EVOLUTION EQUATIONS:
EXISTENCE, REGULARITY AND SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 52
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

ABSTRACT PARABOLIC PROBLEM WITH NON-LIPSCHITZ NONLINEARITY

JAN W. CHOLEWA

Institute of Mathematics, Silesian University
Bankowa 14, 40-007 Katowice, Poland
E-mail: jcholewa@ux2.math.us.edu.pl

TOMASZ DLOTKO

Institute of Mathematics, Silesian University
Bankowa 14, 40-007 Katowice, Poland
E-mail: tdlotko@ux2.math.us.edu.pl

Abstract. An abstract parabolic equation with sectorial operator and continuous nonlinearity is studied in this paper. In particular, the asymptotic behavior of solutions is described within the framework of the theory of global attractors. Examples included in the final part of the paper illustrate the presented ideas.

1. Introduction. A number of parabolic equations originating in Applied Sciences admit the formulation in an abstract form (1) below, where A is a *sectorial operator* in a Banach space X (cf. [HE]) and $F : X^\alpha \to X$, $\alpha \in [0, 1)$, is a continuous map. Usually, to study the solutions to (1) it is assumed that F is Lipschitz continuous on bounded subsets of the *fractional power space* X^α into X (cf. [HE], [HA], [C-D]). Such an assumption is, however, violated in many examples which include e.g. the diffusion equation with strong absorption (21) considered further in this note.

Although the problems with Lipschitz term F have been satisfactorily treated by many authors (cf. [HE], [HA], [C-C-D]), then the behavior of solutions to (1) in the case when Lipschitz condition fails is not so widely described. Following the results of [L-M], [MA] concerning *mild solutions* to (1) we shall thus generalize here earlier results of [C-D] onto the problems of the latter type.

2000 *Mathematics Subject Classification*: Primary 35B40, 35K22.

Key words and phrases: parabolic system, Cauchy problem, global solution, dissipative semigroup, global attractor.

Supported partially by the State Committee for Scientific Research (KBN) grant 2 P03A 046 12.

The paper is in final form and no version of it will be published elsewhere.

2. Local existence result. In the following, consider the problem

$$\dot{u} + Au = F(u),\ t > 0, \qquad u(0) = u_0, \tag{1}$$

under the Assumption (A) below.

ASSUMPTION (A). A is a sectorial, positive operator in a Banach space X, the resolvent of A is compact and, for some $\alpha \in [0,1)$, $F : X^\alpha \to X$ is a continuous function which takes bounded subsets of X^α into bounded subsets of X.

Here positivity of A means that all elements of the spectrum $\sigma(A)$ have strictly positive real parts. Further, $X^\alpha = D(A^\alpha)$ is the domain of the fractional power A^α of the operator A (cf. [HE, p. 29]).

DEFINITION 1. If, for some $\tau > 0$, a function $u \in C([0,\tau), X) \cap C((0,\tau), X^\alpha)$ fulfills in X the Cauchy integral formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds, \text{ for } t \in [0,\tau), \tag{2}$$

then u is called a *local mild X^α-solution* of (1) through $u_0 \in X$.

Based on [L-M, Theorem 1] we obtain the following result.

PROPOSITION 1. *Suppose that Assumption* **(A)** *holds. Then, to each $u_0 \in X^\alpha$ corresponds at least one local mild X^α-solution u of (1). In addition, $u(t,u_0) \to u_0$ in X^α as $t \to 0^+$.*

PROOF. Since the resolvent of A is compact, $e^{-At} : X \to X$ is a compact map for each $t > 0$ (cf. [HA, Lemma 4.2.3]). Specifying in the notation of [L-M], $D = X$, $L = A$, $B(t,v) \equiv F(v)$, and $T(t) \equiv e^{-At}$ one easily observes validity of the assumptions $(C1) - (C5)$ in [L-M, Theorem 1]. The proof is complete.

