities can appear by some local "pinching" of the free boundary.

Let us mention that one has a (partial) global result due to Solonnikov [16], with a "reduced gravitation" (G small), which can be applied in a model interesting for astrophysicists (barotropic Eddington model). Initially, it requires a surface tension on the boundary, but it can be adapted to our situation [13].

In the radiating case, an extra difficulty comes from the high powers in θ for the state functions p, e, χ.

2.2. Some blow-up results for a confined star with positive energy. Let us suppose for a moment that the system (1) has a unique classical solution when the external pressure is zero (external vacuum), and that the fluid remains in a bounded region of $\mathbf{R}^3$, for any $t > 0$, then one has the simple blow-up result, in the spirit of Makino-Perthame [11]:

THEOREM 1. *Let $T > 0$, and (ρ, v, θ, Z) the solution of (1)(3)(4)(5) for $P = 0$, and let:*

$$R(t) = max_{x,y\in\Omega_t}|x-y|,$$

be the maximal spatial extension of Ω_t. Suppose that there exists a positive radius R_m such that:

$$\forall t \geq 0: \quad R(t) \leq R_m.$$

Then, if:

1. *the energy E is positive, large enough,*
2. *the polytropic index satisfies $\gamma \geq \frac{4}{3}$,*
3. *the Stokes hypothesis $3\mu + 2\nu = 0$ is satisfied,*

the solution of (1) has to blow-up in a finite time T_c such that:

$$T_c \leq \frac{1}{2\mathcal{E}}\left(-I_0' + \sqrt{I_0'^2 + 2\mathcal{E}(MR_m^2 - 2I_0)}\right), \tag{6}$$

where $\mathcal{E}$ is the reduced energy:

$$\mathcal{E} = \int_{\Omega_0}\left(\frac{1}{2}\rho_0 v_0^2 + \rho_0 e_0\right) dx - \frac{G}{2}\int_{\Omega_0}\int_{\Omega_0} \frac{\rho_0(x)\rho_0(y)}{|x-y|} \ dx \ dy,$$

and I_0 and I'_0 are the following constants:

$$I_0 = \frac{1}{2}\int_{\Omega_0} |x|^2 \rho_0(x)\ dx, \qquad I'_0 = \int_{\Omega_0} \rho_0(x)\ \ (x \cdot \mathbf{v}_0(x))\ dx.$$

PROOF. If we define the inertia $I(t)$ by:

$$I(t) = \frac{1}{2}\int_{\Omega_t} |x|^2 \rho(x,t)\ dx,$$

then, by using integrations by parts, and, for any regular function f, the formula:

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)\ dx = \int_{\Omega_t} \left(\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f)\right)\ dx,$$

we compute the derivatives:

$$\begin{aligned}
\frac{d}{dt} I(t) &= \int_{\Omega_t} \rho(x \cdot \mathbf{v})\ dx, \\
\frac{1}{2}\frac{d^2}{dt^2} I(t) &= \int_{\Omega_t} \left(\frac{\partial}{\partial t} + \nabla \cdot (\mathbf{v} \cdot)\right) (\rho(x \cdot \mathbf{v}))\ dx, \\
&= \int_{\Omega_t} \left(\rho v^2 + x \cdot \left(\rho \frac{D\mathbf{v}}{Dt}\right)\right)\ dx \\
&= \int_{\Omega_t} \left(\rho v^2 + x \cdot (\nabla \mathbf{s} - \rho \nabla \Phi)\right)\ dx,
\end{aligned}$$

by using the equations of motion.

By a direct computation, one gets:

$$\int_{\Omega_t} x \cdot \nabla \mathbf{s}\ dx = \int_{S_t} P(x \cdot n)\ dS_t + \int_{\Omega_t} 3p\ dx - (3\mu + 2\nu)\int_{S_t} (n \cdot \mathbf{v})\ dS_t.$$

As $P = 0$, and by using the Stokes hypothesis, we get:

$$\int_{\Omega_t} x \cdot \nabla \mathbf{s}\ dx = \int_{\Omega_t} 3p\ dx.$$

The symmetry of Φ gives also:

$$\int_{\Omega_t} x \cdot (\rho \nabla \Phi)\ dx = \frac{G}{2}\int_{\Omega_t}\int_{\Omega_t} \frac{\rho(x,t)\rho(y,t)}{|x-y|}\ dx\ dy = E_g.$$

We obtain finally ("Virial theorem"):

$$\frac{1}{2}\frac{d^2}{dt^2} I(t) = \int_{\Omega_t} (\rho v^2 + 3p)\ dx - E_g.$$

Then, as $p = R\rho\theta + \frac{a}{3}\theta^4$, and:

$$E = \int_{\Omega_t} \left(\frac{1}{2}\rho v^2 + \rho e + \lambda \rho Z\right)\ dx - E_g,$$

we find:

$$\frac{1}{2}\frac{d^2}{dt^2} I(t) - E = \int_{\Omega_t} \left(\frac{1}{2}\rho v^2 + (3R - C_v)\rho\theta\right)\ dx - \int_{\Omega_t} \lambda \rho Z\ dx,$$

so, if $3R-C_v \geq 0$, which is equivalent, with the thermodynamical definitions $R = C_p - C_v$, and $\gamma = \frac{C_p}{C_v}$, to the condition $\gamma \geq \frac{4}{3}$, we have:

$$\frac{1}{2}\frac{d^2}{dt^2}I(t) - E \geq -\int_{\Omega_t} \lambda\rho Z \; dx.$$

As $\phi \geq 0$, and by using the fourth equation (1), and the boudary condition (2), one gets:

$$\frac{d}{dt}\int_{\Omega_t} \rho Z \; dx \leq 0,$$

so we obtain:

$$\frac{1}{2}\frac{d^2}{dt^2}I(t) \geq E - \int_{\Omega_0} \lambda\rho_0 Z_0 \; dx.$$

So, if E is large enough:

$$\mathcal{E} = E - \int_{\Omega_0} \lambda\rho_0 Z_0 \; dx > 0, \qquad I(t) \geq \mathcal{E}\; t^2 + I_0' \; t + I_0,$$

where $I_0' = \int_{\Omega_0} \rho_0(x \cdot v_0) \; dx$ and $I_0 = \frac{1}{2}\int_{\Omega_0} |x|^2\rho_0 \; dx$. But, by using the mass constraint, $I(t) \leq \frac{1}{2}MR_m^2$. So, for any $t \geq 0$, we have the inequality $E\; t^2 + I_0' \; t + I_0 \leq \frac{1}{2}MR_m^2$, which implies the bound (6). ∎

The physical interpretation of the theorem is the following: if the energy is positive ("tendancy to explosion") then either the solution becomes singular in finite time, or the domain expands without bounds into the space.

Now, we are interested in the energy repartition (between kinetic, thermal and gravitational), for a compressible fluid of positive energy, in the simplified situation of the perfect fluid: $\lambda = 0$ and $a = 0$.

We have the following partial result:

Theorem 2. *Let (ρ, v, θ) the solution of (1)(3)(4)(5), with $P = 0$, for a perfect compressible fluid such that the polytropic index γ satisfies $\gamma < \frac{5}{3}$, and the Stokes hypothesis $3\mu + 2\nu = 0$ holds. Suppose also that the (conserved) energy $E = \int_{\Omega_t}\left(\frac{1}{2}v^2 + e + \frac{1}{2}\Phi\right) dx$ is positive, and that the kinetic energy $E_c = \int_{\Omega_t}\frac{1}{2}v^2 dx$ is small enough, i.e.*

$$\exists T_0 > 0,\; \exists \xi \in (0,1) :\; \forall t > T_0,\; E_c \leq \xi E.$$

Then the solution of (1) has to blow-up in a finite time T_c such that:

$$T_c \leq \sqrt{\frac{1}{(1-\xi)E}\left(\int_{\Omega_0} x^2\rho_0(x)\; dx - 2\int_{\Omega_0}(x\cdot v_0)\;\rho_0(x)\; dx + 2E\right)}. \tag{7}$$

Proof. Following Zhoupin Xin[1] [17], we consider the modified functional:

$$I(t) = \int_{\Omega_t} (x-(t+1)v)^2\rho(x,t)\; dx + 2(t+1)^2\int_{\Omega_t}\rho(x,t)\left(e + \frac{1}{2}\Phi\right)\; dx. \tag{8}$$

We have:

$$I(t) = \int_{\Omega_t} x^2\rho(x,t)\; dx - 2(t+1)\int_{\Omega_t}(x\cdot v)\;\rho(x,t)\; dx + 2(t+1)^2\int_{\Omega_t}\rho(x,t)\epsilon(x,t)\; dx, \tag{9}$$

[1]I thank Prof. Song Jiang for communicating me the reference [17].

where $\epsilon = \frac{1}{2}v^2 + e + \frac{1}{2}\Phi$, is the total energy density. From (9), we compute:

$$\frac{d}{dt}I(t) = \sum_{i=1}^{3} \frac{d}{dt}I_i(t).$$

We have:

$$\begin{aligned}
\frac{d}{dt}I_1(t) &= \frac{d}{dt}\left(\int_{\Omega_t} x^2\rho\ dx\right) = 2\int_{\Omega_t}(x\cdot v)\ \rho\ dx,\\
\frac{d}{dt}I_2(t) &= \frac{d}{dt}\left(-2(t+1)\int_{\Omega_t}(x\cdot v)\ \rho\ dx\right)\\
&= -2\int_{\Omega_t}(x\cdot v)\ \rho\ dx + 2(t+1)\bigg(-3\int_{\Omega_t} p\ dx + (2\nu+3\mu)\int_{S_t}(v\cdot n)\ dS\\
&\quad + \int_{\Omega_t}\rho\ (x\cdot\nabla\Phi)\ dx - \int_{\Omega_t}\rho v^2\ dx\bigg).
\end{aligned}$$

By using the Stokes hypothesis together with the following symmetry property of Φ:

$$\int_{\Omega_t}\rho\ (x\cdot\nabla\Phi)\ dx = -\frac{1}{2}\int_{\Omega_t}\rho\ \Phi\ dx,$$

we get:

$$\frac{d}{dt}I_2(t) = -2\int_{\Omega_t}(x\cdot v)\ \rho\ dx - 6(t+1)\int_{\Omega_t} p\ dx - (t+1)\int_{\Omega_t}\rho\ \Phi\ dx - 2(t+1)\int_{\Omega_t}\rho v^2\ dx.$$

One has also:

$$\frac{d}{dt}I_3(t) = \frac{d}{dt}\left(2(t+1)^2\int_{\Omega_t}\rho\epsilon\ dx\right) = 4(t+1)\int_{\Omega_t}\rho\epsilon\ dx.$$

So we get finally:

$$\frac{d}{dt}I(t) = 2(t+1)\int_{\Omega_t}\left(2\rho e - 3p + \frac{1}{2}\Phi\right)\ dx = 2(t+1)\int_{\Omega_t}\left((5-3\gamma)e + \frac{1}{2}\Phi\right)\ dx. \quad (10)$$

If $5-3\gamma<0$, we have $\frac{d}{dt}I(t)\le 0$. So: $I(t)\le I(0)$, where $I(0)>0$ if E_c is small enough. But we have also:

$$I(t) \ge 2(t+1)^2(E-E_c).$$

Finally, if E_c is small enough, say $E_c \le \xi E$, we get finally:

$$(1-\xi)Et^2 \le I(0),$$

which gives the rough bound (7), for the blow-up time T_c. ■

If the analysis is restricted to the spherical or monodimensional geometry, one can get global results and describe, modulo some extra hypothesis, the asymptotic states of the system for large time, covering the possibilities of the physical stellar evolution [1]: asymptotically stable stationary state, expansion, and gravitational collapse.

3. The spherical symmetry. A favorite model for astrophysicists studying (classical) compact stellar objects is the spherical symmetry, which shares some physical properties with the 3d case, and which is extensively considered in the physical (a standard reference is Chandrasekhar [2]) and mathematical literature (Fujita-Yashima, Benabidalla [18], Hoff [19]).

