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INTEGRAL INEQUALITIES AND SUMMABILITY OF SOLUTIONS OF SOME DIFFERENTIAL PROBLEMS

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1. Introduction. The aim of this note is to indicate how inequalities concerning the integral of $|\nabla u|^2$ on the subsets where $|u(x)|$ is greater than k ($k \in \mathbb{R}^+$) can be used in order to prove summability properties of u (joint work with Daniela Giachetti). This method was introduced by Ennio De Giorgi and Guido Stampacchia for the study of the regularity of the solutions of Dirichlet problems.

In some joint works with Thierry Gallouet, inequalities concerning the integral of $|\nabla u|^2$ on the subsets where $|u(x)|$ is less than k ($k \in \mathbb{R}^+$) or where $k \le |u(x)| < k+1$ were used in order to prove estimates in Sobolev spaces larger than $W_0^{1,2}(\Omega)$ for solutions of Dirichlet problems with irregular data.

2. Integral inequalities and summability of u. I recall the following regularity theorem by Guido Stampacchia concerning solutions of linear Dirichlet problems.

Let Ω be a bounded subset of $\mathbb{R}^N$ ($N > 1$). Consider a bounded elliptic matrix $A(x)$ and a function f which belongs to $L^q(\Omega)$, $q > \frac{2N}{N+2}$ and the related boundary value problem

$$u \in H_0^1(\Omega): \qquad -\mathrm{div}(A(x)Du) = f(x).$$

Guido Stampacchia proved that:

$$\begin{cases} q > \frac{N}{2} \Rightarrow u \in L^\infty(\Omega); \\ \frac{2N}{N+2} < q < \frac{N}{2} \Rightarrow u \in L^{q^{**}}(\Omega), \quad q^{**} = (q^*)^* = \frac{qN}{N-2q}. \end{cases}$$

If the matrix A is symmetric, the solution u of the previous equation can be seen as the unique minimum on $H_0^1(\Omega)$ of the functional

$$J(v) = \frac{1}{2}\int_\Omega A(x)\nabla v\,\nabla v - \int_\Omega f\,v, \qquad v \in H_0^1(\Omega).$$

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Thus, the Stampacchia regularity theorem can be stated in the following way: if $f \in L^q(\Omega)$ and $q > \frac{N}{2}$, the minimum u of J belongs to $L^\infty(\Omega)$, if $\frac{2N}{N+2} < q < \frac{N}{2}$, the minimum belongs to $L^{q^{**}}(\Omega)$. The Stampacchia method uses as important tool, in both cases, the test function $u - T_k(u)$, where $T_k(u)$ is the truncation at the levels $+k$ and $-k$.

In order to prove the L^s-estimate of solutions to some variational problems (minima of integral functionals, solutions of nonlinear elliptic equations), our main tool is Lemma 2.3, below. Closely related ideas are contained in the work of Guido Stampacchia, who earlier established regularity results in Marcinkiewicz spaces (and then in L^s, in the linear setting) using integral inequalities (Lemma 2.2).

Recall that ($p = 2$ for sake of simplicity):

Lemma 2.1 [G. Stampacchia]. *Let* $u \in W_0^{1,2}(\Omega)$, $\varphi \in L^r(\Omega)$, $r > \frac{N}{2},$ *satisfy*

$$\int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq \Bigg[\int_{\{x\in\Omega:|u(x)|\geq k\}} \varphi^{\frac{2N}{N+2}}\Bigg]^{\frac{N+2}{N}}.$$

Then u is bounded.

Lemma 2.2 [G. Stampacchia]. *Let* $u \in W_0^{1,2}(\Omega)$, $\varphi \in M^r(\Omega)$, $\frac{2N}{N+2} < r < \frac{N}{2},$ *satisfy*

$$\int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq \Bigg[\int_{\{x\in\Omega:|u(x)|\geq k\}} \varphi^{\frac{2N}{N+2}}\Bigg]^{\frac{N+2}{N}}.$$

Then u belongs to $M^{r^{**}}(\Omega)$.

As important consequence, thanks to the combined use of the previous lemma and the linear interpolation, Guido Stampacchia proved that, if u is the solution of a **linear** elliptic boundary value problem with right hand side $f(x)$, with f in $L^r(\Omega)$, then u belongs to $L^{r^{**}}(\Omega)$.

We proved the following lemma.