DEFINITION 2. A function u is called a *global mild X^α-solution* to (1) if u fulfills requirements of Definition 1 with $\tau = +\infty$.

It is well known (cf. [L-M]) that global solutions exist if F satisfies the sublinear growth condition

$$\|F(v)\|_X \le const.(1 + \|v\|_{X^\alpha}), \text{ for } v \in X^\alpha. \tag{3}$$

We then obtain:

PROPOSITION 2. *Suppose that Assumption (A) is satisfied and (3) holds. Then, for each $u_0 \in X$, there exists at least one global mild X^α-solution to (1). Furthermore, for each bounded set $B \subset X$ and each $t > 0$, $\{u(t,u_0),\ u_0 \in B\}$ is a precompact subset of X.*

PROOF. The assertion is a consequence of [L-M, Theorem 2] (cf. also [L-M, Remark 2]).

3. Existence and stability of global solutions. In the case when an a priori estimate of the local solutions to (1) is known in the norm of some Banach space Y, Proposition 2 may be generalized to the form reported below in Theorem 1. For this purpose we introduce the following hypothesis.

HYPOTHESIS (H). It is possible to choose:

- a Banach space Y, with $X^\alpha \subset Y$,
- a locally bounded function $c : R^+ \to R^+$,
- a nondecreasing function $g : R^+ \longrightarrow R^+$,
- a number $\theta \in [0,1)$,

such that, for $\tau > 0$ and $u_0 \in X^\alpha$, if $u(\cdot, u_0)$ is a local mild X^α-solution to (1) defined on $[0,\tau)$, then

$$\|u(t,u_0)\|_Y \le c(\|u_0\|_{X^\alpha}),\ t \in (0,\tau), \tag{4}$$

$$\|F(u(t,u_0))\|_X \le g(\|u(t,u_0)\|_Y)(1 + \|u(t,u_0)\|^\theta_{X^\alpha}),\ t \in (0,\tau). \tag{5}$$

REMARK 1. There are many examples of parabolic equations for which, because of the fast growth of nonlinear term, the sublinear growth restriction (3) is not satisfied, but the hypothesis (H) holds. Here are such important problems as the 2-D Navier-Stokes system, the Cahn-Hilliard equation, and many reaction-diffusion systems originating in biology (cf. [C-C-D] for details).

THEOREM 1. *Under the Assumptions* (A) *and* (H) *for each* $u_0 \in X^\alpha$ *there exists at least one global mild* X^α*-solution to (1). Moreover, if* $\|u_0\|_{X^\alpha} \le R$ *, then*

$$\|u(t,u_0)\|_{X^\alpha} \le c_1(R),\ t \ge 0. \tag{6}$$

PROOF. Conditions (4) and (5) imply that, for any fixed $u_0 \in X^\alpha$ and as long as $u(\cdot,u_0)$ exists, we have the estimate

$$\|F(u(t,u_0))\|_X \le g\big(c(\|u_0\|_{X^\alpha})\big)(1 + \|u(t,u_0)\|^\theta_{X^\alpha}). \tag{7}$$

Standard calculations show that, for fixed $u_0 \in X^\alpha$,

$$\|u(t,u_0)\|_{X^\alpha} \le M_{u_0}$$

as long as $u(t,u_0)$ exists (cf. [C-D, Theorem 1] for details). Furthermore, (6) holds provided that each solution exists globally in time. We shall now justify this latter supposition.