For technical reasons, let us suppose that, if r is the radial variable, we have $R_0 \leq r \leq R(t)$, where $R_0 > 0$ is a "hard core cut-off" for the star, and we restrict the analysis to the non-radiative situation:

$$e(\theta) = C_v\theta, \quad p(\rho, \theta) = R\rho\theta \quad \chi = \text{Cst}.$$

To take into account the free-boundary in a simple manner, we consider the spherical version of (1) in the lagrangian version, by using a mass variable defined by:

$$x = \int_{R_0}^{r} \rho(s,t)\ s^2\ ds,$$

with $M = \int_{R_0}^{R(t)} \rho(s,t)\ s^2\ ds$. So, the inverse transformation is given by:

$$r(x,t) = \left[R_0^3 + 3\int_0^x \frac{d\xi}{\rho(\xi,t)}\right]^{1/3}.$$

The problem is then posed in the fixed domain $[0, M]$.

If we denote by $u = \frac{1}{\rho}$ the specific volume, the system we have to solve, for (u, v, θ, Z), for each $x \in [0, M]$, is the following:

$$\begin{cases} u_t - (r^2 v)_x = 0, \\ v_t + r^2 p_x = \mu r^2 \left(\dfrac{(r^2 v)_x}{u}\right)_x - G\frac{x}{r^2}, \\ E_t + (r^2 v\ p)_x = \left(\mu \dfrac{r^2 v (r^2 v)_x}{u} - 4\eta \left(r v^2\right)_x + \chi \dfrac{r^4 \theta_x}{u} + \lambda d \dfrac{r^4 Z_x}{u^2}\right)_x \\ Z_t + \phi(\theta, Z) = d\left(\dfrac{r^4 Z_x}{u^2}\right)_x, \end{cases} \tag{11}$$

where E is the total energy: $E = \frac{1}{2}v^2 + C_v\theta + \lambda Z - \frac{Gx}{r}$, μ and η are two viscosity coefficients, satisfying the stability condition:

$$3\mu - 4\eta > 0, \tag{12}$$

and σ is the stress: $\sigma = -\frac{R\theta}{u} + \mu\frac{(r^2 v)_x}{u} - 4\eta\frac{v}{r}$.

The initial data are:

$$(u, v, \theta, Z)(x, 0) = (\frac{1}{\rho_0}, v_0, \theta_0, Z_0)(x), \tag{13}$$

together with the boundary conditions on the sphere $\{|x| = M\}$:

$$\begin{cases} \sigma(M,t) + P = 0, \\ (\theta_x + \mathbf{k}\theta)(M,t) = 0, \\ Z_x(M,t) = 0, \end{cases} \tag{14}$$

and at the core $x = 0$:

$$\begin{cases} v(0,t) = 0, \\ (\theta_x - \mathbf{k}\theta)(0,t) = 0, \\ Z_x(0,t) = 0, \end{cases} \tag{15}$$

where $\mathbf{k}$ is a non-negative coefficient (if $\mathbf{k} = 0$, we recover the above Neumann boundary condition).

We assume finally the compatibility conditions:

$$\begin{cases} v_0(0) = 0, \\ \sigma_0(M) + P = 0, \\ (\theta_{0x} - \mathbf{k}\theta_{0x}(0) = (\theta_{0x} + \mathbf{k}\theta_{0x}(M) = 0, \\ Z_{0x}(0) = Z_{0x}(M) = 0. \end{cases} \tag{16}$$

3.1. Global existence and stationary solution. If $Q_T \equiv [0, M] \times [0, T]$, we consider classical Hölder functional space: If:

$$H_\nu(u) = \sup_{Q_T} |u| + \sup_{x \neq y \ ; \ t \in [0,T]} \frac{|u(x,t) - u(y,t)|}{|x-y|^\nu} + \sup_{x \in [0,M] \ ; \ t \neq s} \frac{|u(x,t) - u(x,s)|}{|t-s|^{\nu/2}},$$

we set:

$$\mathcal{B}^{\nu,\nu/2}(Q_T) = \{u \in C^0(Q_T) : \ H_\nu(u) < \infty\},$$

with the norm $\|u\|_\nu = H_\nu(u)$, together with:

$$\mathcal{B}^{1+\nu} = \{u \in \mathcal{B}^{\nu,\nu/2}(Q_T) : \ u_t, \ u_x \in \mathcal{B}^{\nu,\nu/2}(Q_T)\},$$

$$\mathcal{B}^{2+\nu} = \{u \in \mathcal{B}^{\nu,\nu/2}(Q_T) : \ u_t, \ u_x, \ u_{xx} \in \mathcal{B}^{\nu,\nu/2}(Q_T)\},$$

with natural norms $\|u\|_{1+\nu} = \|u\|_\nu + \|u_t\|_\nu + \|u_x\|_\nu$, and $\|u\|_{2+\nu} = \|u\|_\nu + \|u_t\|_\nu + \|u_x\|_\nu + \|u_{xx}\|_\nu$, where $0 < \nu < 1$.

We finally denote by $\|\cdot\|$ the L^2 norm.

We make the following hypothesis on the data:

$$\begin{cases} 0 < c_0^{-1} \leq u_0(x) \leq c_0 < \infty, \\ 0 < c_0^{-1} \leq \theta_0(x) \leq c_0 < \infty, \\ 0 \leq Z_0(x) \leq 1, \end{cases} \tag{17}$$

where c_0 is a positive constant.

By using an iterative scheme and the Banach fixed point theorem, one can show that a unique solution of the problem (11)-(16) exists, at least on a finite interval $[0, T]$.

Moreover, taking benefit of the presence of the hard core, one proves the global-in-time existence of the solution for the initial boundary value problem (11)-(16).

THEOREM 3. *Let the initial data satisfy the conditions (16), (17), and:*

$$u_0 \in C^{1+\nu}[0, M], \quad v_0, \theta_0, Z_0 \in C^{2+\nu}[0, M],$$
$$\|v_0, \theta_0, Z_0\|_{H^1} \leq C_0, \ \textit{for } 0 < \nu < 1.$$

The problem (11)-(16) has a classical solution $(u, v, \theta, Z) \in \mathcal{B}^{1+\nu}(Q_T) \times (\mathcal{B}^{2+\nu})^3$ *such that for any* $T > 0$ *the following estimates hold:*

$$
\begin{cases}
C_1^{-1} \le u \le C_1, \quad C_2^{-1} \le \theta \le C_2, \quad -C_2 \le v \le C_2, \quad 0 \le Z \le 1, \\
\|u_t, u_x\|^2(t) + \int_0^t \|u_t, u_x\|^2(s) ds \le C_1, \\
\|v_x, \theta_x, Z_x\|^2(t) + \int_0^t \|v_{xx}, \theta_{xx}, Z_{xx}, v_x, \theta_x, Z_x, v_t, \theta_t, Z_t\|^2(s) ds \le C_1, \\
\|u\|_{1+\nu}, \ \|v, \theta, Z\|_{2+\nu} \le C_2,
\end{cases} \tag{18}
$$

where $C_1(C_0)$ *(independent of* T*) and* $C_2(C_0, T)$ *are two positive constants.*

A sketch of the proof (see [15] for the details) is as follows.

By using the philosophy of Kazhikhov-Shelukhin (see [10] [18] [20]), one has to prove upper and lower bounds for the density u.

One gets first an explicit representation for u:

LEMMA 1. *The following formula holds:*

$$
u(x,t) = u_0(x) \left(\frac{r(M,t)}{r_0(M)} \right)^{\beta} \Phi(x,t) \exp\{-\Psi(x,t)\}, \tag{19}
$$

where:

$$
\beta = \frac{4\eta}{\mu},
$$

$$
\Psi(x,t) = \frac{P}{\mu} t + \frac{1}{\mu} \int_0^t \int_x^M \left(\frac{2v^2(y,s)}{r^3(y,s)} + \frac{Gy}{r^4(y,s)} \right) dy \ ds + \frac{1}{\mu} \int_x^M \left(\frac{v(y,t)}{r^2(y,t)} - \frac{v_0(y)}{r_0^2(y)} \right) dy,
$$

$$
\Phi(x,t) = 1 + \frac{R}{\mu u_0(x)} \int_0^t \theta(x,s) \left(\frac{r_0(M)}{r(M,s)} \right)^{\beta} \exp\{\Psi(x,s)\} \ ds.
$$

Now, by using the boundary conditions, one checks the following estimates:

LEMMA 2.

$$
\int_0^M u(x,t) \ dx = |\Omega_t|, \tag{20}
$$

where $|\Omega_t|$ *is the renormalized volume of the gaseous domain:* $|\Omega_t| = 1/3(r^3(M,t) - R_0^3)$;

$$
\int_0^M \left(\frac{1}{2} v^2 + e + \lambda Z \right) \ dx + |\Omega_t| P \le E_0, \tag{21}
$$

where:

$$
E_0 = \int_0^M \left(\frac{1}{2} v_0^2 + e_0 + \lambda Z_0 \right) \ dx + \int_0^M G \cdot x \left(\frac{1}{R_0} - \frac{1}{r_0} \right) \ dx + |\Omega_0| P;
$$

$$
U(t) + \int_0^t (V(s) + W(s)) \ ds \le E_1, \tag{22}
$$

where:

$$
U(t) = \int_0^M \left(\frac{1}{2} v^2 + C_v(\theta - \log\theta - 1) + R(u - \log u - 1) + \lambda Z \right) \ dx,
$$

$$
V(t) = \int_0^M \left(\frac{r^4 \chi \theta_x^2}{u\theta^2} + \frac{2}{3}\mu\zeta \frac{((r^2 v)_x)^2}{\theta u} + \frac{6\mu\zeta(3-4\zeta)}{3-2\zeta} \frac{v^2 u}{\theta r^2} \right) dx,
$$

with $\zeta = \frac{3\mu-4\eta}{4\mu} > 0$,

$$W(t) = \int_0^M \frac{\lambda\phi(\theta, Z)}{\theta}\ dx,$$

$$E_1 = \int_0^M \left(\frac{1}{2}v_0^2 + C_v(\theta_0 - \log\theta_0 - 1) + R(u_0 - \log u_0 - 1) + \lambda Z_0\right)\ dx + \left(1 + 2\frac{R}{P}\right)E_0;$$

$$\int_0^M Z(x,t)\ dx + \int_0^t \int_0^M \phi(\theta, Z)\ dx\ ds = \int_0^M Z_0(x)\ dx; \tag{23}$$

$$0 \le Z(x,t) \le 1. \tag{24}$$

Then, by using convexity arguments, one can prove uniform-in-time upper and lower bounds for u: $0 < C_1^{-1} \le u(x,t) \le C_1$, and, applying the maximum principle to the thermal equation, one gets also a lower bound for θ: $\theta(x,t) \ge C_2^{-1}(T)$.

If we consider the following global quantities:

$$X(t) = \int_0^M r^4\sigma_x^2\ dx, \quad Y(t) = \int_0^M \frac{((r^2v)_x)^2}{u}\ dx, \quad \Xi(t) = \int_0^M \frac{r^4\theta_x^2}{u}\ dx,$$
$$\Gamma(t) = \int_0^M u\sigma^2\ dx, \quad J(t) = \int_0^M W^2\ dx \equiv \int_0^M \left(\frac{1}{2}v^2 + e + \lambda Z\right)^2\ dx,$$
$$H(t) = \int_0^M \frac{r^4 Z_x^2}{u^2}\ dx, \quad \Theta(t) = \int_0^M \phi(\theta, Z)\ Z\ dx, \quad \Delta(t) = \int_0^M \frac{r^4}{u}v^2v_x^2\ dx,$$

one gets first a uniform bound for the chemical part:

Lemma 3. *The following estimate holds:*

$$\frac{1}{2}\int_0^M Z^2\ dx + \int_0^t H(s)\ ds + K\int_0^t \Theta(s)\ ds \le E_0. \tag{25}$$

In fact, if we multiply the last equation in (11) by Z, integrate on $[0, M]$ and use boundary conditions, we get:

$$\frac{1}{2}\frac{d}{dt}\int_0^M Z^2\ dx + H(t) + K\Theta(t) = 0,$$

which implies (25).

