EVOLUTION EQUATIONS:
EXISTENCE, REGULARITY AND SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 52
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

KLEIN-GORDON TYPE DECAY RATES FOR WAVE EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

MICHAEL REISSIG

Faculty of Mathematics and Computer Science
Technical University Bergakademie Freiberg
Bernhard von Cotta Str. 2
09596 Freiberg, Germany
E-mail: reissig@mathe.tu-freiberg.de

KAREN YAGDJIAN

Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki 305, Japan
E-mail: yagdjian@math.tsukuba.ac.jp

Abstract. This work is concerned with the proof of $L_p - L_q$ decay estimates for solutions of the Cauchy problem for the Klein-Gordon type equation $u_{tt} - \lambda^2(t)b^2(t)(\triangle u - m^2 u) = 0$. The coefficient consists of an increasing smooth function λ and an oscillating smooth and bounded function b which are uniformly separated from zero. Moreover, m^2 is a positive constant. We study under which assumptions for λ and b one can expect as an essential part of the decay rate the classical Klein-Gordon decay rate $\frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.

1. Introduction. To prove global existence results for the solutions of the Cauchy problem for nonlinear wave equations so-called $L_p - L_q$ decay estimates for the solutions of the linear wave equation play an essential role [7],[8],[11]. That is the following estimate for the solution $u = u(t,x)$ of the Cauchy problem

$$u_{tt} - \triangle u = 0,\ u(0,x) = 0,\ u_t(0,x) = u_1(x),$$

where $u_1 = u_1(x)$ belongs to $C_0^\infty(\mathbb{R}^n)$ (see [16]): there exist constants C and M depending on p and n such that

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{W_p^M(\mathbb{R}^n)}, \tag{1.1}$$

where $1 < p \le 2$ and $1/p + 1/q = 1$.

2000 *Mathematics Subject Classification*: Primary 35L15, 35Q40; Secondary 35B05.
The paper is in final form and no version of it will be published elsewhere.

In a series of papers [12],[13],[14],[15] the authors considered the question if a similar estimate holds for the solution of a strictly hyperbolic Cauchy problem, where the strictly hyperbolic operator is homogeneous, of second order and has time-dependent coefficients. To explain the results let us choose the Cauchy problem for the model equation

$$u_{tt} - \lambda^2(t) b^2(t) \triangle u = 0,\ u(t_0, x) = u_0(x),\ u_t(t_0, x) = u_1(x), \tag{1.2}$$

where $\lambda = \lambda(t)$ is an increasing function (improving influence on $L_p - L_q$ decay estimates) and $b = b(t)$ is a 1-periodic, non-constant, smooth, and positive function (deteriorating influence on $L_p - L_q$ decay estimates). There exists an interesting action and reaction between both influences. If the growth of λ dominates the oscillating part, then we can prove inequalities similar to (1.1). In opposite to this, if the oscillating part dominates the growth, then we can only prove uniformly for all smooth data with compact support estimates which are very near to the energy inequality for the solution of (1.2) which is obtained by Gronwall's inequality.

Example 1.1. Let us consider the Cauchy problem

$$u_{tt} - \exp(2t^\alpha) b^2(t) \triangle u = 0,\ u(t_0, x) = u_0(x),\ u_t(t_0, x) = u_1(x),$$

where $b = b(t)$ is a 1-periodic, non-constant, smooth and positive function. Then we have:

- in general no $L_p - L_q$ decay estimates if $\alpha < 1/2$,
- $L_p - L_q$ decay estimates if $\alpha > 1/2$,
- the critical case: $L_p - L_q$ decay estimates if $\alpha = 1/2$ and the spatial dimension n is large enough.

Now let us turn to the Cauchy problem for the Klein-Gordon equation (wave equation with a non-vanishing constant mass m)

$$u_{tt} - \triangle u + m^2 u = 0,\ u(0, x) = 0,\ u_t(0, x) = u_1(x). \tag{1.3}$$

The term $m^2 u$ guarantees a $\frac{1}{2}(\frac{1}{p} - \frac{1}{q})$ higher decay-rate in (1.1). We can explain this improvement as follows. One can use the representation of the solution of (1.3) by the aid of Fourier multipliers including the mass in the phase functions (see [5]). After partial Fourier transformation we obtain ($v = \hat{u}$)

$$v_{tt} + (|\xi|^2 + m^2) v = 0,\ v(0, x) = 0,\ v_t(0, x) = \hat{u}_1.$$

For the solution $v = v(t, \xi)$ we have the explicit representation

$$v = v(t, \xi) = \frac{i}{2} \left(e^{-it\langle\xi\rangle_m} \frac{\hat{u}_1(\xi)}{\langle\xi\rangle_m} - e^{it\langle\xi\rangle_m} \frac{\hat{u}_1(\xi)}{\langle\xi\rangle_m} \right),$$

$$u = u(t, x) = \frac{i}{2} F^{-1} \left(e^{-it\langle\xi\rangle_m} \frac{\hat{u}_1(\xi)}{\langle\xi\rangle_m} - e^{it\langle\xi\rangle_m} \frac{\hat{u}_1(\xi)}{\langle\xi\rangle_m} \right),$$

respectively, where $\langle\xi\rangle_m := (|\xi|^2 + m^2)^{1/2}$. For the Fourier multiplier

$$F^{-1} \left(e^{-it\langle\xi\rangle_m} \frac{\hat{u}_1(\xi)}{\langle\xi\rangle_m} \right)$$

we get the $L_p - L_q$ decay estimate

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{W_p^M(\mathbb{R}^n)}, \tag{1.4}$$

where $1 < p \le 2$, $1/p + 1/q = 1$ and M is suitably chosen.

The goal of the present paper is to study $L_p - L_q$ decay estimates for the solutions of the Cauchy problem for Klein-Gordon type equations with time-dependent coefficients. More precisely, we will consider the Cauchy problem

$$u_{tt} - \lambda^2(t)b^2(t) \triangle u + \lambda^2(t)b^2(t)m^2 u = 0,\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \tag{1.5}$$

with C^∞-data having compact support while m^2 is positive and constant. On the one hand we are interested in the interplay oscillations via growth, that is, the influence of $\lambda = \lambda(t)$ and $b = b(t)$. Do we have a similar example to Example 1.1. On the other hand we are interested if the mass term guarantees the better decay rate $\big(\frac{n}{2}(\frac{1}{p} - \frac{1}{q})\big)$.

We will call an equation of *Klein-Gordon type* if in the decay rate of the solution to (1.5) there appears the term $\frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ (see (1.4)). Thus we can formulate the main question:

Under which conditions for $\lambda = \lambda(t)$ and $b = b(t)$ is the differential equation from (1.5) of Klein-Gordon type?

The main results of this paper lead to the following example (compare with Example 1.1).

Example 1.2. Let us consider the Cauchy problem

$$u_{tt} - \exp(2t^\alpha)b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

where $b = b(t)$ is a 1-periodic, non-constant, smooth, positive function and m^2 is a positive constant. Then we have:

- in general no $L_p - L_q$ decay estimates if $\alpha \le 0$,
- $L_p - L_q$ decay estimates if $\alpha > 0$ (see Example 2.2).

Before we begin to derive $L_p - L_q$ decay estimates for the solutions of the Cauchy problem (1.5) we formulate a result which shows that oscillations in the coefficients may destroy $L_p - L_q$ decay estimates. The statement of this theorem can be proved as in [12].

Theorem 1.1. *Consider the Cauchy problem*

$$u_{tt} - b^2(t) \triangle u + m^2 b^2(t)u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

where $b = b(t)$ defined on $\mathbb{R}$ is a 1-periodic, non-constant, smooth, and positive function. Then for every given $b(t)$ there is positive constant m such that there are no constants $q, p, C, L,$ and a nonnegative function f defined on $\mathbb{N}$ such that for every initial data $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ the estimate

$$\|u_t(k,\cdot)\|_{L_q(\mathbb{R}^n)} + \|\nabla u(k,\cdot)\|_{L_q(\mathbb{R}^n)} \le Cf(k)(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)})$$

is fulfilled for all $k \in \mathbb{N}$ while $f(k) \to \infty$, $\ln f(k) = o(k)$ as $k \to \infty$.

2. Klein-Gordon type model equations. We have in mind the functions $\lambda = \lambda(t) = \exp(t^\alpha)$ and $b = b(t)$ from Example 1.1, $\alpha \in (0,1]$. Then $b = b(t)$ satisfies for large t

$$|D_t^k b(t)| \le C_k \left(\lambda(t)(\Lambda(t))^{-\beta}\right)^k, \quad k = 1, 2, \dots, \tag{2.1}$$

where $\beta \le 1$ while $\Lambda(t) := \int_0^t \lambda(\tau) d\tau$. We will consider as a model Cauchy problem of Klein-Gordon type the following one:

$$u_{tt} - \lambda^2(t) b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \tag{2.2}$$

Then we make the assumptions

(A1): the functions $b = b(t)$ and $\lambda = \lambda(t)$ belong to $C^\infty([0,\infty))$;

(A2): the functions $b = b(t)$ and $\lambda = \lambda(t)$ are bounded from below by a positive constant;

(A3): there exist positive constants $d_k,\ k \in \mathbb{N}_0$, such that for $t \in (0,\infty)$

$$\lambda'(t) \ge 0, \quad |D_t^k \lambda(t)| \le d_k \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \lambda(t),\ t > 0, \tag{2.3}$$

where $\Lambda(t) := \int_0^t \lambda(\tau) d\tau$;

(A4): the function $b = b(t)$ is not necessarily periodic, there exist positive constants c_k, $k \in \mathbb{N}_0$, and a nonnegative constant $1/2 \le \beta \le 1$ such that for $t \in (0,\infty)$

$$c_0 \le b^2(t) \le c_1,\ |D_t^k b(t)| \le c_k \left(\lambda(t)(\Lambda(t))^{-\beta}\right)^k, \quad k \ge 1.$$

Under these assumptions we prove $L_p - L_q$ decay estimates for the solutions to (2.2).

Theorem 2.1. *Assume that the conditions (A1) to (A4) are satisfied, in (A4) we suppose $\beta \in (1/2, 1]$, for the Cauchy problem*

$$u_{tt} - \lambda^2(t) b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

Here m^2 is a positive constant and u_0, u_1 are compactly supported smooth data. Then we get with $L = [n(\frac{1}{p} - \frac{1}{q})] + 1$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the $L_p - L_q$ estimate

$$\begin{aligned} &\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \lambda(t) \|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \\ &\le C\sqrt{\lambda(t)}\big(1 + \Lambda(t)\big)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)}\big). \end{aligned}$$

Theorem 2.2. *Assume that the conditions (A1) to (A4) are satisfied, in (A4) we suppose $\beta = 1/2$, for the Cauchy problem*

$$u_{tt} - \lambda^2(t) b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

Then we get with $L = [n(\frac{1}{p} - \frac{1}{q})] + 1$, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the $L_p - L_q$ estimate

$$\begin{aligned} &\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} + \lambda(t) \|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \\ &\le C\sqrt{\lambda(t)}\big(1 + \Lambda(t)\big)^{\frac{C_{0,0}}{m} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)}\big). \end{aligned}$$

The constant $C_{0,0}$ is introduced in Corollary 5.1. It depends on the behaviour of λ and β and its first two derivatives on the interval $[0,\infty)$.