Since the semigroup $\{e^{-At}\}$ is analytic and $Re\sigma(A) > 0$ we have, for some $a > 0$, the estimates (cf. [HE, Theorem 1.4.3]):

$$\begin{aligned} &\|A^\alpha e^{-At}\|_{\mathcal{L}(X,X)} \le c_\alpha \tfrac{e^{-at}}{t^\alpha},\ t > 0,\ \alpha > 0,\\ &\|(e^{-At} - Id)v\|_X \le \tfrac{1}{\varepsilon} C_{1-\varepsilon} t^\varepsilon \|A^\varepsilon v\|_X,\ v \in X^\varepsilon,\ \varepsilon > 0. \end{aligned}$$

Let $u_0 \in X^\alpha$ and $u(\cdot,u_0)$ be a noncontinuable local mild X^α-solution to (1) through u_0 defined on $[0,\tau)$. Suppose further that $\tau < +\infty$. If $0 < \varepsilon < 1 - \alpha$ and $0 < \eta < \bar t < t < \tau$, then from (2) and the above estimates for analytic semigroups we obtain that

$$\begin{aligned} \|u(t,u_0) - u(\bar t,u_0)\|_{X^\alpha} \le{}& \|(e^{-A(t-\bar t)} - Id)A^\alpha e^{-A\bar t}u_0\|_X \\ &+ \int_0^{\bar t} \|(e^{-A(t-\bar t)} - Id)A^\alpha e^{-A(\bar t - s)}F(u(s,u_0))\|_X ds \\ &+ \int_{\bar t}^t \|A^\alpha e^{-A(t-s)}\|_{\mathcal{L}(X,X)} \|F(u(s,u_0))\|_X ds \end{aligned}$$

$$
\begin{aligned}
&\le \tfrac{1}{\varepsilon} C_{1-\varepsilon}(t-\bar{t})^{\varepsilon} c_{\alpha+\varepsilon} \tfrac{e^{-a\bar{t}}}{\bar{t}^{\alpha+\varepsilon}} \|u_0\|_X \\
&\quad + \int_0^{\bar{t}} \frac{1}{\varepsilon} C_{1-\varepsilon}(t-\bar{t})^{\varepsilon} c_{\alpha+\varepsilon} \frac{e^{-a(\bar{t}-s)}}{(\bar{t}-s)^{\alpha+\varepsilon}} \|F(u(s,u_0))\|_X ds \\
&\quad + \int_{\bar{t}}^{t} c_\alpha \frac{e^{-a(t-s)}}{(t-s)^{\alpha}} \|F(u(s,u_0))\|_X ds \\
&\le (t-\bar{t})^{\varepsilon} \frac{c_{\alpha+\varepsilon} C_{1-\varepsilon}}{\varepsilon \eta^{\alpha+\varepsilon}} \|u_0\|_X + (t-\bar{t})^{\varepsilon} \frac{c_{\alpha+\varepsilon} C_{1-\varepsilon}}{\varepsilon(1-\alpha-\varepsilon)} \tau^{1-\alpha-\varepsilon} \sup_{\|v\|_{X^\alpha} \le M_{u_0}} \|F(v)\|_X \\
&\quad + (t-\bar{t})^{\varepsilon} (t-\bar{t})^{1-\alpha-\varepsilon} \frac{c_\alpha}{1-\alpha} \sup_{\|v\|_{X^\alpha} \le M_{u_0}} \|F(v)\|_X \\
&\le (t-\bar{t})^{\varepsilon} const.(\varepsilon, \alpha, \eta, \tau, \|u_0\|_{X^\alpha}, F).
\end{aligned}
$$

Considering Cauchy sequences one shows the existence of the limit $\lim_{t\to\tau^-} \|u(t,u_0)\|_{X^\alpha}$. The latter allows to extend $u(\cdot,u_0)$ onto the interval $[0,\tau+\delta)$ (cf. [L-M, Theorem 1]) which contradicts the maximality of τ.

We have thus justified that, if hypothesis (H) is satisfied, then each local mild X^α-solution to (2) resulting from Proposition 1 may be extended onto the whole half line $[0,+\infty)$. Theorem 1 is thus proved.