By using similar techniques with suitable multiplicators, one gets:

Lemma 4.

$$\int_0^t Y(s)\ ds \le C(T). \tag{26}$$

$$\begin{cases} \Gamma(t) + J(t) + \int_0^t \Xi(s)\ ds + \int_0^t \Delta(s)\ ds \le C(T), \\ \int_0^t \int_0^M u_t^2\ dx\ ds \le C(T), \\ \int_0^t \int_0^M v_t^2\ dx\ ds \le C(T), \end{cases} \tag{27}$$

where $C(T)$ *is a positive constant.*

We have seen that there exists a positive T such that the problem (11) has a unique solution (u,v,θ,Z), for $t\in[0,T]$. It is clear that we can choose this T arbitrarily large, provided that the norm of (u,v,θ,Z) is finite in the prescribed space, i.e. $(u,v,\theta,Z)\in\mathcal{B}^{1+\nu}(Q_T)\times(\mathcal{B}^{2+\nu})^3$. This property is easily checked by inspection.

3.2. Asymptotic behaviour. Let us consider the solutions $(\bar v(x)=0,\bar u(x),\bar\theta(x),\bar Z(x))$ of the stationary version of (11), which reads:

$$\begin{cases} p_x=-G\dfrac{x}{r^4}, \\ \left(r^4\chi\dfrac{\theta_x}{u}\right)_x+\lambda\phi(\theta,Z)=0, \\ \left(r^4 d\dfrac{Z_x}{u^2}\right)_x-\phi(\theta,Z)=0. \end{cases}\tag{28}$$

If $\mathbf{k}=0$, the first equation tells us that $\bar\theta(x)$ cannot be identically zero.

Now, if we integrate the second equation, we find $\bar Z(x)=0$, for each $x\in[0,M]$, which implies in turn that $\theta(x)=\bar\theta$, where $\bar\theta$ is a positive constant: any stationary solution is isothermal and chemically inactive.

Now, if $\mathbf{k}>0$, one gets easily that $\bar\theta(x)$ is identically zero.

For the stationary density, we have:

PROPOSITION 1. *Let:*

$$\begin{cases} \dfrac{d\bar r}{dx}=\dfrac{\bar u}{\bar r^2}, \\ \dfrac{d\bar p}{dx}=-\dfrac{Gx}{\bar r^4}, \end{cases}\tag{29}$$

be the stationary problem for the density $\bar u$ and the lagrangian radius $\bar r$, for $x\in[0,M]$, together with the boundary conditions:

$$\begin{cases} \bar r(0)=R_0, \\ \bar p(M)=P. \end{cases}\tag{30}$$

Then:

(i) *if $\mathbf{k}=0$, this system has a unique solution $\bar u(x)>0$, $\bar r\geq R_0$, for any $\bar\theta>0$, $P>0$, provided that R_0 is large enough,*

(ii) *if $\mathbf{k}>0$, the system has the unique trivial solution $\bar u(x)=0$, $\bar r=R_0$, $\bar\theta=0$.*

The role of the hypothesis on the radius is to avoid possible multiple stationary solutions, and the trivial solution corresponds to the "gravitationally collapsed" solution.

As one expects, for thermally insulated boundaries ($\mathbf{k}=0$), the solution converges toward the associated stationary solution given in proposition 1[2], and for Fourier conditions ($\mathbf{k}>0$), the solution converges toward the trivial stationary solution.

For the dissipative case ($\mathbf{k}>0$), the fluid looses its energy across the exterior boundary, and the gravitation forces the solution to concentrate on the surface of the core.

---

[2]In the literature [1], this corresponds to radial stellar pulsations, the only vibrational modes allowed by the symmetry.

To prove this, one needs stronger estimates than those given in the above section (especially lemma 7), to obtain bounds independent of time [15], and we obtain:

THEOREM 4. (i) *If* $\mathbf{k} = 0$*, and if* R_0 *is large enough, the solution* (u, v, θ, Z) *converges uniformly to the stationary solution* $(\bar{u}, \bar{r}, \bar{v} = 0, \bar{\theta}, \bar{Z} = 0)$ *as* $t \to +\infty$*, where* $\bar{u}$ *is the solution of the stationary problem (29)-(30), and* $\bar{\theta}$ *is a positive constant given by the implicit algebraic equation:*

$$\int_0^M \left(C_v\bar{\theta} - \frac{Gx}{\bar{r}} + P\bar{u} \right) \, dx = \int_0^M \left(\frac{1}{2}v_0^2 + C_v\theta_0 + \lambda Z_0 - \frac{Gx}{r_0} + Pu_0 \right) \, dx, \tag{31}$$

where: $\bar{r}(x) = (R_0^3 + 3\int_0^x \bar{u}(y)\ dy)^{1/3}$*, and:* $r_0(x) = (R_0^3 + 3\int_0^x u_0(y)\ dy)^{1/3}$*.*

(ii) *If* $\mathbf{k} > 0$*, the solution* (u, v, θ, Z) *converges toward the trivial stationary solution* $(\bar{u} = 0, \bar{r} = R_0, \bar{v} = 0, \bar{\theta} = 0, \bar{Z} = 0)$ *as* $t \to +\infty$*.*

In fact, when $\mathbf{k} = 0$, one can check easily that, if R_0 large enough, equation (31) has a unique solution, by using a regular perturbative expansion with respect to R_0. ∎

4. The monodimensional model. We said in the introduction that a difficulty of our problem was the strong non-linearities into the state functions. A good "toy model" for this is the one-dimensional situation, in the spirit of Dafermos, Hsiao, Kawohl, Jiang [27] [21] [20] [22] .

Although rather degenerate, this geometrical situation is physically interresting. In fact, it can be considered as a simplified model for some large-scale structures described in the astrophysical litterature [3] under the name of "Zeldovitch's pancakes".

If x is the mass variable, $u(x, t)$ the specific volume, $v(x, t)$ the velocity, $\theta(x, t)$ the temperature, and $Z(x, t)$ the fraction of reactant, the (lagrangian) system to be solved is now:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma_x + G\left(x - \dfrac{1}{2}M\right) = 0, \\ e_t - \sigma v_x + Q_x - \lambda\phi(\theta, Z) = 0, \\ Z_t - \left(\dfrac{d}{u^2}Z_x\right)_x + \phi(\theta, Z) = 0, \end{cases} \tag{32}$$

for $t \geq 0$ and $x \in [0, M]$, where M is the mass of the slab.

Recall that $p(u, \theta) = R\frac{\theta}{u} + \frac{a}{3}\theta^4$ is the pressure, $e(u, \theta) = C_v\theta + au\theta^4$ is the internal energy, $\sigma(u, v, \theta) = -p + \nu\frac{v_x}{u}$ is the stress, and $Q(u, \theta)$ is the thermo-radiative flux, to be specified below.

The last term in the second equation (32) is the gravitational contribution if $G > 0$ or the Coulomb contribution if $G < 0$. Its specific expression has been chosen in such a way that $x = \frac{1}{2}M$ is a symmetry center for the slab.

We consider, for each x in $[0, M]$, the initial conditions:

$$(u, v, \theta, Z)(x, 0) = (u_0, v_0, \theta_0, Z_0)(x). \tag{33}$$

We take, for each $t \geq 0$, the dynamical boundary conditions:

$$\begin{cases} \sigma(0,t) = -P, \\ \sigma(M,t) = -P, \end{cases} \tag{34}$$

where P is an exterior pressure, modelling the external medium ($P = 0$ corresponds to the vacuum).

We consider also Neumann thermal boundary conditions:

$$\begin{cases} Q(0,t) = 0, \\ Q(M,t) = 0. \end{cases} \tag{35}$$

Then, we consider also the chemical boundary conditions:

$$\begin{cases} Z_x(0,t) = 0, \\ Z_x(M,t) = 0. \end{cases} \tag{36}$$

We suppose also that the data $(u_0, v_0, \theta_0, Z_0)(x)$ have sufficient regularity (see below), and that u_0, θ_0 and Z_0 are positive on $[0, M]$.

Moreover, we impose the following symmetry conditions, for $0 \le x \le \frac{1}{2}M$:

$$\begin{cases} (u, u_0, \theta, \theta_0, Z, Z_0)\left(\dfrac{1}{2}M + x, t\right) = (u, u_0, \theta, \theta_0, Z, Z_0)\left(\dfrac{1}{2}M - x, t\right), \\ (v, v_0)\left(\dfrac{1}{2}M + x, t\right) = -(v, v_0)\left(\dfrac{1}{2}M - x, t\right). \end{cases} \tag{37}$$

Let us describe now the remaining terms in (32).

The flux $Q(u, \theta)$ is given by $Q = -\frac{\chi(u,\theta)}{u}\theta_x$, where the conductivity χ is:

$$\chi = \kappa_1 + \kappa_2 u \theta^q, \tag{38}$$

where the coefficients κ_1, κ_2 and q are positive[3].

We call ν the (constant) viscosity coefficient[4], and $\lambda \ge 0$ and $d \ge 0$ two "chemical" constants.

Finally, the function ϕ mimics the simplest one-order Arrhenius kinetics (see [4]):

$$\phi(\theta, Z) = AZ\theta^{\beta} e^{-\frac{E}{B\theta}}, \tag{39}$$

where $A,\ \beta,\ B,\ E$ are given positive constants.

Our task is now to show that the problem (32)-(37) has a unique global solution, and study its behaviour at large times, under various conditions on the physical parameters.

At this point, we suspect that the exponent q in (38), plays a major role in the a priori estimates.

In fact, several authors [31] [27] [21] [22] [25] [26] considered recently analogous problems for general fluids or solids, under various growth constraints.

Among these conditions the more general are (see [22] [23] [24]):

$$a(1+\theta)^{r+1} \le e(u, \theta) \le a'(1+\theta)^{r+1},$$

$$b(1+\theta)^{r+1} u^{-1} \le p(u, \theta) \le b'(1+\theta)^{r+1} u^{-1},$$

$$c(1+\theta)^q \le \chi(u, \theta) \le c'(1+\theta)^q,$$

[3]The values ($\kappa_2 = 0,\ a = 0$) correspond to the perfect gas.

[4]To clarify the exposition, we consider a unique viscosity coefficient.

where $r \in [0,1]$ and $q \geq r+1$. Clearly, these constraints are not satisfied for radiating fluids for which $r=3$, but a result of our analysis will be that this value is allowed if q is large enough.

We use the notations $(C^r[0,M], \|\cdot\|_r)$ and $(C^{r,r/2}(Q_T), |||\cdot|||_r)$, with $Q_T = [0,M] \times [0,T]$, for the usual Hölder spaces (see [10]).

Our main result for the global existence is the following.

THEOREM 5. *Assume that* $u_0,\ u_{0x},\ v_0,\ v_{0x},\ v_{0xx},\ \theta_0,\ \theta_{0x},\ \theta_{0xx},\ Z_0,\ Z_{0x},\ Z_{0xx}$ *are in* $C^r[0,M]$*, for some* $0<r<1$*. Suppose that* $u_0,\ \theta_0,\ Z_0$ *are positive on* $[0,M]$*, that the compatibility conditions hold between boundary conditions and initial data. Then, if* $q \geq 4$*, there exists a unique solution* $(u(x,t), v(x,t), \theta(x,t), Z(x,t))$ *to the problem (32)-(37) such that* $u(x,t)>0,\ \theta(x,t)>0,\ Z(x,t)>0,$ *on* $[0,M]\times[0,\infty)$*, that:*

$$(u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}, Z, Z_t, Z_x, Z_{xx}) \in (C^{r,r/2}(Q_T))^{16},$$

and that:

$$(u_{tt}, v_{xt}, \theta_{xt}, Z_{xt}) \in (L^2(Q_T))^4.$$

4.1. Global existence. As in [31], [27], [21] or [22], the proof of theorem 1 is based on a priori estimates, and is completed by establishing the following result.

THEOREM 6. *Suppose that the problem (32)-(37) have at least a classical solution:*

$$(u(x,t), v(x,t), \theta(x,t), Z(x,t))\,.$$

Then the functions $(u, v, \theta, Z, v_x, \theta_x, Z_x)$ *can be bounded on* $C^{r,r/2}(Q_T)$*, with* $r = 1/3$*:*

$$|||u|||_{1/3} + |||v|||_{1/3} + |||\theta|||_{1/3} + |||Z|||_{1/3} + |||v_x|||_{1/3} + |||\theta_x|||_{1/3} + |||Z_x|||_{1/3} \leq C,$$

where C *depends only on* T*, the physical parameters of the problem, and the data.*

As usual, we begin with some conservation laws, leading to a priori estimates.

