EVOLUTION EQUATIONS:
EXISTENCE, REGULARITY AND SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 52
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

DATA ASSIMILATION FOR THE TIME-DEPENDENT TRANSPORT PROBLEM

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1. Introduction. In this paper we consider the data assimilation problem for a time-dependent transport problem in a slab when the initial condition is not known. The spaces of traces are introduced, the solvability of the original initial-boundary value transport problem is studied. The properties of the control operator are investigated, the solvability of the data assimilation problem is proved. The class of iterative methods for solving the problem is considered, and the convergence conditions are studied. The results are closely connected with some issues raised in [4], [14], [15].

2. Statement of the problem and functional spaces. Consider the time-dependent initial-boundary value transport problem in a slab [11]:

$$\begin{gathered}\frac{\partial\varphi}{\partial t}+\mu\frac{\partial\varphi}{\partial z}+\sigma(z,\mu,t)\varphi=\textstyle\int_{-1}^{1}K(z,\mu,\mu',t)\varphi(z,\mu',t)d\mu'+f(z,\mu,t),\\ z\in(-H,H),\ \ \mu\in(-1,1),\ \ t\in(0,T),\end{gathered}\tag{1}$$

$$\begin{gathered}\varphi(-H,\mu,t)=0,\quad 0<\mu<1,\ t\in(0,T),\\ \varphi(H,\mu,t)=0,\quad -1<\mu<0,\ t\in(0,T),\end{gathered}\tag{2}$$

$$\varphi(z,\mu,0)=u(z,\mu),\quad z\in(-H,H),\ \ \mu\in(-1,1),\tag{3}$$

where $\varphi(z,\mu,t)$ is an unknown distribution function, $\sigma(z,\mu,t)$, $K(z,\mu,\mu',t)$, $f(z,\mu,t)$, $u(z,\mu)$ are prescribed functions, $T,H<\infty$.

Introduce the notations:

$$\begin{gathered}D=\{(z,\mu,t):z\in(-H,H),\ \mu\in(-1,1),\ t\in(0,T)\},\\ \partial D_-=\{(z,\mu,t):\Gamma_-\cup(t=0)\},\ \partial D_+=\{(z,\mu,t):\Gamma_+\cup(t=T)\},\end{gathered}$$

2000 *Mathematics Subject Classification*: Primary 35B37; Secondary 65K10.
Research supported by Russian Foundation for Basic Research grant 00-01-00611.
The paper is in final form and no version of it will be published elsewhere.

where

$$\Gamma_- = \{(z,\mu,t) : (\mu \in (0,1), z = -H) \cup (\mu \in (-1,0), z = H),\ t \in (0,T)\},$$
$$\Gamma_+ = \{(z,\mu,t) : (\mu \in (-1,0), z = 0) \cup (\mu \in (0,1), z = H),\ t \in (0,T)\}.$$

Here ∂D_- is the "incoming" part of the boundary D, i.e. the part where the initial and boundary conditions are set.

Introduce the spaces $L_2 = L_2(D),\ H_2^1 = H_2^1(D)$ of real-valued functions with the norms

$$\|\varphi\|_{L_2} = \left(\int_D |\varphi|^2 dD\right)^{1/2} = \left(\int_0^T dt \int_{-1}^1 d\mu \int_{-H}^H |\varphi(z,\mu,t)|^2 dz\right)^{1/2},$$
$$\|\varphi\|_{H_2^1} = \|\varphi\|_{L_2} + \left\|\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial z}\right\| L_2.$$

Consider the initial-value function $u(z,\mu)$. We define the space of traces ("boundary values") $L_{2,-}$ as a space of vector-functions $u(z,\mu)$ with the norm

$$\|u\|_{L_{2,-}} = \left[\int_{-1}^1 d\mu \int_{-H}^H \lambda(z,\mu)|u(z,\mu)|^2 dz\right]^{1/2}, \tag{4}$$

where the function $\lambda(z,\mu)$ is defined by

$$\lambda(z,\mu) = \begin{cases} T, & T|\mu| \le H - l_\mu z \\ \dfrac{H - l_\mu z}{|\mu|}, & T|\mu| \ge H - l_\mu z, \end{cases} \tag{5}$$

and

$$l_\mu = \operatorname{sign}\mu = \begin{cases} 1, & \mu > 0 \\ -1, & \mu < 0. \end{cases}$$