THEOREM 2. *Let the Assumptions (A) and (H) be satisfied and V be a subset of X^α. Suppose there exists const. > 0 such that for each $u_0 \in V$ and for each corresponding global mild X^α-solution $u(\cdot,u_0)$ to (2)*

$$
\limsup_{t\to+\infty} \|u(t,u_0)\|_Y < const. \tag{8}
$$

Then, any such solution satisfies the inequality

$$
\limsup_{t\to+\infty} \|u(t,u_0)\|_{X^\alpha} \le const.', \tag{9}
$$

with const.$' > 0$ independent of $u_0 \in V$.

PROOF. Based on (8) we choose for $u_0 \in V$ a positive time t_{u_0} such that, for any $t > \tau > t_{u_0}$,

$$
\sup_{s\in[\tau,t)} \|u(s,u_0)\|_Y \le const. \tag{10}
$$

and *const.* is independent of $u_0 \in V$. We then write the integral equation defining the mild X^α-solution to (1) in the form:

$$
u(t,u_0) = e^{-At}u_0 + \left(\int_0^\tau + \int_\tau^t\right) e^{-A(t-s)} F\big(u(s,u_0)\big) ds. \tag{11}
$$

As a consequence of (5) and estimates in fractional power spaces [HE, p.26], we obtain

$$
\begin{aligned}
\|u(t,u_0)\|_{X^\alpha} \le \quad & \|A^\alpha e^{-At}u_0\|_X + \int_0^\tau \|A^\alpha e^{-A(t-s)}\|_{\mathcal{L}(X,X)} \|F\big(u(s,u_0)\big)\|_X ds \\
& + \int_\tau^t \|A^\alpha e^{-A(t-s)}\|_{\mathcal{L}(X,X)} g(\|u(s,u_0)\|_Y)(1 + \|u(s,u_0)\|_{X^\alpha}^\theta) ds
\end{aligned}
$$

$$\le c_0 e^{-at}\|u_0\|_{X^\alpha} + \sup_{\|v\|_{X^\alpha}\le z_1(u_0)} \|F(v)\|_X \int_{t-\tau}^{t} c_\alpha \frac{e^{-ay}}{y^\alpha}dy$$

$$+ g(const.)(1+\sup_{s\in[\tau,t]}\|u(s,u_0)\|^\theta_{X^\alpha})\int_0^{t-\tau} c_\alpha\frac{e^{-ay}}{y^\alpha}dy, \qquad t>\tau>t_0(u_0),\ u_0\in V, \tag{12}$$

with $z_1(u_0)$ defined as:

$$z_1(u_0) := \sup_{t\in[0,+\infty)}\|u(t,u_0)\|_{X^\alpha}.$$

Let $\tau := \tau_0(\varepsilon) > t_{u_0}$ be such that

$$\sup_{s\in[\tau_0(\varepsilon),+\infty)}\|u(s,u_0)\|_{X^\alpha} \le \limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha} + \varepsilon. \tag{13}$$

Since the first two components of the right hand side in (12) tend to zero as $t \longrightarrow +\infty$, we get:

$$\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha} \le g(const.)c_\alpha\frac{\Gamma(1-\alpha)}{a^{1-\alpha}}(1+\sup_{s\in[\tau_0(\varepsilon),+\infty)}\|u(s,u_0)\|^\theta_{X^\alpha}). \tag{14}$$

Denoting

$$C := g(const.)c_\alpha\frac{\Gamma(1-\alpha)}{a^{1-\alpha}} \tag{15}$$

we obtain from (13) and (14) that

$$\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha} \le C(1+(\varepsilon+\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha})^\theta)$$

and, consequently,

$$\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha} \le C(1+(\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha})^\theta). \tag{16}$$

Condition (16) ensures that $z := \limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha}$ satisfies inequality

$$z \le C(1+z^\theta).$$

The latter yields the estimate

$$\limsup_{t\to+\infty}\|u(t,u_0)\|_{X^\alpha} \le z_0, \tag{17}$$

where $z_0 > 0$ solves the equation $C(1+z^\theta) - z = 0$.

As a consequence of (15), z_0 is independent of $u_0 \in V$. The proof is complete.