LEMMA 5. *The following relations hold, for any* $t \geq 0$*:*

$$\int_0^M Z(x,t)\ dx + \int_0^t \int_0^M \phi(\theta, Z)(x,s)\ ds\ dx = \int_0^M Z_0(x)\ dx; \tag{40}$$

$$\int_0^M \left[\frac{1}{2}v^2 + e + \lambda Z + f(x)u\right] dx = E_0, \tag{41}$$

where $f(x) \equiv P + \frac{1}{2}Gx(M-x)$*, and* $E_0 = \int_0^M \left[\frac{1}{2}v_0^2 + e_0 + \lambda Z_0 + f(x)u_0\right] dx$*;*

$$\theta(x,t),\ Z(x,t) > 0,\ \text{on } [0,M]\times[0,\infty); \tag{42}$$

$$\Phi(t) + \int_0^t \Psi(t)\ dt \leq \mathbf{C}, \tag{43}$$

where:

$$\Phi(t) = \int_0^M \left[R(u - \log\ u - 1) + C_v(\theta - \log\ \theta - 1)\right]\ dx,$$

$$\Psi(t) = \int_0^M \left(\frac{v_x^2}{u\theta} + \chi\frac{\theta_x^2}{u\theta^2} + \lambda\frac{\phi(\theta,Z)}{\theta}\right)\ dx,$$

and $\mathbf{C}$ *is a positive constant, independent of* t *provided* $P>0$*; and*

$$\frac{1}{2}\int_0^M Z^2(x,t)\ dx + \int_0^t\int_0^M \frac{d}{u^2}Z_x^2\ dx\ ds + \int_0^t\int_0^M Z\phi(\theta,Z)(x,s)\ ds\ dx$$
$$= \frac{1}{2}\int_0^M Z_0^2(x,t)\ dx. \qquad (44)$$

PROOF. The relation (40) is obtained by integrating the fourth relation (32) on $[0,M]\times(0,t)$, by using (36).

By multiplying by v the second relation in (32) we find the conservation law:

$$\left(\frac{1}{2}v^2 + e + \lambda Z + G(x-\frac{1}{2}M)r\right)_t = \left(\sigma v - Q + \frac{\lambda d}{u^2}Z_x\right)_x, \qquad (45)$$

where $r = r(x,t)$ is the Lagrangian position, defined by $\frac{\partial}{\partial t}r(x,t) = v(x,t)$.

By using (32), one sees that:

$$\left(\int_0^M G(x-\frac{1}{2}M)r(x,t)\ dx\right)_t = \left(\int_0^M f(x)\ u(x,t)\ dx\right)_t.$$

Now, by integrating (45) on $[0,M]$, and using (33)-(36), we obtain (41).

To get the positivity of θ in (42), we apply the maximum principle to the third equation (32), rewritten as:

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\nu}{u}v_x^2 = \left(\frac{\chi}{u}\theta_x\right)_x + \lambda\phi(u,\theta,Z), \qquad (46)$$

together with (35), and we use the positivity of θ_0.

To get the positivity of Z, we apply the same principle to the fourth equation (32), together with (36), and we use the positivity of Z_0.

To get (43), we multiply (46) by θ^{-1}:

$$e_\theta\frac{\theta_t}{\theta} + p_\theta u_t = \frac{\nu}{u\theta}v_x^2 + \frac{\chi}{u\theta^2}{\theta_x}^2 + \left(\frac{\chi}{u\theta}\theta_x\right)_x + \lambda\frac{\phi(u,\theta,Z)}{\theta}. \qquad (47)$$

A standard thermodynamical computation gives S the entropy:

$$S(u,\theta) = R\log u + C_v\log\theta + \frac{4}{3}au\theta^3 + S_0. \qquad (48)$$

By using the thermodynamical formulae $S_\theta = \frac{e_\theta}{\theta}$, and $S_u = p_\theta$, and by integrating (46) on $[0,M]\times[0,t]$, we get:

$$\int_0^M\int_0^t\left(\frac{\nu}{u\theta}v_x^2 + \frac{\chi}{u\theta^2}{\theta_x}^2 + \lambda\frac{\phi(u,\theta,Z)}{\theta}\right)\ ds\ dx = \int_0^M S(x,t)\ dx - \int_0^M S_0(x)\ dx.$$

So we obtain the identity:

$$\int_0^M\int_0^t\left(\frac{\nu}{u\theta}v_x^2 + \frac{\chi}{u\theta^2}{\theta_x}^2 + \lambda\frac{\phi(u,\theta,Z)}{\theta}\right)\ ds\ dx$$
$$+\int_0^M (R(u-\log u-1) + C_v(\theta-\log\theta-1))\ dx$$
$$= \int_0^M\left(R(u-1) + C_v(\theta-1) + \frac{4}{3}au\theta^3\right)\ dx + \int_0^M\left(Ru_0 + C_v\theta_0 + \frac{4}{3}au{\theta_0}^3\right)\ dx.$$

By using the estimate (41), we bound the two first terms in the first integral of the rhs, when $P > 0$. For the last one, we use Cauchy-Schwarz:

$$\int_0^M u\theta^3\ dx \le \Big(\int_0^M u\theta^2\ dx\Big)^{1/2}\Big(\int_0^M u\theta^4\ dx\Big)^{1/2} \le \Big(\int_0^M u\ dx\Big)^{1/4}\Big(\int_0^M u\theta^4\ dx\Big)^{3/4},$$

and we obtain (43), by using once more (41).

The relation (44) is obtained by multiplying by Z the fourth relation (32) , integrating the result on $[0, M] \times (0, t)$, by using (36) and (40). ■

REMARKS. 1. When $G > 0$ (attractive case) and $P > 0$, (41) gives a bound on $\|u\|_{L^1(0,M)}$ which does not depend on t.

2. When $G < 0$ (repulsive case), a bound on $\|u\|_{L^1(0,M)}$ is given again by (41), provided that $P > -\frac{G}{2}$. ■

Now, following [22], we need some estimates for the mean temperature.

LEMMA 6. *If $q \ge 4$, we have:*

$$\int_0^t \max_{x\in[0,M]} \theta^\alpha(x,s)\ ds \le C, \tag{49}$$

for $1 \le \alpha \le 8$.

PROOF. We have, for any $r \ge 1$:

$$\theta^r(x,s) \le \int_0^M \theta^r(y,s)\ dy + r\int_0^M \theta^{r-1}|\theta_x|\ dx\ ds,$$

when we used the well known fact that:

$$\forall f \in C^0(0,M),\ \exists y(t) \in [0,M]:\quad f(y(t),t) = \int_0^M f(z,t)\ dz.$$

If we suppose that $r \le 4$, we get, by using Cauchy-Schwarz:

$$\theta^r(x,s) \le C + C\int_0^M \frac{\chi^{1/2}}{u^{1/2}\theta}|\theta_x|\frac{u^{1/2}\theta^r}{\chi^{1/2}}\ dx\ ds$$

So:

$$\theta^r(x,s) \le C + C\Big(\int_0^M \frac{\chi}{u\theta^2}\theta_x^2\ dx\Big)^{1/2}\Big(\int_0^M \theta^{2r-q}\ dx\Big)^{1/2}.$$

By using lemma 5, the last integral is bounded if $2r - q = 1$ or 4, and for $1 \le 2r - q \le 4$, by interpolation. So we get finally:

$$\max_{x\in[0,M]} \theta^r(x,s) \le C + C\Big(\int_0^M \frac{\chi}{u\theta^2}\theta_x^2\ dx\Big)^{1/2}, \tag{50}$$

if $5/2 \le r \le 4$.

But we have also:

$$\theta^r(x,s) \le C + C\Big(\int_0^M \frac{\chi}{u\theta^2}\theta_x^2\ dx\Big)^{1/2}\Big(\int_0^M \theta^{2r}\ dx\Big)^{1/2},$$

where the last integral is bounded if $1 \le 2r \le 4$. So (50) holds provided $1/2 \le r \le 2$.

If $2 \le r \le 5/2$, we see that $\theta^{2r-q} \le C\max(1,\theta)$, so $\int \theta^{2r-q}\ dx$ is also bounded in this case.

Finally, by taking the square in (50), putting $2r = \alpha$, and integrating on $[0,t]$, we get (49) as soon as $q \geq 4$.

LEMMA 7. (i) *There is a pair* (u_m, u_M) *of positive numbers, depending only on* T *and the data, such that:*

$$\forall (x,t) \in [0,M] \times [0,T] : \quad u_m \leq u(x,t) \leq u_M. \tag{51}$$

(ii) *One has:*

$$\forall (x,t) \in [0,M] \times [0,T] : \quad 0 \leq Z(x,t) \leq 1. \tag{52}$$

PROOF. (i) We let generically C be a t-dependent constant. Let us consider first the lower bound. From (32), we get:

$$v_t + p_x = \nu (\log u)_{tx} - G(x - \frac{M}{2}). \tag{53}$$

By integrating on $[0,x] \times [0,t]$, we find:

$$-\nu \log u(x,t) + \int_0^t p(x,s)\ ds = -\int_0^x (v(z,t) - v_0(z))\ dz + \int_0^t p(0,s)\ ds - \nu \log u(0,t)$$
$$+\nu \log u_0(0) - \nu \log u_0(x) - t\frac{G}{2}x(x-M).$$

By integrating on $[0,t]$ the boundary condition $\sigma(0,t) = -P$, we have:

$$\int_0^t p(0,s)\ ds - \nu \log u(0,t) + \nu \log u_0(0) = Pt,$$

so we obtain the identity:

$$-\nu \log u(x,t) + \int_0^t p(x,s)\ ds = -\int_0^x (v(z,t) - v_0(z))\ dz + tf(x) - \nu \log u_0(x). \tag{54}$$

So $u(x,t) \geq u_m$, with $u_m = e^{-\frac{1}{\nu}\left(M^{1/2}E_0^{1/2} + \nu \max_{[0,M]} |\log u_0(x)| + t(P + \frac{GM^2}{4})\right)}$.

To find an upper bound, we integrate (54) on $[0,M]$:

$$\int_0^M \nu \log u(x,t)\ dx \leq C_1 + \int_0^M \int_0^t p(x,s)\ ds\ dx,$$

where $C_1 = M^{3/2}E_0^{1/2} + \nu \max_{[0,M]} |\log u_0(x)| + tM^2(P + \frac{GM^2}{12})$. So:

$$\int_0^M \nu \log u(x,t)\ dx \leq C_1 + \int_0^t \int_0^M \left(\frac{R\theta}{u_m} + a\theta^4\right)\ ds\ dx,$$

which, by (41), gives:

$$\int_0^M \log u(x,t)\ dx \leq C_2,$$

where C_2 depends only on t and the data.

As above, we use that $\exists y(t) \in [0,M] :\ \log(y(t),t) = \int_0^M \log u(x,t)\ dx$, and we find:

$$\max_{x\in[0,M]} \log u(x,t) \leq C_3 + \left(\int_0^M (\nu(\log u(x,t))_x - v)^2\ dx\right)^{1/2}.$$

So, we just need an upper bound for the rhs.