Lemma 2.3. *Let* $u \in W_0^{1,2}(\Omega)$, $\varphi \in L^r(\Omega)$, $\frac{2N}{N+2} < r < \frac{N}{2},$ *satisfy*

$$\int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq \Bigg[\int_{\{x\in\Omega:|u(x)|\geq k\}} \varphi^{\frac{2N}{N+2}}\Bigg]^{\frac{N+2}{N}}.$$

Then u belongs to $L^{r^{**}}(\Omega)$.

Sketch of the proof. The previous inequality implies that, for every $k > 0$,

$$k^{2\gamma-1} \int_{\{x\in\Omega:|u(x)|\geq k\}} |\nabla u|^2 \leq k^{2\gamma-1} \Bigg[\int_{\{x\in\Omega:|u(x)|\geq k\}} \varphi^{\frac{2N}{N+2}}\Bigg]^{\frac{N+2}{N}}.$$

Then we use that

$$\sum_{k=0}^{\infty} k^{2\gamma-1} \int\limits_{\{x\in\Omega:|u(x)|\ge k\}} \Psi = \sum_{k=0}^{\infty} k^{2\gamma-1} \sum_{j=k}^{\infty} \int\limits_{\{x\in\Omega:j\le|u(x)|<j+1\}} \Psi$$

$$= \sum_{j=0}^{\infty} \int\limits_{\{x\in\Omega:j\le|u(x)|<j+1\}} \Psi \sum_{k=0}^{j} k^{2\gamma-1} \le c + c\int\limits_{\Omega} \psi\, |u|^{2\gamma}. \ \blacksquare$$

As a consequence of the previous lemma we proved the following regularity results.

THEOREM 2.4. *Consider the nonlinear boundary value problem*

$$u \in H_0^1(\Omega): \qquad -\mathrm{div}(a(x,u,\nabla u)) = f(x),$$

where f belongs to $L^q(\Omega)$, $\frac{2N}{N+2} < q < \frac{N}{2}$. Under standard assumptions on a, u belongs to $L^{q^{**}}(\Omega)$.

Moreover

THEOREM 2.5. *Under standard assumptions on j, the minima u of the functional*

$$(**) \qquad J(v) = \int\limits_{\Omega} j(x,v,\nabla v) - \int\limits_{\Omega} f\, v, \qquad v \in H_0^1(\Omega).$$

belong to $L^{q^{**}}(\Omega)$, if f belongs to $L^q(\Omega)$, $\frac{2N}{N+2} < q < \frac{N}{2}$.

Developments of our method ([3]) can be found in [5] (regularity of minimizing sequences), in [6] (local regularity of minima of functionals), and in [4] and [7] (parabolic equations: global or local case).

3. Integral inequalities and estimates in Marcinkiewicz spaces. Integral inequalities of the type

$$\int\limits_{\{x\in\Omega:|u(x)|<k\}} |\nabla u|^2 \le c_0\, k$$

arise in Dirichlet problems with irregular data (e.g. right hand side measures) and have been used to give estimates in Marcinkiewicz spaces on u and ∇u.

LEMMA 3.1. *Let u be such that, for every $k > 0$,*

$$\int\limits_{\{x\in\Omega:|u(x)|<k\}} |\nabla u|^2 \le c_0\, k.$$

Then u belongs to the Marcinkiewicz space $M^{\frac{N}{N-2}}(\Omega)$ and ∇u belongs to the Marcinkiewicz space $M^{\frac{N}{N-1}}(\Omega)$.

SKETCH OF THE PROOF. The Sobolev inequality implies that

$$c_0\, k \ge \Big(\int\limits_{\Omega} |T_k(u)|^{2^*}\Big)^{\frac{2}{2^*}} \ge \Big(\int\limits_{k<|u|} k^{2^*}\Big)^{\frac{2}{2^*}} = k^2 \mathrm{meas}\{k < |u|\}^{\frac{2}{2^*}}.$$

Thus $u \in M^{\frac{N}{N-2}}(\Omega)$. Moreover

$$\operatorname{meas}\{h < |\nabla u|\} \leq \operatorname{meas}\{h < |\nabla u|,\, |u| \leq k\} + \operatorname{meas}\{k < |u|\} \leq c_1 \frac{k}{h^2} + c_2 \frac{1}{k^{\frac{N}{N-2}}}$$

and the estimate on ∇u follows by minimization on k. ∎

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