Let $\{T(t)\}$ be a semigroup on a metric phase space V. Following [HA], recall that $\{T(t)\}$ is *point dissipative* if there is a bounded subset B_0 of V which *attracts points* of V; i.e.

$$\forall_{v\in V}\ dist_V(T(t)v, B_0) \to 0, \quad \text{as} \quad t\to+\infty.$$

A set $\mathcal{A}\subset V$ is called *positively invariant* if $T(t)\mathcal{A}\subset\mathcal{A}$ for all $t\ge 0$. $\mathcal{A}$ is an *invariant set* if $T(t)\mathcal{A}=\mathcal{A}$ for $t\ge 0$. A compact invariant set $\mathcal{A}$ is a *global attractor* for $\{T(t)\}$ in V if $\mathcal{A}$ *attracts bounded subsets* of V. The latter means that

$$\forall_{B\subset V,\ B\ \text{bounded}}\ \sup_{x\in T(t)B}\inf_{y\in\mathcal{A}} dist_V(T(t)B,\mathcal{A}) \to 0, \quad \text{as} \quad t\to+\infty.$$

COROLLARY 1. *Let the assumptions of Theorem 2 hold. If for $u_0 \in V$ each global mild X^α-solution to (1) is unique and $V \subset X^\alpha$ is closed and positively invariant, then the problem (1) generates a continuous semigroup $T(t): V \to V$, $t \geq 0$, of global mild X^α-solutions which has a global attractor in V.*

PROOF. Setting $T(t)u_0 := u(t, u_0)$, $u_0 \in V$, we shall show that $T(t): V \to V$ is a *continuous semigroup*, that is: (i) $T(0) = Id$ on V, (ii) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$, (iii) For arbitrary $t \geq 0$ the mapping $T(t): V \to V$ is continuous in the X^α-norm.

Fulfillment of conditions (i) and (ii) is a consequence of (2), uniqueness of the solution and the property that $u \in C([0,\sigma), X^\alpha)$ for arbitrary $\sigma > 0$ (cf. [L-M, p. 278]). We need only explain that the condition (iii) is satisfied.

Let $\{u_n\} \subset V$, $u_n \to u_0$ in X^α. Following the proof of Theorem 1 one may show that $\{T(t)u_n,\ n \in N\}$ is equicontinuous on $[\eta, \tau]$ in X^α, for each $0 < \eta < \tau < +\infty$ (cf. [L-M, Lemma 7]). Moreover, since $u_n \to u_0$ in X^α, it is easy to see that $\{T(t)u_n,\ n \in N\}$ is equicontinuous at $t = 0$ in X^α. We shall next observe that $\{T(t)u_n,\ n \in N,\ t \in [\eta, \tau]\}$ is precompact in X^α for each $0 < \eta < \tau < +\infty$. In particular, the maps $T(t)$ are compact in X^α for arbitrarily fixed $t > 0$. Indeed, estimating in (2) with the aid of [HE, Theorem 1.4.3], we find:

$$\|u(t,u_0)\|_{X^{\alpha+\varepsilon}} \leq c_\varepsilon \frac{e^{-at}}{t^\varepsilon}\|u_0\|_{X^\alpha} + \int_0^t c_{\alpha+\varepsilon}\frac{e^{-a(t-s)}}{(t-s)^{\alpha+\varepsilon}}\|F(u(s,u_0))\|_X ds, \tag{18}$$

where $\varepsilon > 0$ and $\alpha + \varepsilon < 1$. Using (6) we obtain

$$\sup_{t\in[0,\tau]} t^\varepsilon \|T(t)u_0\|_{X^{\alpha+\varepsilon}} \leq const.(\tau, R), \qquad \|u_0\|_{X^\alpha} \leq R.$$

This shows that $\{T(t)u_0,\ \|u_0\| \leq R,\ \eta \leq t \leq \tau\}$, as a bounded subset of $X^{\alpha+\varepsilon}$ ($\varepsilon > 0$, $\alpha+\varepsilon < 1$) is precompact in X^α (cf. [HE, Theorem 1.4.8]). In particular, $T(t): X^\alpha \to X^\alpha$ is a compact map for each $t > 0$.