For that purpose, we multiply (53) by $\nu(\log u(x,s))_x - v$, and integrate on $[0,M]\times[0,t]$:

$$\begin{aligned}
&\int_0^M (\nu(\log u(x,t))_x - v)^2 \ dx - \int_0^M (\nu(\log u_0(x))_x - v_0)^2 \ dx \\
&\quad = \int_0^t \int_0^M \left[\left(-\frac{R\theta}{u^2}u_x + (\frac{R}{u} + \frac{4}{3}a\theta^3)\theta_x\right)\right](\nu(\log u(x,s))_x - v) \ dx \ ds \\
&\quad + \int_0^t \int_0^M G\left(x - \frac{M}{2}\right)(\nu(\log u(x,s))_x - v) \ dx \ ds,
\end{aligned}$$

or:

$$\begin{aligned}
&\int_0^M (\nu(\log u(x,t))_x - v)^2 \ dx - \int_0^M (\nu(\log u_0(x))_x - v_0)^2 \ dx \\
&\quad + \nu R \int_0^t \int_0^M \frac{\theta}{u}(\log u(x,s))_x)^2 \ dx \ ds \\
&\quad = \int_0^t \int_0^M \frac{R\theta v}{u}(\log u(x,s))_x \ dx \ ds \\
&\quad + \int_0^t \int_0^M \left(\frac{R}{u} + \frac{4}{3}a\theta^3\right)\theta_x\,(\nu(\log u(x,s))_x - v) \ dx \ ds \\
&\quad + \int_0^t \int_0^M G\left(x - \frac{M}{2}\right)(\nu(\log u(x,s))_x - v) \ dx \ ds, \qquad (55)
\end{aligned}$$

We first bound the first contribution in the rhs:

$$\begin{aligned}
&\int_0^t \int_0^M \frac{R\theta|v|}{u}|(\log u(x,s))_x| \ dx \ ds \\
&\quad \leq \frac{R}{2}\epsilon \int_0^t \int_0^M \frac{\theta}{u}((\log u(x,s))_x)^2 \ dx \ ds + \frac{R}{2\epsilon}\int_0^t \max_{x\in[0,M]} \theta(x,s)\left(\int_0^M v^2 \ dx\right) ds,
\end{aligned}$$

with $\epsilon > 0$.

By using lemma 5 and 7, we have:

$$\left|\int_0^t \int_0^M \frac{R\theta v}{u}(\log u(x,s))_x \ dx \ ds\right| \leq C + \frac{R}{2}\epsilon \int_0^t \int_0^M \frac{\theta}{u}((\log u(x,s))_x)^2 \ dx \ ds.$$

The second contribution gives:

$$\begin{aligned}
&\left|\int_0^t \int_0^M \left(\frac{R}{u} + \frac{4}{3}a\theta^3\right)\theta_x\,(\nu(\log u(x,s))_x - v) \ dx \ ds\right| \\
&\quad \leq \int_0^t \int_0^M \frac{3R + 4au\theta^3}{3u}\frac{\theta u^{1/2}}{\chi^{1/2}}|\nu(\log u(x,s))_x - v|\frac{\chi^{1/2}|\theta_x|}{u^{1/2}\theta} dx \ ds,
\end{aligned}$$

and by using (43) and Cauchy-Schwarz, the rhs is bounded by:

$$C + \frac{1}{18}\int_0^t \int_0^M \frac{(3R\theta + 4au\theta^4)^2}{u\chi}(\nu(\log u(x,s))_x - v)^2 \ dx \ ds,$$

which is dominated by:

$$C + C\int_0^t \int_0^M \left(1 + \theta^{5-q} + \theta^{8-q}\right)(\nu(\log u(x,s))_x - v)^2 \ dx \ ds.$$

Finally, the last contribution gives:

$$\left|\int_0^t \int_0^M G\left(x-\frac{M}{2}\right)(\nu(\log u(x,s))_x - v)\ dx\ ds\right| \\ \leq C + \frac{1}{2}\int_0^t \int_0^M (\nu(\log u(x,s))_x - v)^2\ dx\ ds.$$

By collecting all of these estimates, taking ϵ small enough, and putting:

$$F(t) = \int_0^M (\nu(\log u(x,t))_x - v)^2\ dx,$$

and:

$$K(s) = \max_{x\in[0,M]} \left(1+\theta^{5-q}+\theta^{8-q}\right)(x,s),$$

we get:

$$F(t) \leq C + C\int_0^t K(s)F(s)\ ds.$$

Now, by using lemma 6, and applying Gronwall's lemma, we obtain finally:

$$F(t) \leq C\exp\left(\int_0^t K(s)\ ds\right)\left[1+\int_0^t \exp\left(-\int_0^s K(\tau)\ d\tau\right) ds\right].$$

As the rhs is bounded, by (49), this gives an upper bound for $\int_0^M (\nu(\log u(x,t))_x - v)^2\ dx$, and consequently for u.

2. The proof of (52) is analogous to that of [4], and we omit it. ∎

Lemma 8.

$$\int_0^T \int_0^M v_x^2 dx\ dt \leq C. \tag{56}$$

Proof. By multiplying the second equation (32) by v, and integrating on $[0,M]\times[0,T]$, we have:

$$\frac{1}{2}\int_0^M v(x,T)^2 dx + \int_0^T \int_0^M \nu\frac{v_x^2}{u} dx\ dt \\ = \frac{1}{2}\int_0^M v_0(x)^2 dx + P\int_0^T (v(M,t)-v(0,t))\ dt + \int_0^T\int_0^M pv_x\ dx\ dt \\ - \int_0^T\int_0^M G(x-\frac{M}{2})v\ dx\ dt.$$

By using lemma 5 and 6, the rhs is bounded by:

$$C + C\int_0^T\int_0^M p^2\ dx\ dt + \frac{1}{2}\int_0^T\int_0^M \nu\frac{v_x^2}{u} dx\ dt,$$

and:

$$\int_0^T\int_0^M p^2\ dx\ dt \leq C\int_0^T\int_0^M (\theta^2+\theta^5+\theta^8)\ dx\ dt,$$

where each term is bounded by using lemma 6. ∎

LEMMA 9.

$$\max_{t\in[0,T]}\int_0^M u_x^2 dx \le C. \tag{57}$$

PROOF. As the proof of lemma 7 tells us that:

$$\int_0^M \left(\nu(\log u(x,t))_x - v\right)^2 \ dx \le C,$$

this implies:

$$\nu^2\int_0^M \frac{u_x^2}{u^2}\ dx + \int_0^M v^2\ dx \le 2\nu\int_0^M |v|\frac{|u_x|}{u}\ dx.$$

The result follows by using Young's inequality together with lemma 10. ∎

LEMMA 10.

$$\int_0^T\int_0^M |v_x|^3 dx\ dt \le C. \tag{58}$$

PROOF. By using an argument of Dafermos and Hsiao [27], we set $w(x,t) = \int_0^x v(y,t)\ dy$, and we check that w satisfies the following parabolic problem:

$$\begin{cases} w_t - \frac{\nu}{u}w_{xx} = -p(x,t) + f(x), \\ w(0,t) = 0, \\ w(M,t) = \int_0^M v(y,t)\ dy = 0, \\ w(x,0) = \int_0^x v_0(y)\ dy. \end{cases} \tag{59}$$

Then linear parabolic L^p estimates give in particular

$$\|w_{xx}\|_{L^1([0,T];L^3(0,M))} \le C + \|p\|_{L^1([0,T];L^3(0,M))},$$

and we have just to verify that the rhs is finite. We have:

$$\int_0^T\int_0^M p^3(x,t)\ dx\ dt \le C\int_0^T\int_0^M \left(\theta^3 + 3\theta^6 + 3\theta^9 + \theta^{12}\right)\ dx\ dt.$$

By lemma 6, the first two terms in the rhs are bounded, and, by using lemma 5, we get:

$$\int_0^T\int_0^M \left(\theta^9 + \theta^{12}\right)\ dx\ dt \le C\int_0^T \max_{x\in[0,M]}\left(\theta^5+\theta^8\right)(x,t)\ dt,$$

which is also bounded by lemma 6. ∎

Following [21] [22], we consider the three quantities:

$$\mathbf{X} = \int_0^t\int_0^M (1+\theta^7)\theta_t^2\ dx, \quad \mathbf{Y} = \max_{t\in[0,T]}\int_0^M (1+\theta^8)\theta_x^2\ dx, \quad \mathbf{Z} = \max_{t\in[0,T]}\int_0^M v_{xx}^2\ dx,$$

We have:

$$\begin{aligned}\max_{x\in[0,M]}\theta^{10}(x,t) &\le C\max_{x\in[0,M]}\theta^9(x,t) + C\int_0^M \theta^9|\theta_x|\ dx \\ &\le C + \frac{1}{2}\max_{x\in[0,M]}\theta^{10}(x,t) + C\left(\int_0^M \theta^8\theta_x^2\ dx\right)^{1/2}\left(\int_0^M \theta^{10}\ dx\right)^{1/2}.\end{aligned}$$

So:

$$\max_{x\in[0,M]} \theta^{10}(x,t) \le C + C\mathbf{Y}^{1/2} \max_{x\in[0,M]} \theta^3,$$

and we obtain:

$$\max_{Q_T} \theta \le C(1+\mathbf{Y}^{\frac{1}{14}}).$$

By using the interpolation inequality [32]:

$$\|v_x\|^2_{L^2(0,M)} \le \|v\|_{L^2(0,M)} \|v\|_{H^2(0,M)},$$

we have:

$$\max_{0,T} \int_0^M v_x^2 \ dx \le C(1+\mathbf{Z}^{\frac{1}{2}}).$$

The Sobolev theorem gives:

$$\|v_x\|_{L^\infty(0,M)} \le \|v_x\|_{L^2(0,M)} + 2\|v_{xx}\|_{L^2(0,M)}.$$

By combining the last two estimates:

$$\max_{Q_T} |v_x| \le C(1+\mathbf{Z}^{\frac{3}{8}}),$$

where $Q_T = [0,T]\times[0,M]$.

LEMMA 11. *One has the following inequalities:*

$$\mathbf{Y} \le C(1+\mathbf{Z}^{\frac{7}{8}}), \tag{60}$$

$$\mathbf{X} \le C(1+\mathbf{Z}^{\frac{7}{8}}). \tag{61}$$

PROOF. We consider the function:

$$K(u,\theta) = \int_0^\theta \frac{\chi(u,\xi)}{u} \ d\xi = \kappa_1 \frac{\theta}{u} + \frac{1}{5}\kappa_2 \theta^5,$$

which satisfies $|K_u|, \ |K_{uu}| \ \le C(1+\theta)$. We multiply (46) by K_t, and integrate on Q_t, with $t \le T$:

$$\int_0^t \int_0^M \left(e_\theta \theta_t + \theta p_\theta v_x - \frac{\nu}{u} v_x^2 \right) K_t \ dx \ ds + \int_0^t \int_0^M \frac{\chi}{u} \theta_x K_{tx} \ dx \ ds$$
$$= \int_0^t \int_0^M \lambda \phi(u,\theta,Z) K_t \ dx \ ds. \tag{62}$$

We compute:

$$K_t = K_u v_x + \frac{\chi}{u}\theta_t,$$

$$K_{tx} = \left(\frac{\chi}{u}\theta_x\right)_t + K_u v_{xx} + K_{uu} v_x u_x + \left(\frac{\chi}{u}\right)_u u_x \theta_t.$$

We have to bound each term in (62).

We have first:

$$\int_0^t \int_0^M e_\theta \frac{\chi}{u} \theta_t^2 \ dx \ ds \ge C_1 \mathbf{X}, \tag{63}$$

where C_1 is a positive constant.