Remark 2.1. If in Theorem 2.2 the spatial dimension n is large enough, namely $n > 2C_{0,0}/m$, then there exist p and q such that $L_p - L_q$ decay estimates hold for $u_t/\sqrt{\lambda}$ and for $\sqrt{\lambda}\nabla u$. In opposite to the case $\beta = 1/2$ we obtain in the case $\beta \in (1/2, 1]$ without restrictions $L_p - L_q$ decay estimates for $u_t/\sqrt{\lambda}$ and for $\sqrt{\lambda}\nabla u$.

EXAMPLE 2.1. Let us consider the Cauchy problem

$$u_{tt} - (1+t)^{2l} b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

where $b = b(t)$ is a 1-periodic, non-constant, smooth, positive function and m^2 is a positive constant. Then we have:

- $L_p - L_q$ decay estimates if $l > 1$,
- the critical case: $L_p - L_q$ decay estimates if $l = 1$ and the spatial dimension n is large enough.

EXAMPLE 2.2. Let us consider the Cauchy problem

$$u_{tt} - \exp(2t^{\alpha}) b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

where $b = b(t)$ is a 1-periodic, non-constant, smooth, positive function and m^2 is a positive constant. Then we have:

- in general no $L_p - L_q$ decay estimates if $\alpha \le 0$,
- $L_p - L_q$ decay estimates if $\alpha > 0$.

REMARK 2.2. In a forthcoming paper we will study Example 2.1 for $l < 1$. We expect that in general there are no $L_p - L_q$ decay estimates for the solutions of

$$u_{tt} - (1+t)^{2l} b^2(t)(\triangle u - m^2 u) = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x)$$

(compare Examples 1.1 and 2.1).

3. Tools of the approach

3.1. Zones. We split the set $[0,\infty) \times (\mathbb{R}^n \setminus \{0\})$ in subdomains which will be called *zones*. To do this we define for $\xi \in \mathbb{R}^n$ the function $t = t_{\langle\xi\rangle}$ ($t_{\langle\xi\rangle} = t(\langle\xi\rangle_m)$) by

$$\Lambda(t_{\langle\xi\rangle})^{\beta} \langle\xi\rangle_m = N, \tag{3.1}$$

where β is from assumption (A4) while $N \ge 1$ is a positive constant to be determined later. We define the *pseudodifferential zone*

$$Z_{pd}(N) := \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n \setminus \{0\} : \quad 0 \le t \le t_{\langle\xi\rangle}\} \tag{3.2}$$

and the *hyperbolic zone*

$$Z_{hyp}(N) := \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n \setminus \{0\} : t_{\langle\xi\rangle} \le t\}. \tag{3.3}$$

It is evident that if $N_1 \le N_2$ then $Z_{hyp}(N_2) \subset Z_{hyp}(N_1)$ while $Z_{pd}(N_1) \subset Z_{pd}(N_2)$.

LEMMA 3.1. *Define the function $t = t(p)$ as a solution to $\Lambda(t_{\langle p\rangle})^{\beta}\langle p\rangle_m = N$. Then*

$$|D_p^k t_{\langle p\rangle}| \le C_k \langle p\rangle_m^{-k} \frac{\Lambda(t_{\langle p\rangle})}{\lambda(t_{\langle p\rangle})} \tag{3.4}$$

for all $p \ge 0$ and $k \ge 0$, where the constants C_k are independent of N.

PROOF. We have

$$\frac{dt_{\langle p\rangle}}{dp} = -\frac{2}{\beta} \frac{p}{\langle p\rangle_m^2} \frac{\Lambda(t_{\langle p\rangle})}{\lambda(t_{\langle p\rangle})}.$$

Thus we get the estimate

$$\left|\frac{dt_{\langle p\rangle}}{dp}\right| \le C_1 \langle p\rangle_m^{-1} \frac{\Lambda(t_{\langle p\rangle})}{\lambda(t_{\langle p\rangle})}\,.$$

By induction we can prove the statement for all $k \ge 0$. The constants C_k are independent of N. ∎

3.2. Classes of symbols. For the further considerations we need suitable classes of symbols which are defined only in the hyperbolic zone $Z_{hyp}(N)$.

Definition 3.1. For real numbers r_1, r_2, r_3; $\beta \in (0,1]$; we denote by $S_{m,\beta}\{r_1, r_2, r_3\}$ the set of all symbols $a = a(t,\xi) \in C^\infty(Z_{hyp}(N))$ satisfying

$$|D_t^l D_\xi^\alpha a(t,\xi)| \le C_{\alpha,l} \langle \xi\rangle_m^{r_1-|\alpha|} \lambda(t)^{r_2} \left(\lambda(t)(\Lambda(t))^{-\beta}\right)^{r_3+l} \tag{3.5}$$

for all $(t,\xi) \in Z_{hyp}(N)$, all multi-indices α and all l with constants $C_{\alpha,l}$ independent of N.

Let us summarize some simple rules of the symbolic calculus.

1. $S_{m,\beta}\{r_1, r_2, r_3\} \subset S_{m,\beta}\{r_1 + k, r_2 + k, r_3 - k\}$ for $k \ge 0$;
2. if $a(t,\xi) \in S_{m,\beta}\{r_1, r_2, r_3\}$ and $b(t,\xi) \in S_{m,\beta}\{k_1, k_2, k_3\}$, then $a(t,\xi)b(t,\xi) \in S_{m,\beta}\{r_1 + k_1, r_2 + k_2, r_3 + k_3\}$;
3. if $a(t,\xi) \in S_{m,\beta}\{r_1, r_2, r_3\}$, then $D_t a(t,\xi) \in S_{m,\beta}\{r_1, r_2, r_3 + 1\}$;
4. if $a(t,\xi) \in S_{m,\beta}\{r_1, r_2, r_3\}$, then $D_\xi^\alpha a(t,\xi) \in S_{m,\beta}\{r_1 - |\alpha|, r_2, r_3\}$.

4. Consideration in the pseudodifferential zone $Z_{pd}(N)$. Let us consider (2.2). After partial Fourier transformation we get (keep the same notation for the Fourier transforms)

$$D_t^2 u - \lambda^2(t) b^2(t) \langle \xi\rangle_m^2 u = 0,\ u(0,\xi) = u_0(\xi),\ D_t u(0,\xi) = \frac{1}{i} u_1(\xi).$$

Setting $U = (U_1, U_2)^T := (\lambda(t)\langle\xi\rangle_m u,\ D_t u)^T$ the last equation can be transformed to the system of first order

$$D_t U - \begin{pmatrix} 0 & \lambda(t)\langle\xi\rangle_m \\ \lambda(t)b^2(t)\langle\xi\rangle_m & 0 \end{pmatrix} U - \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = 0.$$

We are interested in the fundamental solution to the Cauchy problem for that system, this is the matrix-valued solution $\mathcal{U} = \mathcal{U}(t,s,\xi)$ to the Cauchy problem

$$D_t \mathcal{U} - \begin{pmatrix} 0 & \lambda(t)\langle\xi\rangle_m \\ \lambda(t)b^2(t)\langle\xi\rangle_m & 0 \end{pmatrix} \mathcal{U} - \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U} = 0, \tag{4.1}$$

$$\mathcal{U}(s,s,\xi) = I \ \text{(identity matrix)}. \tag{4.2}$$

Using the matrizant we obtain for $\mathcal{U}(t,s,\xi)$ the explicit representation

$$\mathcal{U}(t,s,\xi) = I + \sum_{k=1}^{\infty} \int_s^t A(t_1,\xi) \int_s^{t_1} A(t_2,\xi) \ldots \int_s^{t_{k-1}} A(t_k,\xi) dt_k \ldots dt_1,$$

where

$$A(t,\xi) := \begin{pmatrix} 0 & \lambda(t)\langle\xi\rangle_m \\ \lambda(t)b^2(t)\langle\xi\rangle_m & 0 \end{pmatrix} + \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In contrast to the considerations for the wave equations with $m \equiv 0$ we can make use of the advantage that

$$t_{\langle\xi\rangle} \le t_{\langle 0\rangle} =: t_{m,N} \tag{4.3}$$

uniformly for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where we take account of the monotonicity of $\Lambda(t)$. The function $\lambda = \lambda(t)$ is positive. This helps to estimate the norm of the second matrix by a constant maybe depending on N. The norm of the integral over the first matrix can be estimated in $Z_{pd}(N)$ by

$$C\int_s^t \lambda(\tau)\langle\xi\rangle_m d\tau \le C\Lambda(t_{\langle\xi\rangle})\langle\xi\rangle_m \le CN(\Lambda(t_{\langle\xi\rangle}))^{1-\beta} \le CN(\Lambda(t_{m,N}))^{1-\beta},$$

where we used (4.3). Consequently, $\|\mathcal{U}(t,s,\xi)\| \le C_0(N)$ for $(t,\xi) \in Z_{pd}(N)$.

In the same way we estimate $\|D_t^k D_\xi^\alpha \mathcal{U}(t,s,\xi)\|$.

Proposition 4.1. *For every k and α the following estimate holds:*

$$\|D_t^k D_\xi^\alpha \mathcal{U}(t,0,\xi)\| \le C_{\alpha,k,N}\langle\xi\rangle_m^{-|\alpha|}(\lambda(t)\langle\xi\rangle_m)^k$$

for all $(t,\xi) \in Z_{pd}(N)$. The constants $C_{\alpha,k,N}$ depend on N.

5. Consideration in the hyperbolic zone $Z_{hyp}(N)$

5.1. Diagonalization modulo $S_{m,\beta}\{-M,-M,M+1\}$. We carry out a diagonalization process to get estimates for the solution of (4.1), (4.2).