Let $f \in L_p(D),\ u \in L_{2,-},\ \sigma \in L_\infty(D)$. Let also for almost every point $(z,\mu,t) \in D$ the condition [18]

$$\int_{-1}^1 |K(z,\mu,\mu',t)| d\mu' = \int_{-1}^1 |K(z,\mu,\mu',t)| d\mu \le \theta_0, \quad \theta_0 = \text{const} > 0 \tag{6}$$

be satisfied. Moreover, we assume that the function $\sigma(z,\mu,t) = \sigma(z,t)$ does not depend on μ.

The function $\varphi \in H_2^1(D)$ is said to be a *weak solution* to the problem (1)–(3) if

$$\left\|\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial z} + \sigma\varphi - \int_{-1}^1 K\varphi d\mu' - f\right\|_{L_2} = 0, \quad \|\varphi|_{t=0} - u\|_{L_{2,-}} = 0, \tag{7}$$

and (2) is satisfied almost everywhere on Γ_-.

We consider the functional of $\varphi \in H_2^1$ in the form

$$J(\varphi) = \frac{\alpha}{2}\|u\|_{L_{2,-}}^2 + \frac{1}{2}\|\varphi - \widehat{\varphi}\|_{L_2}^2, \tag{8}$$

where $\alpha = const \ge 0$, $\widehat{\varphi} \in L_2 = L_2(D)$ is a prescribed function.

Let us formulate the following *data assimilation problem* [10], [4], [15]: find a pair of functions $u \in L_{2,-},\ \varphi \in H_2^1$ such that they satisfy the problem (1)–(3), while the functional (8) takes the least value on the set of solutions of (1)–(3). This problem is equivalent to the set of equations for $\varphi, \varphi^* \in H_2^1,\ u \in L_{2,-}$ [10], [4], [15]:

$$
(9) \qquad \begin{cases} \dfrac{\partial\varphi}{\partial t} + \mu\dfrac{\partial\varphi}{\partial z} + \sigma(z,\mu,t)\varphi = \int_{-1}^{1} K(z,\mu,\mu',t)\varphi(z,\mu',t)d\mu' + f, \\ \varphi(-H,\mu,t) = 0, \quad 0 < \mu < 1,\ t \in (0,T), \\ \varphi(H,\mu,t) = 0, \quad -1 < \mu < 0,\ t \in (0,T), \\ \varphi(z,\mu,0) = u(z,\mu), \quad z \in (-H,H),\ \mu \in (-1,1), \end{cases}
$$

$$
(10) \qquad \begin{cases} -\dfrac{\partial\varphi^*}{\partial t} - \mu\dfrac{\partial\varphi^*}{\partial z} + \sigma(z,\mu,t)\varphi^* = \int_{-1}^{1} K(z,\mu',\mu,t)\varphi^* d\mu' + \widehat{\varphi} - \varphi, \\ \varphi^*(H,\mu,t) = 0, \quad 0 < \mu < 1,\ t \in (0,T), \\ \varphi^*(-H,\mu,t) = 0, \quad -1 < \mu < 0,\ t \in (0,T), \\ \varphi^*(z,\mu,T) = 0, \quad z \in (-H,H),\ \mu \in (-1,1), \end{cases}
$$

$$
(11) \qquad \alpha(u,v)_{L_{2,-}} - (\varphi^*/_{t=0}, v) = 0, \quad v \in L_{2,-}.
$$

To study the solvability of the system (9)–(11), we formulate first the result on solvability of the original problem (1)–(3).

3. Solvability of the initial-boundary value transport problem. Under the above-formulated restrictions, consider the problem (1)–(3). The following theorem holds.