Now:

$$\left|\int_0^t\int_0^M e_\theta\theta_t K_u v_x\ dx\ ds\right| \le \int_0^t\int_0^M (1+\theta^4)|\theta_t||v_x|\ dx\ ds$$
$$\le \frac{1}{4}C_1\mathbf{X} + C\int_0^t\int_0^M (1+\theta)v_x^2\ dx\ ds.$$

So:

$$\left|\int_0^t\int_0^M e_\theta\theta_t K_u v_x\ dx\ ds\right| \le \frac{1}{4}C_1\mathbf{X} + C + C\mathbf{Z}^{3/4}. \tag{64}$$

Next:

$$\left|\int_0^t\int_0^M \left(\theta p_\theta v_x - \frac{\nu}{u}v_x^2\right) K_u v_x\ dx\ ds\right| \le C\int_0^t\int_0^M ((1+\theta^4)|v_x| + v_x^2)\theta|v_x|\ dx\ ds$$
$$\le C\max_{Q_t} v_x^2 \int_0^t\int_0^M (1+\theta^5)\ dx\ ds + C\max_{Q_t}|v_x|\int_0^t \max_{x\in[0,M]}\theta\int_0^M v_x^2\ dx\ ds.$$

So:

$$\left|\int_0^t\int_0^M \left(\theta p_\theta v_x - \frac{\nu}{u}v_x^2\right) K_u v_x\ dx\ ds\right| \le C + C\mathbf{Z}^{7/8}. \tag{65}$$

Now:

$$\left|\int_0^t\int_0^M \left(\theta p_\theta v_x - \frac{\nu}{u}v_x^2\right)\frac{\chi}{u}\theta_t\ dx\ ds\right|$$
$$\le \frac{1}{8}C_1\mathbf{X} + C\int_0^t\int_0^M (1+\theta^8)v_x^2\ dx\ ds + C\int_0^t\int_0^M v_x^4\ dx\ ds.$$

So, finally, by using lemma 7 and 10:

$$\left|\int_0^t\int_0^M \left(\theta p_\theta v_x - \frac{\nu}{u}v_x^2\right)\frac{\chi}{u}\theta_t\ dx\ ds\right| \le \frac{1}{8}C_1\mathbf{X} + C + C\mathbf{Z}^{3/4}. \tag{66}$$

Let us consider now the various contributions in the second integral of (62). We have:

$$\int_0^t\int_0^M \frac{\chi}{u}\theta_x\left(\frac{\chi^2}{u^2}\theta_x^2\right)_t\ dx\ ds = \frac{1}{2}\int_0^M \left(\frac{\chi^2}{u^2}\theta_x^2\right)(x,t)\ dx - \frac{1}{2}\int_0^M \left(\frac{\chi^2}{u^2}\theta_x^2\right)(x,0)\ dx$$
$$\ge \frac{1}{2}\int_0^M \frac{\chi^2}{u^2}\theta_x^2\ dx - C,$$

so:

$$\int_0^t\int_0^M \frac{\chi}{u}\theta_x\left(\frac{\chi^2}{u^2}\theta_x^2\right)_t\ dx\ ds \ge C_2\mathbf{Y} - C. \tag{67}$$

Now:

$$\left|\int_0^t\int_0^M \frac{\chi}{u}\theta_x K_{uu}v_x u_x\ dx\ ds\right| \le C\max_{Q_T}|v_x|\int_0^t\int_0^M (1+\theta^4)|\theta_x|\theta|u_x|\ dx\ ds$$
$$\le C + C\mathbf{Z}^{3/8}\left(\int_0^t\int_0^M \theta^2\theta_x^2\ dx\ ds\right)^{1/2}\left(\int_0^t\int_0^M (1+\theta^8)u_x^2\ dx\ ds\right)^{1/2},$$
$$\le C + C\mathbf{Z}^{3/8}\mathbf{Y}^{1/2}.$$

So:
$$\left|\int_0^t\int_0^M \frac{\chi}{u}\theta_x K_{uu} v_x u_x \ dx\ ds\right| \leq C + \frac{1}{4}C_2\mathbf{Y} + C\mathbf{Z}^{3/4}. \tag{68}$$
Now:
$$\begin{aligned}\left|\int_0^t\int_0^M \frac{\chi}{u}\theta_x \left(\frac{\chi}{u}\right)_u u_x\theta_t \ dx\ ds\right| &\leq C\int_0^t\int_0^M |\theta_x|(1+\theta^4)|\theta_t||u_x| \ dx\ ds,\\ &\leq \frac{1}{8}C_1\int_0^t\int_0^M (1+\theta^7)\theta_t^2 \ dx\ ds + C\int_0^t\int_0^M \left(\frac{\chi}{u}\theta_x\right)^2 \frac{u^2(1+\theta)}{\chi^2}u_x^2 \ dx\ ds,\\ &\leq \frac{1}{8}C_1\mathbf{X} + C\int_0^t \max_{[0,M]}\left(\frac{\chi}{u}\theta_x\right)^2 \ ds,\end{aligned}$$
by using lemma 9.

Now, we have:
$$\left(\frac{\chi}{u}\theta_x\right)^2 \leq \int_0^M \left(\frac{\chi}{u}\theta_x\right)^2 \ dx + \int_0^M \left|\frac{\chi}{u}\theta_x\right|\left|\left(\frac{\chi}{u}\theta_x\right)_x\right| \ dx,$$
so:
$$\max_{[0,M]}\left(\frac{\chi}{u}\theta_x\right)^2 \leq C\max_{Q_T}(\chi\theta^2)\int_0^M \frac{\chi}{u\theta^2}\theta_x^2 \ dx + \int_0^M \left|\frac{\chi}{u}\theta_x\right|\left|\left(\frac{\chi}{u}\theta_x\right)_x\right| \ dx.$$
So:
$$\int_0^t \max_{[0,M]}\left(\frac{\chi}{u}\theta_x\right)^2 \ ds \leq C\max_{Q_T}(1+\theta^6) + C\left(\int_0^t\int_0^M \chi\theta^2\left|\left(\frac{\chi}{u}\theta_x\right)_x\right|^2 \ dx\ ds\right)^{1/2},$$
and, by using the third equation (32):
$$\leq C\mathbf{Y}^{3/7} + \left(\int_0^t\int_0^M (1+\theta^6)\left[e_\theta^2\theta_t^2 + \theta^2 p_\theta^2 v_x^2 + v_x^4 + \lambda\phi(u,\theta,Z)\right] \ dx\ ds\right)^{1/2}.$$
The integral is bounded by:
$$\begin{aligned}&\Bigg(C(1+\max_{Q_T}\theta^5)\mathbf{X} + \max_{Q_T} v_x^2\cdot\int_0^t\int_0^M (1+\theta^{14}) \ dx\ ds\\ &\qquad + \max_{Q_T}|v_x|(1+\max_{Q_T}\theta^6)\cdot\int_0^t\int_0^M |v_x|^3 \ dx\ ds\Bigg)^{1/2},\\ &\leq C(1+\max_{Q_T}\theta^{5/2})\mathbf{X}^{1/2} + C(1+\max_{Q_T}\theta)\mathbf{Z}^{3/8} + C(1+\max_{Q_T}\theta^3)\mathbf{Z}^{3/16},\end{aligned}$$
and finally, by using once more Young's inegality, we obtain:
$$\left|\int_0^t\int_0^M \frac{\chi}{u}\theta_x \left(\frac{\chi}{u}\right)_u u_x\theta_t \ dx\ ds\right| \leq C + \frac{1}{8}C_1\mathbf{X} + \frac{1}{4}C_2\mathbf{Y} + C\mathbf{Z}^{3/4}. \tag{69}$$
The last contribution is:
$$\begin{aligned}&\left|\int_0^t\int_0^M \lambda\phi(u,\theta,Z)\left[K_u v_x + \frac{\chi}{u}\theta_t\right] \ dx\ ds\right|\\ &\qquad\leq C\int_0^t\int_0^M (1+\theta^2)|v_x| \ dx\ ds + C\int_0^t\int_0^M (1+\theta^4)|\theta_t| \ dx\ ds,\end{aligned}$$

which, by using the definition of ϕ, and lemma 5, is less than:

$$C\left((1+\max_{Q_T}\theta^2)(1+\mathbf{Z}^{3/8})+C+\frac{1}{8C}C_1\mathbf{X}\right),$$

which gives:

$$\left|\int_0^t\int_0^M \lambda\phi(u,\theta,Z)\left[K_u v_x+\frac{\chi}{u}\theta_t\right]\,dx\,ds\right|\le C+\frac{1}{8}C_1\mathbf{X}+C\mathbf{Z}^{3/4}. \tag{70}$$

By collecting the estimates (63)-(70), we obtain:

$$C_1\mathbf{X}+C_2\mathbf{Y}\le C(1+\mathbf{Z}^{7/8}),$$

which ends the proof. ∎

LEMMA 12. *One has the following inequality:*

$$\max_{[0,T]}\int_0^M v_t^2\,dx+\int_0^T\int_0^M v_{xt}^2\,dx\,dt\le C(1+\mathbf{Z}^{\frac{7}{8}}). \tag{71}$$

PROOF. We sketch a formal proof, which can be made rigorous by using adapted mollifiers (see [21] [22]).

If we derivate with respect to t the second equation (32), multiply it by v_t, and integrate on Q_T, we get:

$$\frac{1}{2}\int_0^M v_t^2(x,t)\,dx-\frac{1}{2}\int_0^M v_t^2(x,0)\,dx=\int_0^T\int_0^M v_t\sigma_{xt}\,dx\,dt.$$

By integrating by parts on $[0,M]$, the rhs is:

$$\int_0^T\int_0^M v_{tx}\left[p_t-\nu\frac{v_{tx}}{u}+\nu\frac{v_x^2}{u^2}\right]\,dx\,dt,$$

and we get the majorization:

$$\frac{1}{2}\int_0^M v_t^2(x,t)\,dx+\int_0^T\int_0^M \nu\frac{v_{tx}^2}{u}\,dx\,dt\le C+C\int_0^T\int_0^M\left(p_t^2+v_x^4\right)\,dx\,dt.$$

The integral in the rhs is bounded by:

$$\int_0^T\int_0^M\left((1+\theta^6)\theta_t^2+\theta^2v_x^2+v_x^4\right)\,dx\,dt\le C+C\mathbf{X}+C\mathbf{Z}^{\frac{3}{4}}+C\mathbf{Z}^{\frac{7}{8}},$$

which ends the proof, by using lemma 11. ∎

LEMMA 13. *There exists a constant $C(T)$ such that:*

$$\left(\mathbf{Z},\ \mathbf{X},\ \theta,\ \mathbf{Y},\ \max_{[0,T]}\int_0^M v_x^2\,dx,\ \max_{Q_T}|v_x|,\ \max_{[0,T]}\int_0^M v_t^2\,dx,\ \int_0^T\int_0^M v_{xt}^2\,dx\,dt\right)\le C(T). \tag{72}$$

PROOF. By the second equation (32):

$$v_{xx}=\frac{u}{\nu}\left(v_t+p_x+\frac{\nu}{u^2}u_xv_x\right).$$

So:

$$\int_0^M v_{xx}^2\ dx \le C\int_0^M \left(v_t^2+(1+\theta^6)\theta_x^2+\theta^2u_x^2+u_x^2v_x^2\right)\ dx,$$
$$\le C\int_0^M \left(v_t^2+(1+\theta^6)\theta_x^2+u_x^2v_x^2\right)\ dx.$$

By using lemma 12:

$$\le C+C(1+\mathbf{Z}^{\frac{7}{8}})+C\mathbf{Y}+C\mathbf{Z}^{\frac{3}{4}},$$

so, by lemma 11:

$$\mathbf{Z}\le C(1+\mathbf{Z}^{\frac{7}{8}}).$$

This implies that $\mathbf{Z}$ is bounded, so is $\mathbf{Y}$, and all the other quantities.

Lemma 14. *There exists a constant $C(T)$ such that:*

$$\left(\max_{[0,T]}\int_0^M \theta_t^2\ dx,\ \int_0^T\int_0^M \theta_{xt}^2\ dx\ dt,\ \int_0^T\int_0^M \theta_{xx}^2\ dx\right)\le C(T). \tag{73}$$

Proof. By differentiating equation (46) with respect to t, multiplying by $e_\theta\theta_t$, and integrating on $[0,M]$, we get:

$$\begin{aligned}\frac12\int_0^M (e_\theta\theta_t)^2(x,t)\ dx-\frac12\int_0^M (e_\theta\theta_t)^2(x,0)\ dx+\int_0^t\int_0^M p_\theta v_x e_\theta\theta_t^2\ dx\ ds\\ +\int_0^t\int_0^M \theta_{\theta\theta}v_xe_\theta\theta_t^2\ dx\ ds+\int_0^t\int_0^M \theta p_{\theta u}v_x^2e_\theta\theta_t\ dx\ ds+\int_0^t\int_0^M \theta p_\theta v_{xt}e_\theta\theta_t\ dx\ ds\\ -\int_0^t\int_0^M\left(-\frac{\nu}{u^2}v_x^3+\frac{2\nu}{u}v_xv_{xt}\right)e_\theta\theta_t\ dx\ ds\\ =-\int_0^t\int_0^M\frac{\chi}{u}e_\theta\theta_{xt}^2\ dx\ ds-\int_0^t\int_0^M\left(-\frac{\kappa_1}{u^2}v_x\theta_x+4\kappa_2\theta^3\theta_t\theta_x\right)(e_\theta\theta_t)_x\ dx\ ds\\ -\int_0^t\int_0^M(e_{\theta u}u_x+e_{\theta\theta}\theta_x)\theta_x\ dx\ ds+\int_0^t\int_0^M\lambda\phi(u,\theta,Z)e_\theta\theta_t\ dx\ ds.\end{aligned}$$

So we can write:

$$\frac12\int_0^M (e_\theta\theta_t)^2(x,t)\ dx+\int_0^t\int_0^M\frac{\chi}{u}e_\theta\theta_{xt}\ dx\ ds=\text{other terms}.$$

Using the same techniques as above, one can bound the rhs (see [27]), then we get:

$$\max_{[0,T]}\int_0^M\theta_t^2\ dx,\ \int_0^T\int_0^M\theta_{xt}^2\ dx\ dt\le C(T).$$

Now, by using equation (46), we see that:

$$\theta_{xx}=\frac{u}{\chi}\left(e_\theta\theta_t+\theta p_\theta v_x-\frac{\nu}{u}v_x^2-\left(\frac{\chi}{u}\right)_x\theta_x-\lambda\phi(u,\theta,Z)\right).$$

As all the coefficients in the rhs are L^2, the last estimate in (73) follows. ∎

The Hölder regularity of the solution is now proved in the same manner as in [22], which ends the proof of theorems 1 and 6.