Let us define the matrices

$$M^{-1}(t) := \frac{1}{\sqrt{\lambda(t)b(t)}}\begin{pmatrix} 1 & 1 \\ -b(t) & b(t)\end{pmatrix}, \quad M(t) := \frac{1}{2}\sqrt{\frac{\lambda(t)}{b(t)}}\begin{pmatrix} b(t) & -1 \\ b(t) & 1\end{pmatrix}. \tag{5.1}$$

Substituting $\mathcal{U} = M^{-1}\mathcal{V}$ some calculations transform (4.1) into the first-order system

$$D_t\mathcal{V} - \lambda(t)b(t)\langle\xi\rangle_m \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\mathcal{V} - \frac{D_t\lambda(t)}{\lambda(t)}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\mathcal{V} - \frac{1}{2}\frac{D_t(\lambda(t)b(t))}{\lambda(t)b(t)}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\mathcal{V} = 0.$$

We denote

$$\tau_1(t,\xi) := -\lambda(t)b(t)\langle\xi\rangle_m + \frac{D_t\lambda(t)}{\lambda(t)},$$
$$\tau_2(t,\xi) := \lambda(t)b(t)\langle\xi\rangle_m + \frac{D_t\lambda(t)}{\lambda(t)}.$$

With some positive number c we have

$$|\tau_2(t,\xi) - \tau_1(t,\xi)| \ge c\lambda(t)\langle\xi\rangle_m. \tag{5.2}$$

The matrix

$$-\lambda(t)b(t)\langle\xi\rangle_m\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}$$

belongs to $S_{m,\beta}\{1,1,0\}$. But the matrix

$$\frac{D_t(\lambda(t)b(t))}{\lambda(t)b(t)}I$$

belongs even to $S_{m,\beta}\{0,0,1\}$. Thus we got the diagonalization mod $S_{m,\beta}\{0,0,1\}$ in the form

$$D_t\mathcal{V} - \mathcal{D}(t,\xi)\mathcal{V} + B(t,\xi)\mathcal{V} = 0, \tag{5.3}$$

where

$$\mathcal{D}(t,\xi) := \begin{pmatrix} \tau_1(t,\xi) & 0 \\ 0 & \tau_2(t,\xi) \end{pmatrix}, \quad B(t,\xi) := -\frac{1}{2}\frac{D_t(\lambda(t)b(t))}{\lambda(t)b(t)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We will carry out further steps of perfect diagonalization, namely diagonalization modulo $S_{m,\beta}\{-M,-M,M+1\}$ for some given nonnegative integer M. The next proposition shows that this is possible for every nonnegative integer M.

Proposition 5.1. *For a nonnegative integer M there exist matrix-valued functions $N_M(t,\xi) \in S_{m,\beta}\{0,0,0\}$, $F_M(t,\xi) \in S_{m,\beta}\{-1,-1,2\}$, $R_M(t,\xi) \in S_{m,\beta}\{-M,-M,M+1\}$ such that the following operator-valued identity holds in $Z_{hyp}(N)$:*

$$(D_t - \mathcal{D}(t,\xi) + B(t,\xi))N_M(t,\xi) = N_M(t,\xi)(D_t - \mathcal{D}(t,\xi) + F_M(t,\xi) - R_M(t,\xi)), \tag{5.4}$$

where the matrix F_M is diagonal while the matrix $N_M \in S_{m,\beta}\{0,0,0\}$ is invertible and its inverse matrix $N_M^{-1}(t,\xi) \in S_{m,\beta}\{0,0,0\}$ too, provided that the parameter N is sufficiently large.

Proof. We look for $N_M = N_M(t,\xi)$ and $F_M = F_M(t,\xi)$, $M \geq 1$, having the representations

$$N_M(t,\xi) = \sum_{r=0}^{M} N^{(r)}(t,\xi),\ F_M(t,\xi) = \sum_{r=0}^{M-1} F^{(r)}(t,\xi),$$

where $N^{(0)} := I,\ B^{(0)} := B,\ F^{(r)} := \operatorname{diag}(B^{(r)}), F^{(0)}(t,\xi) \equiv 0,$

$$\begin{aligned} N^{(r+1)} &:= \begin{pmatrix} 0 & B_{12}^{(r)}/(\tau_1-\tau_2) \\ B_{21}^{(r)}/(\tau_2-\tau_1) & 0 \end{pmatrix}, \\ B^{(r+1)} &:= (D_t - \mathcal{D} + B)\Big(\sum_{\mu=0}^{r+1} N^{(\mu)}\Big) - \Big(\sum_{\mu=0}^{r+1} N^{(\mu)}\Big)\Big(D_t - \mathcal{D} + \sum_{\mu=1}^{r} F^{(\mu)}\Big) \end{aligned}$$

for $r = 0, 1, \dots, M-1$. Using (5.1) we have $N^{(1)} \in S_{m,\beta}\{-1,-1,1\}$. For $B^{(1)}$ we obtain the relation

$$B^{(1)} = B + [N^{(1)}, \mathcal{D}] + D_tN^{(1)} + BN^{(1)}.$$

The sum of the first two matrices vanishes, while the last two summands belong to $S_{m,\beta}\{-1,-1,2\}$ due to the rules of the symbolic calculus from Subsection 3.2. Hence $B^{(1)} \in S_{m,\beta}\{-1,-1,2\}$.

Supposing $B^{(r)} \in S_{m,\beta}\{-r,-r,r+1\}$ we apply the principle of induction to show the statement for $B^{(r+1)}$. On the one hand we have from the construction

$$N^{(r+1)} \in S_{m,\beta}\{-(r+1),-(r+1),r+1\} \quad \text{and} \quad F^{(r)} \in S_{m,\beta}\{-r,-r,r+1\}.$$

On the other hand,

$$\begin{aligned} &B^{(r+1)} \\ &= B^{(r)} + [N^{(r+1)}, \mathcal{D}] - F^{(r)} + D_tN^{(r+1)} + BN^{(r+1)} + N^{(r+1)}\sum_{\mu=0}^{r} F^{(\mu)} - \Big(\sum_{\mu=0}^{r+1} N^{(\mu)}\Big)F^{(r)}. \end{aligned}$$

Moreover, we have $B^{(r)} + [N^{(r+1)}, \mathcal{D}] - F^{(r)} = 0$. The sum of the other terms and consequently $B^{(r+1)}$ belong to $S_{m,\beta}\{-(r+1), -(r+1), r+2\}$.

Thus we have shown $N^{(r)} \in S_{m,\beta}\{-r,-r,r\}$, that is with Definition 3.3.1

$$\|N^{(r)}(t,\xi)\| \le C_r \left(\frac{1}{(\Lambda(t))^{\beta}\langle\xi\rangle_m}\right)^r \le C_r \left(\frac{1}{N}\right)^r, \quad (t,\xi) \in Z_{hyp}(N), \quad r = 0,\dots,M.$$

This implies

$$\Big\|\sum_{r=1}^{M} N^{(r)}(t,\xi)\Big\| \ \le \ \sum_{r=1}^{M} C_r \left(\frac{1}{N}\right)^r,$$

where C_r is independent of N. A sufficiently large N provides $\|N_M - I\| \le 1/2$ in $Z_{hyp}(N)$ and consequently the statements concerning N_M and N_M^{-1}. Finally let us define $R_M := -N_M^{-1} B^{(M)}$. This matrix belongs obviously to $S_{m,\beta}\{-M,-M,M+1\}$. The proposition is proved. ■

5.2. Estimates for the fundamental solution. Let us consider the system $(D_t - \mathcal{D} + F_M - R_M)\mathcal{W} = 0$. Let $E_2 = E_2(t,r,\xi)$ be the matrix-valued function

$$E_2(t,r,\xi) = \begin{pmatrix} \exp\left(i\int_r^t \left\{-\lambda(s)b(s)\langle\xi\rangle_m + \frac{1}{i}\frac{\lambda'(s)}{\lambda(s)}\right\} d\,s\right) & 0 \\ 0 & \exp\left(i\int_r^t \left\{\lambda(s)b(s)\langle\xi\rangle_m + \frac{1}{i}\frac{\lambda'(s)}{\lambda(s)}\right\} d\,s\right) \end{pmatrix}.$$

Hence

$$E_2(t,r,\xi) = \frac{\lambda(t)}{\lambda(r)} \begin{pmatrix} \exp\left(-i\int_r^t \lambda(s)b(s)\langle\xi\rangle_m \, d\,s\right) & 0 \\ 0 & \exp\left(i\int_r^t \lambda(s)b(s)\langle\xi\rangle_m \, d\,s\right) \end{pmatrix}. \tag{5.5}$$

Let us define the matrix-valued function

$$\mathcal{R}_M(t,r,\xi) \ = -F_M(t,\xi) + E_2(r,t,\xi) R_M(t,\xi) E_2(t,r,\xi).$$

LEMMA 5.1. *The matrix-valued function $\mathcal{R}_M = \mathcal{R}_M(t,r,\xi)$ satisfies for every l and α the estimate*

$$\begin{aligned}\|\partial_t^l \partial_\xi^\alpha \left(\mathcal{R}_M(t,r,\xi) + F_M(t,\xi)\right)\| &\le C_{M,l,\alpha} (\lambda(t)\langle\xi\rangle_m)^l \Lambda(t)^{|\alpha|} \\ &\quad \times \lambda(t)\,(\Lambda(t))^{-\beta} \left(\Lambda^{\beta}(t)\langle\xi\rangle_m\right)^{-M}\end{aligned} \tag{5.6}$$

with constants $C_{M,l,\alpha}$ independent of N.

COROLLARY 5.1. *The matrix-valued function $\mathcal{R}_M(t,r,\xi)$ satisfies for every given l and $\alpha, |\alpha| \le \beta(M-1)$, in $Z_{hyp}(N)$ the estimate*

$$\|\partial_t^l \partial_\xi^\alpha \mathcal{R}_M(t,r,\xi)\| \le C_{M,l} (\lambda(t)\langle\xi\rangle_m)^l \langle\xi\rangle_m^{-|\alpha|} \frac{\lambda(t)}{\Lambda^{2\beta}(t)\langle\xi\rangle_m}$$

with constants $C_{M,l}$ independent of N.

PROOF. Applying Proposition 5.1 and Lemma 5.1 gives

$$\begin{aligned}\|\partial_t^l\partial_\xi^\alpha \mathcal{R}_M(t,r,\xi)\| &\le \|\partial_t^l\partial_\xi^\alpha\left(\mathcal{R}_M(t,r,\xi)+F_M(t,\xi)\right)\| + \|\partial_t^l\partial_\xi^\alpha F_M(t,\xi)\| \\ &\le C_{M,l}(\lambda(t)\langle\xi\rangle_m)^l\Lambda(t)^{|\alpha|}\lambda(t)\,(\Lambda(t))^{-\beta}\left(\Lambda^\beta(t)\langle\xi\rangle_m\right)^{-M} \\ &\quad + C_{M,l}(\lambda(t)\langle\xi\rangle_m)^l\langle\xi\rangle_m^{-|\alpha|}\frac{\lambda(t)}{\Lambda^{2\beta}(t)\langle\xi\rangle_m}\left(\frac{1}{\Lambda^\beta(t)\langle\xi\rangle_m}\right)^l.\end{aligned}$$

In $Z_{hyp}(N)$ we can estimate for $|\alpha|\le\beta(M-1)$ a part of the first term of the right-hand side in the following way:

$$\begin{aligned}&\Lambda(t)^{|\alpha|}\lambda(t)\,(\Lambda(t))^{-\beta}\left(\Lambda^\beta(t)\langle\xi\rangle_m\right)^{-M} \\ &\le \langle\xi\rangle_m^{-|\alpha|}\frac{\lambda(t)}{\Lambda^{2\beta}(t)\langle\xi\rangle_m}\,(\Lambda(t)\langle\xi\rangle_m)^{|\alpha|}\left(\frac{1}{\Lambda^\beta(t)\langle\xi\rangle_m}\right)^{M-1} \\ &\le \langle\xi\rangle_m^{-|\alpha|}\frac{\lambda(t)}{\Lambda^{2\beta}(t)\langle\xi\rangle_m}\left(\frac{1}{\Lambda^\beta(t)\langle\xi\rangle_m}\right)^{M-1-|\alpha|/|\beta|} \\ &\le \langle\xi\rangle_m^{-|\alpha|}\frac{\lambda(t)}{\Lambda^{2\beta}(t)\langle\xi\rangle_m}.\end{aligned}$$

In the last inequality we used the definition of $Z_{hyp}(N)$, especially $N\ge 1$. This definition helps to estimate the second term of the right-hand side, too. The corollary is proved. ∎

With the aid of $\mathcal{R}_M$ we define the matrix-valued function

$$\begin{aligned}&Q_M(t,t_{\langle\xi\rangle},\xi) \\ &:= \sum_{j=1}^{\infty} i^j\int_{t_{\langle\xi\rangle}}^{t}\mathcal{R}_M(t_1,t_{\langle\xi\rangle},\xi)dt_1\int_{t_{\langle\xi\rangle}}^{t_1}\mathcal{R}_M(t_2,t_{\langle\xi\rangle},\xi)dt_2\ldots\int_{t_{\langle\xi\rangle}}^{t_{j-1}}\mathcal{R}_M(t_j,t_{\langle\xi\rangle},\xi)dt_j\end{aligned}$$

for $t\ge t_{\langle\xi\rangle}$.