The above considerations allow us to use the Ascoli-Arzela theorem. Let $\{\xi_k\}_{k\in N}$ be a sequence of nonnegative numbers dense in $[0, +\infty)$. If $\{u_{n'}\}$ is any subsequence of $\{u_n\}$, then based on the above considerations one may choose a diagonal subsequence $\{u_{n''}\}$ of $\{u_{n'}\}$ such that $T(\xi_k)u_{n''}$ is convergent in X^α for each $k \in N$. Since the family $\{T(\cdot)u_{n''}\}$ is equicontinuous on compact subintervals of $(0, +\infty)$, there is an element $v \in C((0,+\infty), X^\alpha)$ such that $T(t)u_{n''} \to v(t)$ uniformly on compact subintervals of $(0, +\infty)$. Since also $\{T(\cdot)u_{n''}\}$ is equicontinuous at 0, $v(h) \to u_0$ as $h \to 0^+$, which ensures that $v \in C([0,+\infty), X^\alpha)$. Passing to the limit in the integral equation (2) written for initial data $u_{n''}$ we deduce the formula:

$$v(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(v(s))ds,\ t \geq 0.$$

Therefore, by uniqueness,

$$v(t) = T(t)u_0,\ t \geq 0.$$

The above considerations show that from each subsequence $\{u_{n'}\}$ of $\{u_n\}$ one may choose a subsequence $\{u_{n''}\}$ such that, for any $t \geq 0$, $T(t)u_{n''} \to T(t)u_0$ in X^α. Consequently, $T(t)u_n$ converges to $T(t)u_0$ in X^α, which ensures that the maps $T(t): V \to V$, $t \geq 0$, are continuous.

Finally, under condition (8), the continuous semigroup $\{T(t)\}$ is point dissipative in V whereas the estimate (6) guarantees that orbits of bounded sets are bounded. By [LA, Theorem 2.2], $\{T(t)\}$ possesses a global attractor in V. The proof is complete.

For slowly growing nonlinearities the existence of a global attractor will be shown without assuming the condition (H). We then have:

COROLLARY 2. *Let the Assumption* (A) *hold, $\alpha \in (0,1)$, and let $F : X^\alpha \to X$ satisfy the condition*

$$\|F(v)\|_X \le const.(1 + \|v\|^\theta_{X^\alpha}), \quad for \;\; v \in X^\alpha, \tag{19}$$

with some $\theta \in [0,1)$ and const. independent on v. Let further, for $u_0 \in W$, each global mild X^α-solution to (1) be unique, where $W \subset X$ is closed and positively invariant. Then the problem (1) generates a C^0-semigroup $S(t) : W \to W$, $t \ge 0$, of global mild X^α-solutions which has a global attractor in W.

PROOF. Based on Proposition 2 the existence of a compact C^0-semigroup $\{S(t)\}$ on W is straightforward (cf. [L-M, Remark 2]). It thus suffices to prove that $\{S(t)\}$ is point dissipative.

For $u_0 \in W$ the integral equation written for $S(1)u_0$ has the form

$$S(t)S(1)u_0 = e^{-At}S(1)u_0 + \int_0^t e^{-A(t-s)}F(S(s)S(1)u_0)ds, \quad t > 0,$$

where $S(1)u_0 \in X^\alpha$. As in Theorem 2 we obtain that the orbit of each point enters to and remains inside a fixed ball in X^α. Therefore, $\{S(t)\}$ is point dissipative in W, which completes the proof.