4.2. Asymptotic behaviour. As the physics taking place in our model is rather complex (radiation, gravity, chemistry, free boundary), so is the problem of the precise asymptotic behaviour, and we get only partial results.

In fact, for technical reasons, we have to limit the analysis to three simplified situations of physical interest:

1. The non-gravitating Dirichlet problem.
2. The free-boundary chemical problem, in which one discards gravity and radiation.
3. The gravitating Eddington's model.

Before considering these simplified models, let us begin with some results valid in the general case.

As usual the first thing to do is to identify all the possible asymptotic states.

Let us consider the regular static solutions $\big(\bar u(x), \bar v(x) = 0, \bar\theta(x), \bar Z(x)\big)$ of (32), satisfying the system:

$$\begin{cases} \bar p_x + G\Big(x - \dfrac{1}{2}M\Big) = 0, \\ \bar Q_x - \lambda\phi(\bar\theta, \bar Z) = 0, \\ \Big(\dfrac{d}{\bar u^2}\bar Z_x\Big)_x - \phi(\bar\theta, \bar Z)) = 0. \end{cases} \tag{74}$$

The corresponding energy of this solution is given by:

$$\bar E(\bar\theta) = \int_0^M \big(e(\bar u, \bar\theta) + f(x)\bar u\big)\ dx. \tag{75}$$

We obtain easily the following result:

Lemma 15. *Any admissible static solution of (74) is defined by:*

$$\begin{cases} \bar u(x) = \dfrac{R\bar\theta}{P - \frac{a}{3}\bar\theta^4 + \frac{1}{2}Gx(M-x)}, \\ \bar\theta(x) = \bar\theta, \\ \bar Z(x) = 0, \end{cases} \tag{76}$$

where $\bar\theta$ is a positive constant. Such a solution exists if and only if the temperature satisfies:

$$P > \frac{a}{3}\bar\theta^4. \tag{77}$$

Proof. By integrating the third equation (74) on $[0, M]$, we see that:

$$\int_0^M \phi(\bar\theta, \bar Z)\ dx = 0.$$

As $\phi \geq 0$, we find that in fact $\phi(\bar\theta, \bar Z) = 0$, and so $\bar Z$ must be zero, at least in the gravitational case (by the first equation (74), $\bar\theta = 0$ cannot be a solution if $G \neq 0$).

By putting into the second equation (74), we see that $\bar Q(x) = \bar Q$, where $\bar Q$ is a constant, which is zero, by symmetry properties. This implies that $\bar\theta(x) = \bar\theta$, another constant.

By integrating the first equation (74), we get finally:

$$\bar{u}(x) = \frac{R\bar{\theta}}{P - \frac{a}{3}\bar{\theta}^4 + \frac{1}{2}Gx(M-x)}. \tag{78}$$

One sees that this solution is admissible ($\bar{u}(x)$ finite) only if (77) is satisfied, i.e. if the exterior pressure P "dominates" the radiation. ∎

If the condition (77) is not satisfied, no static solution can exist, and one cannot expect the system to converge toward an equilibrium.

If (77) holds, the energy of the solution is then:

$$\bar{E}(\bar{\theta}) = \int_0^M \big(e(\bar{u},\bar{\theta}) + f(x)\bar{u}\big)\ dx = M(C+R)\bar{\theta} + \frac{4}{3}a\bar{\theta}^4 \int_0^M \bar{u}\ dx.$$

We get finally:

$$\bar{E}(\bar{\theta}) = M(C+R)\bar{\theta} + \frac{8aR\bar{\theta}^5}{3\tau G}\log\left|\frac{\frac{M}{2}+\tau}{\frac{M}{2}-\tau}\right|, \tag{79}$$

where $\tau(\bar{\theta}) = \sqrt{\frac{2}{G}\left(P + \frac{GM^2}{8} - \frac{a}{3}\bar{\theta}^4\right)}$. Clearly, $\bar{\theta} \to \bar{E}(\bar{\theta})$ is increasing from $[0,\theta^*[$ to $[0,+\infty[$, so given $\bar{E}$, one gets a unique corresponding $\bar{\theta}$. ∎

Now, we have first a useful representation lemma for the specific volume (see [10]):

Lemma 16. *One has the formula:*

$$u(x,t) = \frac{1}{Y(x,t)B(x,t)D(x,t)}\left[u_0(x) + \frac{R}{\nu}\int_0^t Y(x,s)B(x,s)D(x,s)\theta(x,s)\ ds\right], \tag{80}$$

where:

$$B(x,t) = \exp\left(\frac{1}{\nu}\int_0^x (v_0(y) - v(y,t))\ dy\right),$$

$$D(x,t) = \exp\bigg(-\frac{a}{3\nu}\int_0^t \theta^4(x,s)\ ds\bigg), \qquad Y(x,t) = \exp\bigg(\frac{1}{\nu}tf(x)\bigg).$$

Lemma 17. (i) *Let θ_i and θ_s be the two positive roots, with $0 < \theta_i \le \theta_s$, of the equation:*

$$\theta - \log\theta - 1 = \frac{\mathbf{C}}{C_v},$$

where $\mathbf{C}$ is a constant (independent of t) (see lemma 15). For each $t \ge 0$, there is a $y(t) \in [0,M]$ such that:

$$0 < \theta_i \le \theta(y(t),t) \le \theta_s. \tag{81}$$

Moreover:

$$0 < \theta_i \le \frac{1}{M}\int_0^M \theta(x,t)\ dx \le \theta_s, \quad \forall t \ge 0. \tag{82}$$

(ii) *Let u_i and u_s be the two positive roots, with $0 < u_i \le u_s$, of the equation:*

$$u - \log u - 1 = \frac{\mathbf{C}}{R}.$$

For each $t \geq 0$, there is a $z(t) \in [0, M]$ such that:

$$0 < u_i \leq u(z(t), t) \leq u_s. \tag{83}$$

Moreover:

$$0 < u_i \leq \frac{1}{M} \int_0^M u(x,t)\ dx \leq u_s, \quad \forall t \geq 0. \tag{84}$$

PROOF. It is sufficient to prove the first part of the lemma. By lemma 15, we have:

$$\int_0^M C_v \left(\theta(x,t) - \log\theta(x,t) - 1\right)\ dx \leq \mathbf{C},$$

and by applying the mean value theorem, there is a $y(t) \in [0, M]$ such that:

$$\theta(x,t) - \log\theta(x,t) - 1 \leq \frac{\mathbf{C}}{C_v},$$

which gives (81).

Using Jensen's inequality for the convex function $x \to x - \log x - 1$, we get:

$$\Theta(t) - \log\Theta(t) - 1 \leq \frac{\mathbf{C}}{C_v},$$

where $\Theta(t) = \frac{1}{M}\int_0^M \theta(x,t)\ dx$, which gives (82). ∎

By using the representation of u, one has:

LEMMA 18. *If $P > 0$, there exist two positive constants u^+ and u^-, independent of t, such that:*

$$u^- \leq u(x,t) \leq u^+, \tag{85}$$

for any $t \geq 0$, and $x \in [0, M]$.

PROOF. From now on, we denote by C and $\mathbf{C}_j$ various time-independent constants.

By lemma 15, it is clear that:

$$0 < \frac{1}{\mathbf{C}_1} \leq B(x,t) \leq \mathbf{C}_1, \tag{86}$$

where $\mathbf{C}_1 = \exp(2ME_0)$ is independent of t.

Now, we have, as usual:

$$\theta^2(x,t) = \theta^2(y(t),t) + 2\int_{y(t)}^x \theta(x,t)\theta_x(x,t)\ dx,$$

where $y(t)$ is defined in lemma 21. Then:

$$|\theta^2(x,t) - \theta^2(y(t),t)| \leq CV^{1/2}(t),$$

with $V(t) = \int_0^M \frac{\chi}{u\theta^2}\theta_x^2\ dx$. By using lemma 16, we get:

$$\frac{1}{2}\theta_i^4 - CV(t) \leq \theta^4(x,t) \leq 2\theta_s^4 + CV(t),$$

so, by lemma 15:

$$\frac{1}{2}\theta_i^4(t-s) - C \leq \int_s^t \theta^4(x,\tau)\ d\tau \leq 2\theta_s^4(t-s) + C,$$

for any $0 \le s \le t$. So:

$$Ce^{-\frac{2a}{3\nu}\theta_s^4 t} \le D(x,t) \le Ce^{-\frac{a}{6\nu}\theta_i^4 t}, \tag{87}$$

and, for $0 \le s \le t$:

$$Ce^{\frac{a}{6\nu}\theta_i^4(t-s)} \le \frac{D(x,s)}{D(x,t)} \le Ce^{\frac{2a}{3\nu}\theta_s^4(t-s)}. \tag{88}$$

Now, by using lemma 16, we decompose $u(x,t) = u_1(x,t) + u_2(x,t)$, with:

$$u_1(x,t) = \frac{u_0(x)}{Y(x,t)B(x,t)D(x,t)},$$

$$u_2(x,t) = \frac{R}{\nu}\int_0^t \frac{Y(x,s)B(x,s)D(x,s)}{Y(x,t)B(x,t)D(x,t)}\theta(x,s)\ ds.$$

By using (86) and (87):

$$Ce^{\frac{t}{\nu}\left(\frac{a}{6}\theta_i^4 - f(x)\right)} \le u_1(x,t) \le Ce^{\frac{t}{\nu}\left(\frac{2a}{3}\theta_s^4 - f(x)\right)}, \tag{89}$$

where the two bounds are exponentially decreasing, provided that $P > \frac{2a}{3}\theta_s^4$.

By using the same methods, we have also, for any $\epsilon > 0$:

$$\theta_i - \epsilon\mathbf{C}_2 - \frac{1}{4\epsilon}V(t) \le \theta(x,t) \le \theta_s + \epsilon\mathbf{C}_2 + \frac{1}{4\epsilon}V(t),$$

where $\mathbf{C}_2 = \frac{E_0}{\kappa_1(Pa)^{1/2})}$. By taking $\epsilon = \frac{\theta_i}{2\mathbf{C}_2}$, we get:

$$\frac{1}{2}\theta_i - \mathbf{C}_3 V(t) \le \theta(x,t) \le \mathbf{C}_4(1 + V(t)),$$

where $\mathbf{C}_3 = \frac{\mathbf{C}_2}{2\theta_i}$ and $\mathbf{C}_4$ are time independent.