LEMMA 5.2. *The matrix-valued function* $Q_M(t,t_{\langle\xi\rangle},\xi)$ *satisfies for all* α, $|\alpha|\le\beta(M-1)$, $\beta\in(1/2,1]$, *the estimates*

$$\|\partial_\xi^\alpha Q_M(t,t_{\langle\xi\rangle},\xi)\|\le C_M\langle\xi\rangle_m^{-|\alpha|} \tag{5.7}$$

with a constant C_M *independent of* N.

SKETCH OF PROOF. Using Corollary 5.1 for $|\alpha|=0$ and $l=0$ we obtain

$$\begin{aligned}\|Q_M(t,t_{\langle\xi\rangle},\xi)\| &\le \exp\left(\int_{t_{\langle\xi\rangle}}^{t}C_{0,0}\frac{\lambda(s)}{\Lambda^{2\beta}(s)\langle\xi\rangle_m}ds\right) \\ &= \exp\left(C_{0,0}\frac{1}{1-2\beta}\frac{\Lambda^{1-2\beta}(t)}{\langle\xi\rangle_m}-C_{0,0}\frac{1}{1-2\beta}\frac{\Lambda^{1-2\beta}(t_{\langle\xi\rangle})}{\langle\xi\rangle_m}\right) \\ &\le \exp\left(C_{0,0}\frac{1}{2\beta-1}\frac{1}{\Lambda^{2\beta-1}(t_{\langle\xi\rangle})\langle\xi\rangle_m}\right) \\ &\le \exp\left(C_{0,0}\frac{1}{2\beta-1}\frac{1}{m}\right).\end{aligned}$$

This proves the statement for $\alpha=0$. Now let us form $|\alpha|$ derivatives, $|\alpha|\le\beta(M-1)$. Then

$$\partial_\xi^\alpha Q_M(t,t_{\langle\xi\rangle},\xi) = \sum_{j=1}^{\infty} i^j \partial_\xi^\alpha\Big(\int_{t_{\langle\xi\rangle}}^{t}\mathcal{R}_M(t_1,t_{\langle\xi\rangle},\xi)\int_{t_{\langle\xi\rangle}}^{t_1}\mathcal{R}_M(t_2,t_{\langle\xi\rangle},\xi)\ldots\int_{t_{\langle\xi\rangle}}^{t_{j-1}}\mathcal{R}_M(t_j,t_{\langle\xi\rangle},\xi)\,dt_j\ldots dt_1\Big).$$

Straightforward calculations lead to the statement for $|\alpha|\le\beta(M-1)$. But the main point is that Lemma 3.1 and Corollary 5.1 allow to follow the lines of the proof of Lemmas 3.2.15 and 3.2.16 from [15]. ∎

Now let us turn to the case $\beta=1/2$. Without new difficulties one can prove the next result.

LEMMA 5.3. *The matrix-valued function $Q_M(t,t_{\langle\xi\rangle},\xi)$ satisfies for all α, $|\alpha|\le\beta(M-1)$, $\beta=1/2$, the estimates*

$$\|\partial_\xi^\alpha Q_M(t,t_{\langle\xi\rangle},\xi)\|\le C_M(\Lambda(t))^{C_{0,0}/m}\langle\xi\rangle_m^{-|\alpha|} \tag{5.8}$$

with a constant C_M independent of N, where $C_{0,0}$ is the constant from Corollary 5.1.

The matrix-valued function $\mathcal{W}(t,t_{\langle\xi\rangle},\xi)=E_2(t,t_{\langle\xi\rangle},\xi)(I+Q_M(t,t_{\langle\xi\rangle},\xi))$ solves the Cauchy problem

$$(D_t-\mathcal{D}+F_M-R_M)\mathcal{W}=0,\ \mathcal{W}(t_{\langle\xi\rangle},t_{\langle\xi\rangle},\xi)=I,\ t\ge t_{\langle\xi\rangle}.$$

Applying the transformations which bring the system for the fundamental solution to the above one, we obtain that

$$\mathcal{U}(t,0,\xi) = M^{-1}(t)N_M(t,\xi)E_2(t,t_{\langle\xi\rangle},\xi)(I+Q_M(t,t_{\langle\xi\rangle},\xi))N_M(t_{\langle\xi\rangle},\xi)^{-1}M(t_{\langle\xi\rangle})\mathcal{U}(t_{\langle\xi\rangle},0,\xi)$$

for $t\ge t_{\langle\xi\rangle}$.

To estimate the derivatives of $\mathcal{U}(t,0,\xi)$ with respect to ξ we have to estimate all factors. We get estimates for $N_M(t,\xi)$, and $N_M(t_{\langle\xi\rangle},\xi)^{-1}$ from Proposition 5.1 and Lemma 3.1. Using Lemma 3.1 and condition (A4) it follows that $M(t_{\langle\xi\rangle})$ belongs to $S_{m,\beta}\{0,0,0\}$. From Lemma 5.2 we have estimates for $Q_M(t,t_{\langle\xi\rangle},\xi)$. Finally, derivatives of $\mathcal{U}(t_{\langle\xi\rangle},t_0,\xi)$ can be estimated by Proposition 4.1. Hence it remains to estimate

$$E_2(t,t_{\langle\xi\rangle},\xi)\ =\ E_2(t,0,\xi)E_2(0,t_{\langle\xi\rangle},\xi).$$

For $E_2(0,t_{\langle\xi\rangle},\xi)$ we have the explicit representation

$$E_2(0,t_{\langle\xi\rangle},\xi)=\frac{\lambda(0)}{\lambda(t_{\langle\xi\rangle})}\begin{pmatrix}\exp\left(i\langle\xi\rangle_m\int_0^{t_{\langle\xi\rangle}}\lambda(s)b(s)\,ds\right) & 0\\ 0 & \exp\left(-i\langle\xi\rangle_m\int_0^{t_{\langle\xi\rangle}}\lambda(s)b(s)\,ds\right)\end{pmatrix}.$$

A careful calculation shows that $\|\partial_\xi^\alpha E_2(0,t_{\langle\xi\rangle},\xi)\|\le C_\alpha\langle\xi\rangle_m^{-|\alpha|}$. Summarizing we obtain the next results.

PROPOSITION 5.2. *Let us suppose that the assumptions (A1) to (A4) are satisfied with $\beta\in(1/2,1]$ in (A4). Then the fundamental solution $\mathcal{U}=\mathcal{U}(t,0,\xi)$ can be represented as* *follows:*

$$\mathcal{U}(t,0,\xi) = \mathcal{U}^-(t,0,\xi)\exp\Big(-i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds\Big) + \mathcal{U}^+(t,0,\xi)\exp\Big(i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds\Big),$$

where the matrix-valued amplitudes $\mathcal{U}^-$ *and* $\mathcal{U}^+$ *satisfy for* $|\alpha|\le\beta(M-1)$ *the estimates*

$$\|\partial_\xi^\alpha\mathcal{U}^\pm(t,0,\xi)\|\le C_M\sqrt{\lambda(t)}\langle\xi\rangle_m^{-|\alpha|},\qquad (t,\xi)\in Z_{hyp}(N).$$

Proposition 5.3. *Let us suppose that the assumptions (A1) to (A4) are satisfied with* $\beta=1/2$ *in (A4). Then the fundamental solution* $\mathcal{U}=\mathcal{U}(t,0,\xi)$ *can be represented as follows:*

$$\mathcal{U}(t,0,\xi) = \mathcal{U}^-(t,0,\xi)\exp\Big(-i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds\Big) + \mathcal{U}^+(t,0,\xi)\exp\Big(i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds\Big),$$

where the matrix-valued amplitudes $\mathcal{U}^-$ *and* $\mathcal{U}^+$ *satisfy for* $|\alpha|\le\beta(M-1)$ *the estimates*

$$\|\partial_\xi^\alpha\mathcal{U}^\pm(t,0,\xi)\|\le C_M\sqrt{\lambda(t)}(\Lambda(t))^{C_{0,0}/m}\langle\xi\rangle_m^{-|\alpha|},\qquad (t,\xi)\in Z_{hyp}(N),$$

where $C_{0,0}$ *is the constant from Corollary 5.1.*

6. Solutions to the Cauchy problems. Summarizing all the calculations of the previous sections we arrive at the following results.