4. Applications

EXAMPLE 1 (General semilinear initial boundary value problem). We first describe a large class of problems for which Proposition 1 is applicable. There will be the *initial boundary value problems* of the type:

$$\begin{cases} u_t = -Au + f(x, d^{m_0}u), \;\; (t,x) \in R^+ \times \Omega, \\ B_0 u = B_1 u = \ldots = B_{m-1}u = 0 \;\; \text{on} \;\; \partial\Omega, \\ u(0,x) = u_0(x) \;\; \text{in} \;\; \Omega, \end{cases} \tag{20}$$

with $2m$-th order *uniformly strongly elliptic operator* A (cf. [FR, p. 2]) and continuous function $f : \overline{\Omega} \times R^{d_0} \to R$. Here Ω is a bounded domain in R^n and $d^{m_0}u$, $m_0 \le 2m-1$, denotes the vector $(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^{m_0} u}{\partial x_n^{m_0}})$ of the spatial partial derivatives of u of order less or equal m_0; consequently, $d_0 = \frac{(n+m_0)!}{n!(m_0)!}$.

Whenever $(A, \{B_j\}, \Omega)$ forms a regular elliptic boundary value problem, equation (20) admits abstract formulation (1) with A sectorial, positive in $X = L^p(\Omega)$ ($p \in (1, +\infty)$) and such that the resolvent of A is compact (cf. [FR, p. 101]).

If we set $\alpha \in [0,1)$ and $2m\alpha - \frac{n}{p} > m_0$, then $X^\alpha \subset C^{m_0}(\overline{\Omega})$ (cf. [HE, p. 39]). For such a choice of parameters, if u varies in a bounded subset of X^α, then the argument $d^{m_0}u$ will vary in a bounded subset of R^{d_0}. Since f is uniformly continuous on compact sets

$\overline{\Omega} \times [-M, M]^{d_0}$, the values of

$$F(u) = f(\cdot, d^{m_0} u(\cdot))$$

will be bounded in $C^0(\overline{\Omega})$, and hence also in $L^p(\Omega)$. Therefore, the assumptions of Proposition 1 hold and the existence of local mild X^α-solutions to (20) follows.

Example 2. Consider now special case of (20), the diffusion equation with strong absorption term:

$$\begin{cases} u_t = \Delta u - \lambda |u|^\theta, \ \lambda > 0, \ \theta \in (0,1), \\ u = 0 \ \text{ on } \ \partial\Omega, \\ u(0,x) = u_0(x) \ \text{ in } \ \Omega. \end{cases} \tag{21}$$

For $\alpha \in [0,1)$, $p \in [2, +\infty)$ satisfying $2\alpha > \frac{n}{p}$, Proposition 2 ensures the existence of a global mild X^α-solution $u(\cdot, u_0)$ through each $u_0 \in L^p(\Omega)$. Following [L-M, Theorem 2], such a solution is a limit of a sequence $u_{z_n} = u(\cdot, z_n)$ of global mild X^α-solutions such that $\|z_n - u_0\|_{L^p(\Omega)} \to 0$ and $\{z_n\} \subset X^\alpha$.

Let $\partial\Omega$ be of class $C^{2+\varepsilon}$, $\varepsilon > 0$, and define

$$D^+ = \{\phi \in C^{2+\varepsilon}(\overline{\Omega});\ \phi_{|\partial\Omega} = 0, \Delta\phi_{|\partial\Omega} = 0, \phi \geq 0\}.$$

As shown in [DL, Theorem 1], for $z \in D^+$, the problem (21) possesses a unique positive Hölder solution $v_z = v_z(t,x)$. That is, there exists a unique $v_z \geq 0$ satisfying (21) in the classical sense and such that $v_z \in C^{1+\frac{\varepsilon}{2}, 2+\varepsilon}([0,\tau] \times \overline{\Omega})$ for each $\tau > 0$. It is clear that v_z may be treated as an element of $C^1((0,+\infty), X) \cap C([0,+\infty), X^\alpha)$. In particular, v_z is a global mild X^α-solution to (21).