So we find first, for the upper bound:

$$u_2(x,t) \le \mathbf{C}_5\int_0^t e^{\frac{1}{\nu}(f(x) - \frac{2a}{3}\theta_s^4)(s-t)}(1 + V(s))\ ds \le \mathbf{C}_5\left(1 + \int_0^t V(s)\ ds\right) \le \mathbf{C}_6.$$

The lower bound is obtained as follows, following [23]:

$$u_2(x,t) \ge \frac{R}{\nu\mathbf{C}_1}\int_0^t e^{\mathbf{C}_6(s-t)}(\frac{1}{2}\theta_i - \mathbf{C}_3 V(s))\ ds,$$

where $\mathbf{C}_6 = \frac{1}{\nu}(P - \frac{a}{6}\theta_i^4) > 0$. So:

$$u_2(x,t) \ge \mathbf{C}_7\left(\frac{\theta_i}{2\mathbf{C}_6}(1 - e^{-\mathbf{C}_6 t}) - \mathbf{C}_3 e^{-\mathbf{C}_6\frac{t}{2}}\int_0^{t/2} V(s)\ ds - \mathbf{C}_3\int_{t/2}^t V(s)\ ds\right),$$

and the rhs is bounded, for t large enough, by a positive constant. So we get:

$$\mathbf{C}_7 \le u_2(x,t) \le \mathbf{C}_6, \tag{90}$$

and taking into account (89) and (90), we obtain $\mathbf{C}_7 \le u(x,t) \le 1 + \mathbf{C}_6$, which ends the proof. ∎

The large time properties of the solution of (32)-(37) are the following:

Theorem 7. *Let (u, v, θ, Z) be the solution of (32)-(37), and $r(x,t)$ the associated lagrangian position, solution of:*

$$\begin{cases} \dfrac{dr}{dt} = v(x,t), \ \textit{for } x \in [0,M], \ t \geq 0, \\ r(x,0) = r_0(x). \end{cases} \tag{91}$$

(i) *If $P > 0$, the specific volume $u(x,t)$ and the chemical fraction $Z(x,t)$ are bounded uniformly on $[0,M] \times [0,\infty)$:*

$$\begin{cases} 0 < u^- \leq u(x,t) \leq u^+, \ \textit{for } x \in [0,M], \ t \geq 0, \\ 0 \leq Z(x,t) \leq 1, \ \textit{for } x \in [0,M], \ t \geq 0. \end{cases} \tag{92}$$

Moreover, the solution admits a slab of finite extension r_∞ as a spatial attractor:

$$r(x,t) \leq r_\infty, \tag{93}$$

where $r_\infty \leq r_0(M) + \frac{E_0}{P}$.

(ii) *If $P = 0$ ("vacuum"), there exist two positive numbers T_1 and C_1 such that:*

$$\int_0^M u(x,t)\ dt \geq C_1(t + T_1), \tag{94}$$

for $t > T_1$.

Proof. The estimate (92) for Z follows from the maximum principle applied to the chemical equation.

By integrating the second equation (32), on $[0,M]$, we have:

$$r(x,t) \leq r_0(M) + \int_0^M u(x,t)\ dx \leq r_0(M) + \frac{E_0}{P}.$$

By integrating the second equation (32), on $[0,x]$, we have:

$$\left(\int_0^x u(y,t)\ dy\right)_t = \sigma + \frac{1}{2}Gx(x - M).$$

By multiplying by u:

$$u\left(\int_0^x u(y,t)\ dy\right)_t + pu = \nu u_t - f(x), \tag{95}$$

where $f(x) = \frac{1}{2}Gx(x - M)u$. Then, by integrating on $[0,t] \times [0,M]$ the first term in the lhs, we have:

$$\begin{aligned} &\int_0^t \int_0^M u\left(\int_0^x u(y,s)\ dy\right)_s dx\ ds \\ &\qquad = \int_0^M \left\{ -\int_0^t u_s \int_0^x v(y,s)\ dy\ ds + u \int_0^x v(y,s)\ dy\Big|_0^t \right\} dx. \end{aligned}$$

As $u_t = v_x$, we can integrate by parts in x the first term in the rhs, and we obtain:

$$\int_0^t \int_0^M u\left(\int_0^x u(y,s)\ dy\right)_s dx\ ds \geq \int_0^t \int_0^M v^2\ dx\ ds - C\left(1 + \int_0^M u(x,t)\ dx\right).$$

By putting into (95):

$$\int_0^t \int_0^M (v^2 + pu + f(x)u)\ dx\ ds \le C\Big(1 + \int_0^M u(x,t)\ dx\Big).$$

So:

$$\Big(1 + \int_0^M u(x,t)\ dx\Big) \ge C^{-1} R \int_0^t \int_0^M u\ dx\ ds.$$

But we find easily a lower bound for u by integrating the second equation (32) on $[0,x] \times [0,t]$.

We have:

$$\int_0^x (v(y,t) - v_0(y))\ dy = \int_0^t \sigma\ ds - \int_0^t G\Big(x - \frac{M}{2}\Big)\ dt.$$

So:

$$\nu \log \frac{u(x,t)}{u_0(x)} = -\int_0^t (p + f(x))\ ds + \int_0^x (v(y,t) - v_0(y))\ dy,$$

and, by Cauchy-Schwarz:

$$\nu \log \frac{u(x,t)}{u_0(x)} \ge -\int_0^x (|v| + |v_0|)\ dy \ge -2(2M)^{1/2} E_0^{1/2},$$

with $E_0 = \int_0^t \int_0^M (\frac{1}{2}v^2 + C_v\theta + au\theta^4 + f(x)u + \lambda Z)\ dx\ ds$.

Then we get finally:

$$u(x,t) \ge u^-,$$

where $u^- = \min_{x\in[0,M]} u_0(x) \exp(-2(2M)^{1/2} E_0^{1/2})$. This implies:

$$\Big(1 + \int_0^M u(x,t)\ dx\Big) \ge C^{-1} R M u^- t,$$

which ends the proof. ■

Now, to obtain more precise asymptotics, we consider successively the three particular situations described above.

4.2.1. The radiating Dirichlet problem. We simplify the system (32) as follows:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma_x = 0, \\ e_t - \sigma v_x + Q_x - \lambda\phi(\theta, Z) = 0, \\ Z_t - \Big(\dfrac{d}{u^2} Z_x\Big)_x + \phi(\theta, Z) = 0, \end{cases} \tag{96}$$

with Dirichlet conditions:

$$v(0,t) = v(M,t) = 0, \tag{97}$$

and we keep the other conditions (33) and (35)-(37). In this case, one checks easily, by using the first equation (96), that the constant stationary state is given by:

$$\bar{u} = \frac{1}{M} \int_0^M u_0(x)\ dx. \tag{98}$$

Then, the asymptotic behaviour of the solution of this problem is the following [14]:

THEOREM 8. *Suppose that the initial conditions u_0, v_0, θ_0, Z_0 satisfy:*

$$u_0, u_0', v_0, v_0', v_0'', \theta_0, \theta_0', \theta_0'', Z_0, Z_0', Z_0'' \in C^\beta[0, M],$$

with $\beta \in [0, 1]$, and that:

$$u_0(x), \theta_0(x), Z_0(x) \ > 0,$$

for $x \in [0, M]$. If q is large enough ($q \geq 6$), then the solution (u, v, θ, Z) converges toward the static solution $(\bar{u}, 0, \bar{\theta}, 0)$, in $H^1(0, M)$, when $t \to \infty$, where $\bar{u}$ is given by (98), and $\bar{\theta}$ is found by solving the quartic equation:

$$a\bar{u}\bar{\theta}^4 + C_v\bar{\theta} = \frac{E_0}{M}, \tag{99}$$

where $E_0 = \int_0^M \left(\frac{1}{2}v_0^2 + e_0 + \lambda Z_0\right)\, dx$. Moreover, there exist three positive numbers γ, T_0, and C_0 such that:

$$\|u - \bar{u}\|_{H^1(0,M)} + \|v\|_{H^1(0,M)} + \|\theta - \bar{\theta}\|_{H^1(0,M)} + \|Z\|_{H^1(0,M)} \leq C_0 e^{-\gamma t}, \tag{100}$$

for $t \geq T_0$.

Physically, the result tells us that the static equilibrium is stable provided that the radiating part of the thermal dissipation is strong enough. ∎

The proof of the theorem relies on bounds for high Sobolev norms. The complication with respect to [23] comes from the high powers in θ of $p(u, \theta)$, $e(u, \theta)$, and $\kappa(u, \theta)$, together with the anisotropic behaviours of the gaseous and radiating contributions of e and p.

4.2.2. The free-boundary chemical problem. We consider the problem (32)-(37), with $a = 0$ (no radiation), and $G = 0$ (no gravitation).

The regular static solutions $\left(\bar{u}(x), \bar{v}(x) = 0, \bar{\theta}(x), \bar{Z}(x)\right)$ is given by:

$$\begin{cases} \bar{u} = \dfrac{R\bar{\theta}}{P}, \\ \bar{\theta}(x) = \bar{\theta}, \\ \bar{Z}(x) = 0, \end{cases} \tag{101}$$

where $\bar{\theta}$ is a positive constant. Then, we have the following result:

THEOREM 9. *If $P > 0$, and if the initial conditions u_0, v_0, θ_0, Z_0 satisfy:*

$$u_0, u_0', v_0, v_0', v_0'', \theta_0, \theta_0', \theta_0'', Z_0, Z_0', Z_0'' \in C^\beta[0, M],$$

with $\beta \in [0, 1]$, and:

$$u_0(x), \theta_0(x), Z_0(x) \ > 0,$$

for $x \in [0, M]$, the solution (u, v, θ, Z) converges toward the static solution $(\bar{u}, 0, \bar{\theta}, 0)$, in $H^1(0, M)$, when $t \to \infty$, where $\bar{u}$ and $\bar{\theta}$ are given by:

$$\begin{cases} \bar{u} = \dfrac{RE_0}{MP(C_v + R)}, \\ \bar{\theta} = \dfrac{E_0}{M(C_v + R)}, \end{cases} \tag{102}$$

where $E_0 = \int_0^M \left(\frac{1}{2}v_0^2 + e_0 + Pu_0 + \lambda Z_0\right)\, dx$.

This result is proved in [30], by adapting to the reacting case the results of Nagasawa [29].

4.2.3. The Eddington's model. This model is a barotropic approximation used in astrophysics to simulate radiation at low cost!

The idea is the following. If we write $p_{gas} = \frac{R\theta}{u}$ and $p_{rad} = \frac{a}{3}\theta^4$, and if we set $\beta(x,t) = \frac{p}{p_{rad}}$, some observations show that $\beta(x,t)$ is slowly varying. The Eddington's model consists in supposing that β is a pure constant. This leads to the following barotropic equation for the pressure:

$$p(u) = \frac{A}{u^\gamma},$$

with $A = \left(\frac{3\beta^3 R^4}{a(\beta-1)^4}\right)^{1/3}$, and $\gamma = 4/3$. The temperature is then: $\theta = \left(\frac{3R^4}{a(\beta-1)}\right)^{1/3} \frac{1}{u^{\gamma-1}}$. The problem to solve is then:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma_x + G\left(x - \dfrac{1}{2}M\right) = 0, \end{cases} \tag{103}$$

for $t \geq 0$ and $x \in [0, M]$, where M is the mass of the slab, and $p(u) = \frac{A}{u^\gamma}$.

We consider the initial conditions:

$$(u, v)(x, 0) = (u_0, v_0)(x), \tag{104}$$

together with the boundary conditions (34), with $P > 0$, and the symmetry conditions (37).

By using the technique of Okada [28], one can prove the following result:

Theorem 10. (i) *Suppose the problem ((103),(104),(34),(37)) has at least a classical solution* $(u(x,t), v(x,t))$. *Then the functions* (u, v, v_x) *can be bounded on* $C^{r,r/2}(Q_T)$, *with* $r = 1/3$*:*

$$|||u|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} \leq C,$$

where C *depends only on* T*, the physical parameters of the problem, and the data.*

(ii) *If* $P > 0$*, and if the initial conditions* u_0, v_0 *satisfy:*

$$u_0, u_0', v_0, v_0', v_0'' \in C^\beta[0, M],$$

with $\beta \in [0, 1]$*, and:*

$$u_0(x) > 0,$$

for $x \in [0, M]$*, the solution* (u, v) *converges toward the static solution* $(\bar{u}, 0)$*, in* $H^1(0, M)$*, when* $t \to \infty$*, where* $\bar{u}$ *is given by:*

$$\bar{u}(x) = \left(\frac{A}{f(x)}\right)^{1/\gamma}, \tag{105}$$

where $f(x) = P + \frac{1}{2}Gx(M - x)$.

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