Theorem 6.1. *Under the assumptions (A1) to (A4), we suppose* $\beta\in(1/2,1]$ *in (A4), let us consider the Cauchy problem*

$$u_{tt}+\lambda^2(t)b^2(t)\langle\xi\rangle_m^2u=0,\ \ u(0,\xi)=u_0(\xi),\ \ u_t(0,\xi)=u_1(\xi).$$

Then the solution can be written as

$$\begin{aligned} u(t,\xi) &= a_0^-(t,0,\xi)u_0(\xi)\exp\Big(-i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds\Big)\\ &+\ a_0^+(t,0,\xi)u_0(\xi)\exp\Big(i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds\Big)\\ &+\ a_1^-(t,0,\xi)u_1(\xi)\exp\Big(-i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds\Big)\\ &+\ a_1^+(t,0,\xi)u_1(\xi)\exp\Big(i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds\Big), \end{aligned}$$

where we have

$$\begin{aligned} |a_0^\pm(t,0,\xi)| &\le C\frac{1}{\lambda(t)},\ (t,\xi)\in Z_{pd}(N),\\ |a_1^\pm(t,0,\xi)| &\le C\frac{1}{\lambda(t)\langle\xi\rangle_m},\ (t,\xi)\in Z_{pd}(N),\\ |\partial_\xi^\alpha a_0^\pm(t,0,\xi)| &\le C_M\frac{1}{\sqrt{\lambda(t)}}\langle\xi\rangle_m^{-|\alpha|},\ (t,\xi)\in Z_{hyp}(N),\\ |\partial_\xi^\alpha a_1^\pm(t,0,\xi)| &\le C_M\frac{1}{\sqrt{\lambda(t)}}\langle\xi\rangle_m^{-(|\alpha|+1)},\ (t,\xi)\in Z_{hyp}(N), \end{aligned}$$

for all α, $|\alpha| \le \beta(M-1)$. *Moreover, we obtain*

$$\begin{aligned} u_t(t,\xi) &= b_0^-(t,0,\xi)u_0(\xi)\exp\Big(-i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds\Big) \\ &+\ b_0^+(t,0,\xi)u_0(\xi)\exp\Big(i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds\Big) \\ &+\ b_1^-(t,0,\xi)u_1(\xi)\exp\Big(-i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds\Big) \\ &+\ b_1^+(t,0,\xi)u_1(\xi)\exp\Big(i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds\Big), \end{aligned}$$

where we have

$$\begin{aligned} |b_0^\pm(t,0,\xi)| &\le C\langle\xi\rangle_m,\ (t,\xi)\in Z_{pd}(N), \\ |b_1^\pm(t,0,\xi)| &\le C,\ (t,\xi)\in Z_{pd}(N), \\ |\partial_\xi^\alpha b_0^\pm(t,0,\xi)| &\le C_M\sqrt{\lambda(t)}\langle\xi\rangle_m^{-(|\alpha|-1)},\ (t,\xi)\in Z_{hyp}(N), \\ |\partial_\xi^\alpha b_1^\pm(t,0,\xi)| &\le C_M\sqrt{\lambda(t)}\langle\xi\rangle_m^{-|\alpha|},\ (t,\xi)\in Z_{hyp}(N), \end{aligned}$$

for all α, $|\alpha| \le \beta(M-1)$.

THEOREM 6.2. *Under the assumptions (A1) to (A4), we suppose* $\beta = 1/2$ *in (A4), let us consider the Cauchy problem*

$$u_{tt} + \lambda^2(t)b^2(t)\langle\xi\rangle_m^2 u = 0,\ \ u(0,\xi) = u_0(\xi),\ \ u_t(0,\xi) = u_1(\xi).$$

Then the solution $u = u(t,\xi)$ *and its derivative* $u_t = u_t(t,\xi)$ *possess the same representations as in the previous theorem. The amplitudes satisfy the same estimates if we replace* C_M *by* $C_M(\Lambda(t))^{C_{0,0}/m}$, *where* $C_{0,0}$ *is the constant from Corollary 5.1.*

7. Littman-type lemmas. To derive $L_p - L_q$ decay estimates for Fourier multipliers in the next section we need the following two Littman-type lemmas.

PROPOSITION 7.1. *Let us suppose that the function* $a = a(t,\xi)$ *has uniformly for all* $t \in [t_{m,N},\infty)$ *(we choose* $t_{m,N}$ *from (4.3)) a support (with respect to* ξ*) contained in a compact set* $\mathcal{K} \subset \mathbb{R}^n$ *. Moreover, assume that*

$$|\partial_\xi^\alpha a(t,\xi)| \le C\langle\xi\rangle_m^{-|\alpha|} \qquad \textit{for} \quad |\alpha| \le n+1,\ \ (t,\xi)\in[t_{m,N},\infty)\times\mathcal{K}.$$

Then

$$\|F^{-1}(e^{i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds}a(t,\xi))\|_{L_\infty(\mathbb{R}^n)} \le C\Lambda(t)^{-\frac{n}{2}} \quad \textit{for all} \quad t\in[t_{m,N},\infty), \tag{7.1}$$

where the constant C *depends on* $\sup\{|\xi| : \xi\in\mathcal{K}\}$ *only.*

PROOF. We have to estimate

$$\sup_{x\in\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{ix\cdot\xi + i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds}a(t,\xi)d\xi\Big|$$

for all $t \in [t_{m,N},\infty)$. There are two unbounded parameters, the scalar t (and, consequently, the function $\tau = \tau(t) := \int_0^t \lambda(s)b(s)ds$ is unbounded) and the vector $x \in \mathbb{R}^n$ in the last integral. We are going to get an estimate which is independent of $x \in \mathbb{R}^n$ and $t \in [t_{m,N},\infty)$.

Let us denote $\Phi(t,x,\xi) := x + \frac{\xi}{\langle\xi\rangle_m}\tau$. Then there exist constants δ_1 and δ_2 such that $|\Phi(t,x,\xi)| \geq \delta_2\tau$ for $|x| \geq \delta_1\tau$. With

$$L\, e^{ix\cdot\xi+i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds} = e^{ix\cdot\xi+i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds}, \qquad L = \frac{1}{|\Phi|^2}\sum_{r=1}^n \Phi_r\frac{\partial}{\partial\xi_r},$$

we obtain for an arbitrary $N \leq n+1$ the inequality

$$\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi+i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds} a(t,\xi)d\xi\right| \leq C_N\Lambda(t)^{-N} \tag{7.2}$$

for all $\{(t,x) : t\in[t_{m,N},\infty),\ |x|\geq\delta_1\tau\}$. Here we need N derivatives of $a = a(t,\xi)$ with respect to ξ.

For $|x| \leq \delta_1\tau$ we rewrite with $y := x/\tau$ and the inverse function $t = t(\tau)$

$$\begin{aligned}\sup_{|x|\leq\delta_1\tau}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi+i\langle\xi\rangle_m\int_0^t\lambda(s)b(s)ds} a(t,\xi)d\xi\right| &= \sup_{|x|\leq\delta_1\tau}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi+i\langle\xi\rangle_m\tau} a(t(\tau),\xi)d\xi\right| \\ &= \sup_{|y|\leq\delta_1}\left|\int_{\mathcal{K}} e^{i\tau(y\cdot\xi+\langle\xi\rangle_m)} a(t(\tau),\xi)d\xi\right|.\end{aligned}$$

For the stationary points of the phase function we get the relation

$$y + \frac{\xi}{\langle\xi\rangle_m} = 0. \tag{7.3}$$

The Hessian H_φ of the phase function $\varphi = \varphi(y,\xi) = y\cdot\xi + \langle\xi\rangle_m$ has the elements $(H_\varphi)_{jk} = \delta_{jk} - \xi_j\xi_k\langle\xi\rangle_m^{-2}$. Thus the stationary points are non-degenerate ones. If $|y| \geq 1$, then there is no stationary point. If $|y| < 1$, then a stationary point is given by

$$\xi = -\frac{m}{\sqrt{1-|y|^2}}y, \qquad \xi\in\mathcal{K}.$$

Without loss of generality one can choose $\mathcal{K}$ as a ball of the radius R. Then one has

$$|y| \leq \frac{R}{\sqrt{R^2+m^2}}.$$

Therefore we choose for y some direction $e_0 = y_0/|y_0|$ and consider only points y belonging to the segment $[0, R/\sqrt{R^2+m^2}]$ of this direction. We are going to get an estimate independent of any direction. To simplify notations we set $y_0 = (-1,0,\dots,0)$. Thus we can restrict ourselves to the consideration of the integral

$$\int_{|\xi|\leq R} e^{i\tau(-z\xi_1+\langle\xi\rangle_m)} a(t(\tau),\xi)d\xi, \qquad z\in[0,R/\sqrt{R^2+m^2}]$$

with the critical point

$$\xi = \left(\frac{m}{\sqrt{1-z^2}}z, 0,\dots,0\right)$$

smoothly depending on $z\in[0,R/\sqrt{R^2+m^2}]$. We are going to get an estimate uniform with respect to $z\in[0,R/\sqrt{R^2+m^2}]$.

The consideration of the asymptotic behaviour of the integral depending on the large parameter τ and parameter z is quite standard and follows with the arguments used in the method of stationary phase (see for instance, [3], [18]). We fix a point $z = z_0 \in [0,R/\sqrt{R^2+m^2}]$ and will get an estimate independent of z_0. In the small neighbourhood of the non-degenerate critical point $\xi^0 = (mz_0/\sqrt{1-z_0^2}, 0, \dots, 0)$ we use Morse lemma. Then there are a neighbourhood U of $0 \in \mathbb{R}^n$ (independent of z_0) and a ball V (independent of z_0) and a diffeomorphism $\mathcal{H}_{z_0} : V + \xi^0 \longrightarrow U$ $(y = \mathcal{H}_{z_0}(\xi))$ depending smoothly on the parameter $z_0 \in [0, R/\sqrt{R^2+m^2}]$ such that for $\varphi(z_0, \xi) = -z\xi_1 + \langle \xi \rangle_m$ one has

$$\varphi \circ \mathcal{H}_{z_0}^{-1}(y) = \varphi(z_0, \xi^0) + \frac{1}{2}\left(y_1^2 + \ldots + y_n^2\right) \quad \text{for all} \quad y \in U.$$

Moreover, the Jacobian of the diffeomorphism is uniformly bounded, that is there is a constant C such that

$$\left|\frac{\mathcal{D}\mathcal{H}_{z_0}^{-1}(y)}{\mathcal{D}y}\right| \le C \quad \text{for all} \quad z_0 \in [0, R/\sqrt{R^2+m^2}].$$

For the integral under consideration we write

$$\begin{aligned}\int_{|\xi|\le R} e^{i\tau(-z_0\xi_1+\langle\xi\rangle_m)} a(t(\tau),\xi)d\xi &= \int_{|\xi|\le R} e^{i\tau(-z_0\xi_1+\langle\xi\rangle_m)} \chi(\xi) a(t(\tau),\xi)d\xi \\ &\quad + \int_{|\xi|\le R} e^{i\tau(-z_0\xi_1+\langle\xi\rangle_m)} (1-\chi(\xi)) a(t(\tau),\xi)d\xi,\end{aligned}$$

where the cut-off function $\chi \in C_0^\infty(\xi_0 + V)$ and $\chi(\xi) \equiv 1$ if $\xi \in C_0^\infty(\xi_0 + V/2)$. For the last integral it is easily seen that for every given $N \le n+1$ there is constant C_N such that

$$\int_{|\xi|\le R} e^{i\tau(-z_0\xi_1+\langle\xi\rangle_m)} (1-\chi(\xi)) a(t(\tau),\xi)d\xi = C_N \tau^{-N} \quad \text{for all} \quad z_0 \in [0, R/\sqrt{R^2+m^2}].$$