THEOREM 1. *For $u \in L_{2,-},\ f \in L_2(D)$, there exists a unique function $\varphi \in H_2^1(D)$ such that the equations* (7) *are satisfied and the estimates*

$$
(12) \qquad c_1(\|u\|_{L_{2,-}} + \|f\|_{L_2}) \le \|\varphi\|_{H_2^1} \le c_2(\|u\|_{L_{2,-}} + \|f\|_{L_2})
$$

hold, where the constants c_1, c_2 do not depend on f, φ, u.

PROOF. The equations of characteristic curves corresponding to (1) are as follows:

$$
\frac{dt}{d\tau} = 1, \quad \frac{dz}{d\tau} = \mu, \quad \frac{d\mu}{d\tau} = 0.
$$

Hence, the characteristic curves are the lines $t = \tau + c_1,\ z = \mu\tau + c_2,\ \mu = c_3,\ c_i =$const, $i = 1,2,3$. Consider the characteristic curves entering D at the points of ∂D_- and going out of D at the points of ∂D_+. Let $(z,\mu,t) \in \partial D_-$, then by $l(z,\mu,t)$ we denote the value of τ, for which the characteristic curve, going through the point (z,μ,t), goes out of D (the characteristic curves are assumed to enter D when $\tau = 0$). The function $l(z,\mu,t)$ is said to be the "length" of the characteristic curve in D. Thus, each point $(z,\mu,t) \in D$ may be defined by a point $(z',\mu',t') \in \partial D_-$ and some value of τ. Then, the domain D may be treated as a Cartesian product

$$
D = \{(z',\mu',t',\tau) :\ (z',\mu',t') \in \partial D_-,\ 0 < \tau < l(z',\mu',t')\}.
$$

If the function $\varphi(z,\mu,t)$ is summable on D, then, as follows from [9],

$$
\int_D \varphi dD = \int_{\partial D_-} \int_0^{l(z',\mu',t')} \varphi(z',\mu',t',\tau) d\tau d\nu^-,
$$

where

$$
d\nu^- = \begin{cases} d\mu' dz', & t = 0 \\ |\mu| d\mu' dt', & (z',\mu',t') \in \Gamma_-. \end{cases}
$$

Using the results of [16], it is readily seen that the function $l(z,\mu,t)$ has the form:

$$l(z,\mu,t)=\begin{cases}\lambda(z,\mu), & t=0\\ (T-t), & (z,\mu,t)\in\Gamma_-,\end{cases}$$

where the function $\lambda(z,\mu)$ is defined by formula (5). The function $l(z,\mu,t)$ is upper bounded: $l(z,\mu,t)\leq T$, but it vanishes at the points $t=T,\ (z,\mu,t)\in\Gamma_-$ and $t=0,\ (z,\mu,t)\in\Gamma_+$.

According to [9], [1], the function $l(z,\mu,t)$ should be the weight function when constructing the boundary norm:

$$\|v\|_l=\left(\int_{\partial D_-}l(z,\mu,t)|v|^2d\nu^-\right)^{1/2}.$$

Following [1], [9] and taking into account the boundedness of $l(z,\mu,t)$, it is easy to show that the function $\varphi\in H_2^1(D)$ has the trace $v\equiv\varphi\big|_{\partial D_-}$ such that

$$\|\varphi\big|_{\partial D_-}\|_l\leq c_1\|\varphi\|_{H_p^1},\quad c_1=\text{const}>0.$$

Conversely, if $\|v\|_l<\infty$, then there exists a function $\varphi\in H_2^1(\Omega)$, such that $v=\varphi\big|_{\partial D_-}$ and

$$\|\varphi\|_{H_2^1}\leq c_2\|\varphi\big|_{\partial D_-}\|_l,\quad c_2=\text{const}>0,$$

where the constant c_1,c_2 do not depend on φ,v.