Choose $u_0 \in W := cl_{L^p(\Omega)} D^+$. Following [L-M, Theorem 2], a global mild X^α-solution through u_0 may be thus obtained as the limit of the sequence $\{v_{z_n}(t,\cdot)\}$ of Hölder solutions through $z_n \in D^+$, where $z_n \to u_0$ in $L^p(\Omega)$. If $z_n, w_n \in D^+$ and v_{z_n}, v_{w_n} are corresponding nonnegative Hölder solutions to (21), then

$$(v_{z_n} - v_{w_n})_t = \Delta(v_{z_n} - v_{w_n}) - \lambda(v_{z_n}^\theta - v_{w_n}^\theta). \tag{22}$$

Multiply (22) in $L^2(\Omega)$ by $(v_{z_n} - v_{w_n})$, integrate by parts and use the condition

$$sgn(v_{z_n}^\theta - v_{w_n}^\theta) = sgn(v_{z_n} - v_{w_n}),$$

to get an estimate

$$\|v_{z_n}(t,\cdot) - v_{w_n}(t,\cdot)\|_{L^2(\Omega)} \leq \|z_n - w_n\|_{L^2(\Omega)},\ z_n, w_n \in D^+,\ t \geq 0.$$

This shows the uniqueness of the limit solution $v(\cdot, u_0)$ through $u_0 \in W$. In particular the problem (21) generates on W a C^0-semigroup $\{T(t)\}$ of global mild X^α-solutions. As a consequence of Corollary 2, $\{T(t)\}$ possesses a global attractor $\mathcal{A}$ in W. Indeed, since

$$\|F(v)\|_{L^p(\Omega)} = \||v|^\theta\|_{L^p(\Omega)} = \|v\|^\theta_{L^{p\theta}(\Omega)},\ v \in L^p(\Omega),$$

condition (19) follows as a result of Sobolev inclusions $X^\alpha \subset L^p(\Omega) \subset L^{p\theta}(\Omega)$ where $\theta \in (0,1)$ and $\alpha \in [0,1)$.

Remark 2. We remark that $W = cl_{L^p(\Omega)} D^+$ is the cone of nonnegative elements of $L^p(\Omega)$. Also, in this example the attractor $\mathcal{A}$ is trivial. Moreover, bounded subsets of W are absorbed by $\{0\}$ in a finite time (cf. [DL, Theorem 2]). We finally recall that, as a result of [PA 1, Theorem 5.2], mild solutions considered above in Example 2 are actually *strong solutions* of the abstract sectorial differential equation corresponding to (21). That is, $u(t, u_0)$ is strongly continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t > 0$, and $u(t) \to u_0$ in X as $t \to 0^+$.

References

[C-C-D] A. N. de Carvalho, J. W. Cholewa and T. Dlotko, *Examples of global attractors in parabolic problems*, Hokkaido Math. J. 27 (1998), 77–103.

[C-D] J. W. Cholewa and T. Dlotko, *Global attractor for sectorial evolutionary equation*, J. Differential Equations 125 (1996), 27–39.

[DL] T. Dlotko, *Parabolic equation modelling diffusion with strong absorption*, Atti Sem. Mat. Fis. Univ. Modena 38 (1990), 61–70.

[FR] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[HA] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, R. I., 1988.

[HE] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.

[LA] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.

[L-M] J. H. Lightbourne and R. H. Martin, *Relatively continuous nonlinear perturbations of analytic semigroups*, Nonlinear Analysis TMA 1 (1977), 277–292.

[MA] R. H. Martin, *Invariant sets and a mathematical model involving semilinear differential equations*, in: Nonlinear Equations in Abstract Spaces, Proc. Inter. Symp. Univ. of Texas Arlington, Academic Press, New York 1978, 135–148.

[PA 1] A. Pazy, *A class of semi-linear equations of evolution*, Israel J. Math. 20 (1975), 23–36.

[PA 2] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.