For the first one we write

$$\begin{aligned}&\int_{|\xi|\le R} e^{i\tau(-z_0\xi_1+\langle\xi\rangle_m)} \chi(\xi) a(t(\tau),\xi)d\xi \\ &= \int_U e^{i\tau(\varphi(z_0,\xi^0)+\frac{1}{2}|y|^2)} \chi\big(\mathcal{H}_{z_0}^{-1}(y)\big) a\big(t(\tau), \mathcal{H}_{z_0}^{-1}(y)\big) \left|\frac{\mathcal{D}\mathcal{H}_{z_0}^{-1}(y)}{\mathcal{D}y}\right| dy \\ &= e^{i\tau\varphi(z_0,\xi^0)} \int_{\mathbb{R}^n} e^{i\tau\frac{1}{2}|y|^2} \chi\big(\mathcal{H}_{z_0}^{-1}(y)\big) a\big(t(\tau), \mathcal{H}_{z_0}^{-1}(y)\big) \left|\frac{\mathcal{D}\mathcal{H}_{z_0}^{-1}(y)}{\mathcal{D}y}\right| dy.\end{aligned}$$

Hence, we obtain for a smooth function $u(\tau, y, z_0)$ having compact support with respect to y uniformly with $z_0 \in [0, R/\sqrt{R^2+m^2}]$, $\tau \in [\tau_0, \infty)$, the representation

$$\int_{\mathbb{R}^n} e^{i\tau\frac{|y|^2}{2}} u(\tau, y, z_0)\, dy = (2\pi)^{\frac{n}{2}} e^{i\frac{\pi n}{4}} \tau^{-\frac{n}{2}} \sum_{k=0}^{N-1} \frac{\tau^{-k}}{k!} \left(\frac{i}{2}\triangle_y\right)^k u(\tau, 0, z_0) + S_N(u, \tau, y, z_0),$$

where

$$|S_N(u,\tau,y,z_0)| \le C_\varepsilon (N!)^{-1} \tau^{-\frac{n}{2}-N} \left\| \left(\frac{1}{2}\triangle_y\right)^N u(\tau, y, z_0) \right\|_{H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n_y)}$$

for any $\varepsilon > 0$. The special choice $N = 1$ completes the proof of the proposition. ∎

PROPOSITION 7.2. *Let $\phi = \phi(s)$ be a C^∞-function having compact support in $\{s \in \mathbb{R}\ ;\ s \in [c_0, c_1]\ \}$, $c_0 > 0$. Then for $t \in (0, t_{m,N}]$ and large τ*

$$\|F^{-1}\left(e^{i\tau\Lambda(t)|\xi|\sqrt{1+(\frac{m\,\Lambda(t)}{\tau|\xi|})^2}}\phi\Big(\sqrt{|\xi|^2+(m\,\Lambda(t)/\tau)^2}\Big)\right)\|_{L_\infty(\mathbb{R}^n)}$$
$$\le C(1+\tau\Lambda(t))^{-\frac{n-1}{2}}\sum_{|\alpha|\le n}\|D_\xi^\alpha\ \phi\Big(\sqrt{|\xi|^2+(m\,\Lambda(t)/\tau)^2}\Big)\|_{L_\infty(\mathbb{R}^n)}.$$

PROOF. For $\tau \ge \tau_0,\ \tau_0$ large, we obtain $c_0(\tau_0) \le |\xi| \le c_1(\tau_0)$, $c_0(\tau_0) > 0$, on the support of function ϕ uniformly for $t \in (0, t_{m,N}]$. One can write

$$F^{-1}\left(e^{i\tau\Lambda(t)|\xi|\sqrt{1+(\frac{m\,\Lambda(t)}{\tau|\xi|})^2}}\phi\Big(\sqrt{|\xi|^2+(m\,\Lambda(t)/\tau)^2}\Big)\right)$$
$$= F^{-1}\left(e^{i\tau\Lambda(t)|\xi|}\left\{e^{i\tau\Lambda(t)|\xi|(\sqrt{1+(\frac{m\,\Lambda(t)}{\tau|\xi|})^2}-1)}\phi\Big(\sqrt{|\xi|^2+(m\,\Lambda(t)/\tau)^2}\Big)\right\}\right).$$

It is easy to see that on the support of ϕ for all $t \in (0, t_{m,N}]$ and for all $\tau \ge \tau_0$

$$\left|D_\xi^\alpha\left(e^{i\tau\Lambda(t)|\xi|\Big(\sqrt{1+(\frac{m\,\Lambda(t)}{\tau|\xi|})^2}-1\Big)}\phi\Big(\sqrt{|\xi|^2+(m\,\Lambda(t)/\tau)^2}\Big)\right)\right| \le C_\alpha.$$

Then by means of the result of [9] we complete the proof of proposition in the way used to prove Lemma 4 [1]. ■

8. $L_p - L_q$ decay estimates for Fourier multipliers. The representations for the solutions from Theorems 6.1 and 6.2 suggest the study of the model Fourier multiplier

$$F^{-1}\left(e^{i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds}a(t,\xi)F(u_0)(\xi)\right),\ u_0 \in C_0^\infty(\mathbb{R}^n).$$

THEOREM 8.1. *Suppose that the following assumptions are satisfied for the amplitude function $a = a(t,\xi)$:*

$$|a(t,\xi)| \le C\frac{1}{\lambda(t)},\ (t,\xi) \in Z_{pd}(N),$$
$$|\partial_\xi^\alpha a(t,\xi)| \le C_M\frac{1}{\sqrt{\lambda(t)}}\langle\xi\rangle_m^{-|\alpha|}, |\alpha| \le \beta(M-1),\ (t,\xi) \in Z_{hyp}(N).$$

If $M \ge (n+1)/\beta + 1$, then we have the decay estimate

$$\Big\|F^{-1}\Big(e^{i\int_0^t\lambda(s)b(s)\langle\xi\rangle_m ds}a(t,\xi)F(u_0)(\xi)\Big)\Big\|_{L_q(\mathbb{R}^n)}$$
$$\le C\frac{1}{\sqrt{\lambda(t)}}(1+\Lambda(t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{W_p^L(\mathbb{R}^n)},$$

where $L = [n(\frac{1}{p}-\frac{1}{q})] + 1$.

PROOF. a) $t \in (0, t_{m,N}]$: Let $\chi = \chi(s) \in C^\infty(\mathbb{R}^1)$ be a function with $\chi(s) = 0$ for $s \leq N$, $\chi(s) = 1$ for $s \geq 2N$ and $0 \leq \chi(s) \leq 1$. We begin to estimate

$$F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(1-\chi(K(t)\langle\xi\rangle_m))\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big),$$

where $K(t) := \Lambda(t)^\beta$. Using the transformations $K(t)\xi = \eta$ and $K(t)z = x$ we get

$$\begin{aligned}
I &= \Big\|F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(1-\chi(K(t)\langle\xi\rangle_m))\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big)\Big\|^q_{L_q(\mathbb{R}^n)} \\
&= K(t)^{n+(2r-n)q}\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\frac{e^{iz\cdot\eta+i\frac{1}{K(t)}\int_0^t\lambda(s)b(s)(|\eta|^2+K^2(t)m^2)^{1/2}ds}}{(|\eta|^2+m^2K^2(t))^r} \\
&\times(1-\chi((|\eta|^2+K^2(t)m^2)^{1/2}))a\Big(t,\frac{\eta}{K(t)}\Big)F(u_0)\Big(\frac{\eta}{K(t)}\Big)d\eta\Big|^q dz \\
&= K(t)^{n+(2r-n)q}\Big\|F^{-1}\Big(\frac{e^{i\frac{1}{K(t)}\int_0^t\lambda(s)b(s)(|\eta|^2+K^2(t)m^2)^{1/2}ds}}{(|\eta|^2+m^2K^2(t))^r} \\
&\quad\times(1-\chi((|\eta|^2+K^2(t)m^2)^{1/2}))a\Big(t,\frac{\eta}{K(t)}\Big)\Big) \\
&* F^{-1}\Big(F(u_0)(\frac{\eta}{K(t)})\Big)\Big\|^q_{L_q(\mathbb{R}^n)}.
\end{aligned}$$

With the notations

$$T_t := F^{-1}\left(\frac{e^{i\frac{1}{K(t)}\int_0^t\lambda(s)b(s)(|\eta|^2+K^2(t)m^2)^{1/2}ds}}{(|\eta|^2+m^2K^2(t))^r}(1-\chi((|\eta|^2+K^2(t)m^2)^{1/2}))a\Big(t,\frac{\eta}{K(t)}\Big)\right)$$

the norm I can be written in the form

$$I = K(t)^{n+(2r-n)q}\Big\|T_t * F^{-1}\Big(F(u_0)\Big(\frac{\eta}{K(t)}\Big)\Big)\Big\|^q_{L_q(\mathbb{R}^n)}.$$

The distributions $F(T_t)$ belong to M_p^q for all $2r \leq n(\frac{1}{p}-\frac{1}{q})$ (see [6]). This follows from the facts that for $t \in (0, t_{m,N}]$ the functions $1-\chi((|\eta|^2+K^2(t)m^2)^{1/2})$ have a uniformly compact support with respect to η, from $|a(t,\frac{\eta}{K(t)})| \leq C$ on this support, from

$$\begin{aligned}
&\text{meas}\,\{\eta : (|\eta|^2+m^2K^2(t))^{-r} \geq l\} \leq \text{ meas}\,\{\eta : |\eta|^{-2r} \geq l\} \\
&= \text{meas}\,\{\eta : |\eta| \leq l^{-\frac{1}{2r}}\} \leq Cl^{-\frac{n}{2r}}
\end{aligned}$$

and from Theorem 1.11 [6]. Consequently,

$$\begin{aligned}
&\Big\|F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(1-\chi(K(t)\langle\xi\rangle_m))\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big)\Big\| \\
&\qquad\leq CK(t)^{2r-n(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)} \qquad (8.1)
\end{aligned}$$

for all $2r \leq n(\frac{1}{p}-\frac{1}{q})$. To study

$$F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}\chi(K(t)\langle\xi\rangle_m)\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big),$$

we choose a nonnegative C^∞-function $\phi = \phi(s)$ having compact support in $\{s \in \mathbb{R}^1 : 1/2 \le s \le 2\}$. We set $\phi_k(s) := \phi(2^{-k}s)$ while $\phi_0(s) := 1 - \sum_{k=1}^{\infty} \phi_k(s)$. One can find such a function that $\sum_{k=-\infty}^{+\infty} \phi_k(s) = 1$. Hence, supp $\phi_0 \subset \{s \in \mathbb{R}^1 : s \le 2\}$. Using the same ideas as above one can prove for $0 \le k \le k_0$ that

$$\begin{aligned}&\Big\| F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(\chi\phi_k)(K(t)\langle\xi\rangle_m)\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big)\Big\|_{L_q(\mathbb{R}^n)} \\ &\le C_k\, K(t)^{2r-n(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)}.\end{aligned} \tag{8.2}$$