Under the condition $|\sigma(z,t)|\leq\sigma_1=$const $<\infty$ and by the substitution $\varphi=\exp(\gamma t)\tilde{\varphi}$, where $\gamma=$const$>\sigma_1$, the problem (1)–(3) may be reduced to the problem for the function $\tilde{\varphi}$, with the coefficient $\tilde{\sigma}(z,t)=\gamma+\sigma(z,t)\geq\gamma-\sigma_1>0$ instead of $\sigma(z,t)$. Taking γ sufficiently large, under the condition (6) one can obtain

$$\int_{-1}^{1}\left|\frac{K(z,\mu,\mu',t)}{\sigma(z,t)}\right|d\mu'\leq k_0,\quad k_0=\text{const}<1. \tag{13}$$

Therefore, in further proof, we assume the condition (13) be satisfied and $\sigma(z,t)\geq\sigma_0>0$.

Following the technique of [3], [1], we should, first, show that for the space $H_2^1(D)$ the norm $\|\cdot\|_{H_2^1}$ is equivalent to the norm

$$[\varphi]_2=\|\varphi\big|_{\partial D_-}\|_l+\|A\varphi-S\varphi\|_{L_2},$$

where $A\varphi=\dfrac{\partial\varphi}{\partial t}+\mu\dfrac{\partial\varphi}{\partial z}+\sigma\varphi$, $S\varphi=\int_{-1}^{1}K(z,\mu,\mu',t)\varphi(z,\mu',t)d\mu'$.

The inequality $[\varphi]_2\leq c\|\varphi\|_{H_p^1}$ is easily obtained, taking into account the above-formulated result on traces of H_2^1. To derive the backward inequality, we consider the function $\varphi\in H_2^1$ and introduce the notation $f=Au-Su$. Then, by integrating along the characteristic curve we get the formula:

$$\varphi(z',\mu',t',\tau)=\exp\left(-\int_0^\tau\sigma(z',t',s)ds\right)\varphi(z',\mu',t',0)+$$
$$\int_0^\tau\exp\left(-\int_\xi^\tau\sigma(z',t',s)ds\right)(f+S\varphi)(z',\mu',t',\xi)d\xi,$$
$$(z',\mu',t')\in\partial D_-,\ \ 0<\tau<l(z',\mu',t').$$

Hence,

$$\|\varphi\|_{B_2} \le \|\varphi\big|_{\partial D_-}\|_l + \|(f+S\varphi)/\sigma\|_{B_2}, \tag{14}$$

where $\|\varphi\|_{B_2} = \|\sigma^{1/2}\varphi\|_{L_2}$.

From (13), it follows that $\|S\varphi/\sigma\|_{B_2} \le k_0\|\varphi\|_{B_2}$. Then, along with (14) and following [1], we find successively

$$(1-k_0)\|\varphi\|_{B_2} \le \|\varphi\big|_{\partial D_-}\|_l + \|f/\sigma\|_{B_2} \le c[\varphi]_2,$$
$$\|\varphi\|_{L_2} \le c[\varphi]_2,$$
$$\left\|\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial z}\right\|_{L_2} \le \|\sigma\varphi - S\varphi\|_{L_2} + \|(A-S)\varphi\|_{L_2} \le c[\varphi]_2,$$
$$\|\varphi\|_{H_2^1} \le c[\varphi]_2.$$

Using the norm equivalence, and the well-known results on properties of the solution to the time-dependent transport problem with regular prescribed functions [5], [16], we come to the conclusion of the theorem. ∎

Related issues concerning the properties of boundary values (traces) of solutions to the transport equations have been studied in the papers by V. S. Vladimirov [18], T. A. Germogenova [7], [8], C. Bardos [5], V. I. Agoshkov [1], [3], [2], M. Cessenat [6], S. Ukai [17], W. Greenberg, C. Van der Mee, and V. Protopopescu [9], S. Mischler [12].

4. The control operator and solvability of data assimilation problem. Consider the data assimilation problem (9)–(10). Following [4], [14], we obtain the equation for the control u. In view of Theorem 1, for the solutions φ, φ^* of (9), (10) the following representations are valid:

$$\varphi = G_0 u + G_1 f,$$
$$\varphi^* = G_1^{(T)}(\widehat{\varphi} - \varphi),$$

where $G_1 \in \mathcal{L}(L_2, H_2^1)$, $G_0 \in \mathcal{L}(L_{2,-}, H_2^1)$, $G_1^{(T)} \in \mathcal{L}(L_2, H_2^1)$ are linear bounded operators.

By eliminating φ, φ^*, we obtain the equation for u:

$$Lu = F, \tag{15}$$

where the control operator L and the right-hand side F are defined by the equalities:

$$(Lu, v)_{L_{2,-}} = \alpha(u,v)_{L_{2,-}} + (T_0 G_1^{(T)} G_0 u, v) = \alpha(u,v)_{L_{2,-}} + (G_0 u, G_0 v)_{L_2},$$
$$(F, v)_{L_{2,-}} = (-T_0 G_1^{(T)} G_1 f + T_0 G_1^{(T)}\widehat{\varphi}, v) = (-G_1 f + \widehat{\varphi}, G_0 v)_{L_2}, \quad v \in L_{2,-}.$$

The following statements are valid.

Lemma 1. *Let $\alpha = 0$. Then the operator L maps $L_{2,-}$ into $L_{2,-}$ with the domain of definition $D(L) = L_{2,-}$; it is bounded, self-adjoint and positive.*

Lemma 2. *If $\alpha = 0$, the range $R(L)$ of the operator L is dense in $L_{2,-}$.*

LEMMA 3. *Let $\sigma \geq \sigma_0 = const > 0$, $\theta_0 > \sigma_0$. Then for every $u \in L_{2,-}$ the following estimates are valid:*

$$\alpha(u,u)_{L_{2,-}} \leq (Lu,u)_{L_{2,-}} \leq \left(\alpha + \frac{\sigma_0}{(\theta_0-\sigma_0)^2}\right)(u,u)_{L_{2,-}}.$$

PROOF. Lemmas 1 and 2 are obtained from the definition of the operator L, following [4], [14]. Lemma 3 follows from the equality:

$$(Lu,u)_{L_{2,-}} = \alpha(u,v)_{L_{2,-}} + (G_0u, G_0u)_{L_2}, \quad u \in L_{2,-}.$$

Let $u \in L_{2,-}$ and $\varphi = G_0 u$ be the solution of (9) for $f = 0$. Then, from the proof of Theorem 1 (see (14)), it follows that for $k_0 = \theta_0/\sigma_0$ we have

$$(1-k_0)\|\sigma^{1/2}\varphi\|_{L_2} \leq \|\varphi\big|_{t=0}\|_{L_{2,-}},$$

hence,

$$(1-k_0)^2\sigma_0(G_0u, G_0u)_{L_2} \leq (u,u)_{L_{2,-}},$$

or

$$(G_0u, G_0u)_{L_2} \leq \frac{\sigma_0}{(\theta_0-\sigma_0)^2}(u,u)_{L_{2,-}}.$$

This implies the assertion of Lemma 3. ∎

From Lemmas 1-3, we deduce the following theorem.

THEOREM 2. *Let $f, \widehat{\varphi} \in L_2$. Then for $\alpha > 0$ the data assimilation problem* (9)–(11) *has a unique solution $\varphi, \varphi^* \in H_2^1$, $u \in L_{2,-}$, and*

$$\|\varphi\|_{H_2^1} + \|\varphi_0^*\|_{H_2^1} + \|u\|_{L_{2,-}} \leq c\,(\|f\|_{L_2} + \|\widehat{\varphi}\|_{L_2}), \quad c = const > 0.$$

The proof of this theorem is based on the properties of the control operator L introduced above with the use of the reasoning of [4], [15].

5. Iterative algorithms. In order to solve the problem (9)–(11) we consider a class of iterative methods of the form:

$$\frac{d\varphi^k}{dt} + A\varphi^k = S\varphi^k + f,\ t \in (0,T);\ \varphi^k/_{t=0} = u^k, \tag{16}$$

$$-\frac{d\varphi^{*k}}{dt} + A^*\varphi^{*k} = S^*\varphi^{*k} + \widehat{\varphi} - \varphi^k,\ t \in (0,T);\ \varphi^{*k}/_{t=T} = 0, \tag{17}$$

$$(\xi^k, v)_{L_{2,-}} = \alpha(u^k, v)_{L_{2,-}} - (\varphi^{*k}/_{t=0}, v), \quad v \in L_{2,-}. \tag{18}$$

$$u^{k+1} = u^k - \alpha_{k+1}B_k\xi^k + \beta_{k+1}C_k(u^k - u^{k-1}) \text{ in } L_{2,-}, \tag{19}$$

where φ^k, φ^{*k}, u^k are iterative sequences, α_{k+1}, β_{k+1} are iterative parameters, and B_k, $C_k :\ L_{2,-} \to L_{2,-}$ are some operators.