To estimate for $k \ge k_0$ the L_q-norm of these multipliers we use the transformation $K(t)\xi = 2^k\eta$, derive $L_1 - L_\infty$, $L_2 - L_2$ estimates, respectively, and apply an interpolation argument (see [10]). We get

$$\begin{aligned} I_k &= \sup_{x\in\mathbb{R}^n}\Big| \int_{\mathbb{R}^n} e^{ix\cdot\xi + i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(\chi\phi_k)(K(t)\langle\xi\rangle_m)\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}d\xi\Big| \\ &= K(t)^{2r-n}2^{k(n-2r)} \\ &\times \sup_{y\in\mathbb{R}^n}\Big| F^{-1}\Big(e^{i2^k|\eta|\frac{1}{K(t)}\int_0^t \lambda(s)b(s)(1+\frac{K^2(t)m^2}{2^{2k}|\eta|^2})^{1/2}ds}\chi((2^{2k}|\eta|^2 + K^2(t)m^2)^{1/2}) \\ &\times \phi\Big(\Big(|\eta|^2 + \frac{K^2(t)m^2}{2^{2k}}\Big)^{1/2}\Big)\frac{a(t,\frac{2^k\eta}{K(t)})}{(|\eta|^2 + \frac{m^2K^2(t)}{2^{2k}})^r}\Big)\Big|. \end{aligned}$$

With a sufficiently large k_0 we have $|\eta| \in [1/4, 2]$ on supp ϕ and $\chi((2^{2k}|\eta|^2 + K^2(t)m^2)^{1/2}) \equiv 1$ for all $k \ge k_0$ and $t \in (0, t_{m,N}]$. Consequently we are able to apply Proposition 7.2. Here we use that $K(t) = O(\Lambda(t)^{\beta/2})$ if t tends to 0. It gives

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n}\Big| F^{-1}\Big(e^{i2^k|\eta|\frac{1}{K(t)}\int_0^t \lambda(s)b(s)(1+\frac{K^2(t)m^2}{2^{2k}|\eta|^2})^{1/2}ds}\chi((2^{2k}|\eta|^2 + K^2(t)m^2)^{1/2}) \\ &\times \phi\Big(\Big(|\eta|^2 + \frac{K^2(t)m^2}{2^{2k}}\Big)^{1/2}\Big)\frac{a(t,\frac{2^k\eta}{K(t)})}{(|\eta|^2 + \frac{m^2K^2(t)}{2^{2k}})^r}\Big)\Big| \\ &\le (1 + 2^k\Lambda(t)^{1-\beta/2})^{-\frac{n-1}{2}} \sum_{|\alpha|\le n}\|D_\eta^\alpha\Big(\frac{\phi((|\eta|^2 + \frac{K^2(t)m^2}{2^{2k}})^{1/2})a(t,\frac{2^k\eta}{K(t)})}{(|\eta|^2 + \frac{m^2K^2(t)}{2^{2k}})^r}\Big)\|_{L_\infty(\mathbb{R}^n)}. \end{aligned}$$

Due to Proposition 7.2 we need in $Z_{hyp}(N)$ the assumption for the derivatives $\partial_\xi^\alpha a(t,\xi)$ of the amplitude function for $|\alpha| \le n$. This means we have to guarantee that $n \le \beta(M-1)$. But this follows from the assumption for M. Therefore we have to choose in Proposition 5.1 the number of steps of perfect diagonalization $M \ge n/\beta + 1$. Then

$$\Big\| D_\eta^\alpha a\Big(t, \frac{2^k\eta}{K(t)}\Big)\Big\|_{L_\infty(\mathbb{R}^n)} \le C_M\Big(\frac{2^k}{K(t)}\Big)^{|\alpha|}\Big(\frac{2^{2k}|\eta|^2}{K^2(t)} + m^2\Big)^{-|\alpha|} \le C_M$$

for $|\alpha| \le n$. Using Lemma 3 from [1] leads to

$$\begin{aligned} I_k &\le C_M K(t)^{2r-n}2^{k(n-2r)} \\ &\quad\times\Big\| F^{-1}\Big(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}(\chi\phi_k)(K(t)\langle\xi\rangle_m)\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\Big)\Big\|_{L_\infty(\mathbb{R}^n)} \end{aligned}$$

$$\leq \quad C_M\, K(t)^{2r-n} 2^{k(n-2r)} (1+2^k \Lambda(t)^{1-\beta})^{-\frac{n-1}{2}} \|u_0\|_{L_1(\mathbb{R}^n)}$$

for $t \in (0, t_{m,N}]$ and $k \geq k_0$. The corresponding $L_2 - L_2$ estimate is

$$\left\| F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} (\chi\phi_k)(K(t)\langle\xi\rangle_m) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right) \right\|_{L_2(\mathbb{R}^n)}$$
$$\leq C\, K(t)^{2r} 2^{-2kr} \|u_0\|_{L_2(\mathbb{R}^n)}$$

for $t \in (0, t_{m,N}]$ and $k \geq k_0$. An interpolation argument leads to

$$\left\| F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} (\chi\phi_k)(K(t)\langle\xi\rangle_m) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right) \right\|_{L_q(\mathbb{R}^n)}$$
$$\leq \quad C_M K(t)^{2r-n(\frac{1}{p}-\frac{1}{q})} 2^{k(n(\frac{1}{p}-\frac{1}{q})-2r)} (1+2^k\Lambda(t)^{1-\beta})^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L_p(\mathbb{R}^n)}$$
$$\leq C_M K(t)^{2r-n(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L_p(\mathbb{R}^n)} \tag{8.3}$$

if $2r \geq n(\frac{1}{p}-\frac{1}{q})$. The inequalities (8.1) to (8.3) imply together with Lemmas 1 and 2 from [1] the $L_p - L_q$ estimate

$$\left\| F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right) \right\|_{L_q(\mathbb{R}^n)} \leq C\|u_0\|_{L_p(\mathbb{R}^n)} \tag{8.4}$$

for $t \in (0, t_{m,N})$ and $2r = n(\frac{1}{p}-\frac{1}{q})$.

The estimate (8.4) coincides with the estimate from [10] for $m = 1$ and $t \in (0, t_{m,N}]$.

b) $t \in [t_{m,N}, \infty)$. Let $\psi = \psi(s) \in C^\infty(\mathbb{R}^1)$ be a function with $\psi(s) = 0$ for $s \leq \sqrt{2}m$, $\psi(s) = 1$ for $s \geq 2m$ and $0 \leq \psi(s) \leq 1$. We begin to estimate

$$F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} (1-\psi(\langle\xi\rangle_m)) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right).$$

We have now the advantage that the amplitude function

$$(1-\psi(\langle\xi\rangle_m)) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}$$

has a compact support with respect to ξ uniformly for $t \in [t_{m,N}, \infty)$. As a Littman-type lemma we apply Proposition 7.1. The same approach as in the second part of a) gives

$$\begin{aligned} &\left\| F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} (1-\psi(\langle\xi\rangle_m)) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right) \right\|_{L_q(\mathbb{R}^n)} \\ &\leq \quad C_r \frac{1}{\sqrt{\lambda(t)}} \Lambda(t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L_p(\mathbb{R}^n)} \end{aligned} \tag{8.5}$$

for $r \geq 0$ and all $t \in [t_{m,N}, \infty)$.

Thus we can use in the following the information that $|\xi| \geq m$.

To estimate

$$F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} \psi(\langle\xi\rangle_m) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right)$$

we split it into

$$I_k = F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} \psi(\langle\xi\rangle_m) \phi_k(K(t)\langle\xi\rangle_m) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} F(u_0)(\xi) \right), k \geq 0,$$

where $\phi = \phi(s)$ is a nonnegative function having compact support in $\{s \in \mathbb{R}^1 : 2^{m_0} \leq s \leq 2^{m_0+2}\}$, $2^{m_0} > m$. If we consider I_k for $k \leq k_0$, in general k_0 is large, then we are able to apply Proposition 7.1 and obtain the $L_p - L_q$ estimates

$$\sum_{m=0}^{k_0} \|I_m(t,\cdot)\|_{L_q(\mathbb{R}^n)} \leq C_r \frac{1}{\sqrt{\lambda(t)}} \Lambda(t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L_p(\mathbb{R}^n)} \tag{8.6}$$

for all $r \geq 0$ (pay attention that $t \geq t_{m,N}$!). Moreover we use here that there exists a constant t_0 such that the amplitudes of I_k, $k \leq k_0$ vanish for all $t \geq t_0$. For the consideration of I_k, $k \geq k_0 + 1$, we use in opposite to the considerations for $t \in (0, t_{m,N}]$ the Klein-Gordon type transformation $K(t)\langle\xi\rangle_m = 2^k\langle\eta\rangle_m$ which is acting only in the radial direction. Let us analyze properties of this one to one transformation.

- It can be described by

$$\xi_l = \frac{f(|\eta|, 2^k, K(t))}{|\eta|}\eta_l, \; f(|\eta|, 2^k, K(t)) = \sqrt{\frac{2^{2k}}{K(t)^2}\langle\eta\rangle_m^2 - m^2}.$$

  Here we use $\xi_l = |\xi| g_l(\phi, \theta_1, \ldots, \theta_{n-2})$, $\eta_l = |\eta| g_l(\phi, \theta_1, \ldots, \theta_{n-2})$ and the above transformation.
- We have $x \cdot \xi = \frac{f(|\eta|, 2^k, K(t))}{|\eta|} x \cdot \eta$.
- We have

$$\nabla_\eta f(|\eta|, 2^k, K(t)) = \frac{2^k}{K(t)} \frac{\eta}{\sqrt{1 - \frac{m^2}{\langle\xi\rangle_m^2}}} \; .$$

  The additional information $|\xi| \geq m$ implies $|\nabla_\eta f(|\eta|, 2^k, K(t))| \leq C\frac{2^k}{K(t)}$.
- In the same way one shows that

$$|\nabla_\eta \xi_l| \leq C\frac{2^k}{K(t)}, \text{ where we use } \quad |\eta| \in [c_0(m_0), c_1(m_0)], \; c_0(m_0) > 0.$$

  Consequently, $|\det J_\eta(\xi)| \leq C(\frac{2^k}{K(t)})^n$.