Let us introduce the notations:

$$\gamma_1 = \frac{\sigma_0}{(\theta_0-\sigma_0)^2}, \tag{20}$$

$$\rho = 2/(\alpha+\gamma_1), \quad \tau_{opt} = 2/(2\alpha+\gamma_1), \tag{21}$$

$$\theta = 1 + 2\alpha/\gamma_1, \tag{22}$$

$$\tau_k = 2[\gamma_1(\theta - \cos w_k\pi)]^{-1}, \tag{23}$$

$$\alpha_{k+1} = \begin{cases} \tau_{opt}, & k=0 \\ \dfrac{4}{\gamma_1}\dfrac{T_k(\theta)}{T_{k+1}(\theta)}, & k>0 \end{cases} ; \quad \beta_{k+1} = \begin{cases} 0, & k=0 \\ \dfrac{T_{k-1}(\theta)}{T_{k+1}(\theta)}, & k>0 \end{cases}, \tag{24}$$

where $\{w_k\}_{k=1}^s$ is a segment of the T-sequence [11], and T_k is the Chebyshev polynomial of the first kind of degree k.

The following convergence theorem holds.

THEOREM 3. (I) *If $\alpha_k = \tau$, $B_k = E$ (the identity operator), $\beta_k = 0$, a sufficient condition for the iterative process* (16)–(19) *to converge is the condition* $0 < \tau < \rho$. *For the value of $\tau = \tau_{opt}$ determined by formula* (21), *the convergence rate estimates hold*

$$\|\varphi - \varphi^k\|_{H_2^1} \le c_1\varepsilon, \quad \|\varphi^* - \varphi^{*k}\|_{H_2^1} \le c_2\varepsilon, \quad \|u - u^k\|_{L_{2,-}} \le c_3\varepsilon, \tag{25}$$

where $\varepsilon = 1/\theta^k$, θ is given by formula (22) *and the constants c_1, c_2, c_3 do not depend on the number of iteration and the functions $\varphi, \varphi^k, \varphi^*, \varphi^{*k}, u, u^k$, $k > 0$.*

(II) *If $B_k = E$, $\beta_k = 0$, $\alpha_k = \tau_k$, where the parameters τ_k are determined by formula* (23) *and repeated cyclically with period s, then the iterative process* (16)–(19) *is convergent. After $k = ls$ iterations of the algorithm the error estimates* (25) *hold for $\varepsilon = (T_s(\theta))^{-l}$.*

(III) *If $B_k = C_k = E$ and α_{k+1}, β_{k+1} are determined by formula* (24), *the algorithm* (16)–(19) *converges and the estimates* (25) *hold for $\varepsilon = (T_k(\theta))^{-1}$.*

PROOF. Following the arguments of [15], we can show that the iterative process (16)–(19) is equivalent to the following iterative algorithm

$$u^{k+1} = u^k - \alpha_{k+1}B_k(Lu^k - F) + \beta_{k+1}C_k(u^k - u^{k-1}), \tag{26}$$

for solving the control equation (15). The lower and upper bounds of the spectrum of the self-adjoint operator L are given from Lemma 3 by

$$m = \alpha, \quad M = \frac{\sigma_0}{(\theta_0 - \sigma_0)^2},$$

respectively. If we use the explicit form of m and M and employ the simple iterative method and the Chebyshev acceleration methods (s-cyclic and two-step) in the form (26) to solve the equation (15), we arrive at the statements of the theorem on the basis of the well-known results conserning the convergence of these methods [11]. ∎

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