Now we have all tools to follow the lines from the second part of a). We have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi + i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds} \psi(\langle\xi\rangle_m)\phi_k(K(t)\langle\xi\rangle_m) \frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}} d\xi \right| \\
&= K(t)^{2r} \sup_{y\in\mathbb{R}^n} \Bigg| \int_{\mathbb{R}^n} e^{iy\cdot\eta + i\int_0^t \lambda(s)b(s)\frac{2^k\langle\eta\rangle_m}{K(t)} ds} \psi\left(\frac{2^k\langle\eta\rangle_m}{K(t)}\right) \\
&\times \phi(\langle\eta\rangle_m) \frac{a(t,\xi(\eta))}{(2^k\langle\eta\rangle_m)^{2r}} |\det J_\eta(\xi)| d\eta \Bigg| \\
&\leq C 2^{k(n-2r)} K(t)^{2r-n} \sup_{y\in\mathbb{R}^n} \Bigg| F^{-1}\Bigg(e^{i\int_0^t \lambda(s)b(s)\frac{2^k\langle\eta\rangle_m}{K(t)} ds} \psi\left(\frac{2^k\langle\eta\rangle_m}{K(t)}\right) \\
&\times \phi(\langle\eta\rangle_m) \frac{a(t,\xi(\eta))}{\langle\eta\rangle_m^{2r}} \Bigg) \Bigg|.
\end{aligned}$$

But $D_\eta^\alpha \psi(\frac{2^k\langle\eta\rangle_m}{K(t)}) = 0$ for $\frac{2^k\langle\eta\rangle_m}{K(t)} \notin [\sqrt{2}m, 2m]$. Moreover, η takes values from a compact set. Hence, due to Proposition 7.1 we have

$$\sup_{y\in\mathbb{R}^n}\left|F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\frac{2^k\langle\eta\rangle_m}{K(t)}ds}\psi(\frac{2^k\langle\eta\rangle_m}{K(t)})\phi(\langle\eta\rangle_m)\frac{a(t,\xi(\eta))}{\langle\eta\rangle_m^{2r}}\right)\right| \le C_r\left(\frac{2^k\Lambda(t)}{K(t)}\right)^{-\frac{n}{2}}\frac{1}{\sqrt{\lambda(t)}}.$$

Using (8.6) and the assumptions for $a = a(t,\xi)$ we finally obtain as in the second part of a)

$$\begin{aligned}&\left\|F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}\psi(\langle\xi\rangle_m)\frac{a(t,\xi)}{\langle\xi\rangle_m^{2r}}F(u_0)(\xi)\right)\right\|_{L_q(\mathbb{R}^n)}\\ &\le C_M\frac{1}{\sqrt{\lambda(t)}}\,K(t)^{2r-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Lambda(t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)}\\ &\le C_M\Lambda(t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L_p(\mathbb{R}^n)}, \end{aligned}\tag{8.7}$$

if $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) \le 2r$. Here we need all derivatives $\partial_\xi^\alpha a(t,\xi)$ with $|\alpha| \le n+1$. This gives the condition of the theorem for M.

c) Let us summarize the information given from the estimates (8.4), (8.5) and (8.7).

In (8.4) we choose $2r = n(\frac{1}{p}-\frac{1}{q})$ and obtain

$$\left\|F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}a(t,\xi)F(u_0)(\xi)\right)\right\|_{L_q(\mathbb{R}^n)} \le C\|u_0\|_{W_p^L(\mathbb{R}^n)},$$

$L = [n(\frac{1}{p}-\frac{1}{q})]+1$, $t \in (0, t_{m,N}]$.

In (8.7) we choose $2r = \frac{n}{2}(\frac{1}{p}-\frac{1}{q})$ and obtain with the same L

$$\left\|F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}a(t,\xi)F(u_0)(\xi)\right)\right\|_{L_q(\mathbb{R}^n)} \le C\frac{1}{\sqrt{\lambda(t)}}\Lambda(t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{W_p^L(\mathbb{R}^n)}$$

for $t \in [t_{m,N}, \infty)$. Both these inequalities imply together with (8.5)

$$\left\|F^{-1}\left(e^{i\int_0^t \lambda(s)b(s)\langle\xi\rangle_m ds}a(t,\xi)F(u_0)(\xi)\right)\right\|_{L_q(\mathbb{R}^n)} \le C\frac{1}{\sqrt{\lambda(t)}}(1+\Lambda(t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{W_p^L(\mathbb{R}^n)}$$

for $t \in (0,\infty)$, where $L = [n(\frac{1}{p}-\frac{1}{q})]+1$. The statement of the theorem is proved. ∎

9. Klein-Gordon type decay rates. Summarizing the results from the previous sections we are able to prove Theorems 2.1 and 2.2 and can discuss Examples 2.1 and 2.2.

Proof of Theorem 2.1. After application of Theorem 8.1 to the explicit representations for ∇u and u_t obtained from Theorem 6.1 we get immediately the $L_p - L_q$ estimates

$$\|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\frac{1}{\sqrt{\lambda(t)}}\big(1+\Lambda(t)\big)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^L(\mathbb{R}^n)}\big)\,;$$

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\sqrt{\lambda(t)}\big(1+\Lambda(t)\big)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}\big).$$

Here $L = [n(\frac{1}{p}-\frac{1}{q})]+1$. But this implies the statement of the theorem. ∎

Proof of Theorem 2.2. After application of Theorem 8.1 to the explicit representations for ∇u and u_t obtained from Theorem 6.2 we get immediately the $L_p - L_q$ estimates

$$\|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\frac{1}{\sqrt{\lambda(t)}}\big(1+\Lambda(t)\big)^{\frac{C_{0,0}}{m}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}\big);$$

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\sqrt{\lambda(t)}\big(1+\Lambda(t)\big)^{\frac{C_{0,0}}{m}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}\big).$$

Here $L = [n(\frac{1}{p}-\frac{1}{q})]+1$. But this implies the statement of the theorem. ∎

We are able to describe classes of differential equations of Klein-Gordon type. If we compare our results with those for $m^2 \equiv 0$, then the decay functions coincide, but the essential part of the decay rates is $\frac{1}{2}(\frac{1}{p}-\frac{1}{q})$ higher in the Klein-Gordon case.

Discussion of Example 2.1. Let us consider for $l \ge 1$ the Cauchy problem

$$u_{tt} - (1+t)^{2l}b^2(t)(\triangle u - m^2 u) = 0,\ \ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

Here $b = b(t)$ is a 1-periodic, non-constant, smooth and positive function. Then the conditions (A1) to (A4) are satisfied, where $\beta = l/(l+1)$ in (A4). Thus we have to distinguish two cases.

Case $l > 1$: Applying Theorem 2.1 we have

$$\|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t^l)^{-1/2}(1+t^{l+1})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}),$$

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t^l)^{1/2}(1+t^{l+1})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}).$$

Case $l = 1$: Applying Theorem 2.2 we have

$$\|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t^l)^{-1/2}(1+t^{l+1})^{\frac{C_{0,0}}{m}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}),$$

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t^l)^{1/2}(1+t^{l+1})^{\frac{C_{0,0}}{m}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}).$$

Pay attention that in the case $m \equiv 0$ corresponding $L_p - L_q$ decay estimates cannot hold for all $l \ge 0$, see [13].

Discussion of Example 2.2. Let us consider for $\alpha \in (0,1]$ the Cauchy problem

$$u_{tt} - \exp(2t^\alpha)b^2(t)(\triangle u - m^2 u) = 0,\ \ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

Here $b = b(t)$ is a 1-periodic, non-constant, smooth and positive function. All assumptions (A1) to (A4) are satisfied. The condition (A4) is satisfied for each $\beta < 1$. Applying Theorem 2.1 we have

$$\|\nabla u(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t)^{\frac{1-\alpha}{2}}(1+\Lambda(t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}),$$

$$\|u_t(t,\cdot)\|_{L_q(\mathbb{R}^n)} \le C\,(1+t)^{\frac{\alpha-1}{2}}(1+\Lambda(t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}(\|u_0\|_{W_p^{L+1}(\mathbb{R}^n)} + \|u_1\|_{W_p^{L}(\mathbb{R}^n)}).$$

Here we use $\sqrt{\frac{\lambda(t)}{\Lambda(t)}} \sim t^{\frac{\alpha-1}{2}}$ for large t. Pay attention that in the case $m \equiv 0$ corresponding $L_p - L_q$ decay estimates cannot hold for $\alpha < 1/2$, see [13].

REMARK 9.1. At the end of this paper we want to remember different constants appearing in our approach. First we determine for our given Cauchy problem (1.5) the constant $\beta \in [1/2, 1]$ from (2.2). Then we determine the natural number $M \geq (n+1)/\beta+1$. The number M determines the number of steps of perfect diagonalization carried out in Proposition 5.1. Finally we have to choose $N \geq 1$ large enough, such that the matrix N_M from Proposition 5.1 is invertible. In this way one can choose all constants needed for our approach and follow all considerations. In Theorems 2.1 and 2.2 we need the constant L to describe the regularity of the data u_0 and u_1 which we have to suppose.

References

[1] P. BRENNER, *On $L_p - L_q$ estimates for the wave-equation*, Math. Zeitschrift 145 (1975), 251–254.

[2] P. BRENNER, *On the existence of global smooth solutions of certain semi-linear hyperbolic equations*, Math. Zeitschrift 167 (1979), 99–135.

[3] A. GRIGIS and J. SJÖSTRAND, *Microlocal Analysis for Differential Operators. An Introduction*, London Mathematical Society Lecture Note Series, Vol.196, University Press, Cambridge, 1994.

[4] F. HIROSAWA, *Energy decay for degenerate hyperbolic equations of Klein-Gorden type with dissipative term*, manuscript.

[5] L. HÖRMANDER, *Remarks on the Klein-Gorden equation*, Journées Equations aux Dérivées Partielles, Saint Jean de Monts, 1987.

[6] L. HÖRMANDER, *Translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.

[7] S. KLAINERMAN, *Global existence for nonlinear wave equations*, Comm. on Pure and Appl. Math. 33 (1980), 43-101.

[8] LI TA-TSIEN, *Global classical solutions for quasilinear hyperbolic systems*, John Wiley & Sons, 1994.

[9] W. LITTMAN, *Fourier transformations of surface carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. 69 (1963), 766–770.

[10] H. PECHER, *L_p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen. I*, Math. Zeitschrift 150 (1976), 159-183.

[11] R. RACKE, *Lectures on Nonlinear Evolution Equations*, Aspects of Mathematics, Vieweg, Braunschweig/Wiesbaden, 1992.

[12] M. REISSIG and K. YAGDJIAN, *An example for the influence of oscillations on $L_p - L_q$ decay estimates*, to appear.

[13] M. REISSIG and K. YAGDJIAN, *One application of Floquet's theory to $L_p - L_q$ estimates*, Math. Meth. Appl. Sci. 22 (1999), 937–951.

[14] M. REISSIG and K. YAGDJIAN, *$L_p - L_q$ estimates for the solutions of strictly hyperbolic equations of second order with time dependent coefficients*, accepted for publication in Mathematische Nachrichten.

[15] M. REISSIG and K. YAGDJIAN, *$L_p - L_q$ estimates for the solutions of hyperbolic equations of second order with time dependent coefficients - Oscillations via growth* -, Preprint 98-5, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg.

[16] R. Strichartz, *A priori estimates for the wave-equation and some applications*, J. Funct. Anal. 5 (1970), 218-235.

[17] W. v. Wahl, L^p*-decay rates for homogeneous wave-equations*, Math. Zeitschrift 120 (1971), 93-106.

[18] K. Yagdjian, *The Cauchy problem for hyperbolic operators. Multiple characteristics. Micro-local approach*, Math. Topics, Vol. 12, Akademie Verlag, Berlin, 1997.