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GLOBAL EXISTENCE OF SOLUTIONS OF THE EINSTEIN-BOLTZMANN EQUATION IN THE SPATIALLY HOMOGENEOUS CASE

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We consider the Einstein-Boltzmann system [1, 2]:

$$p^\alpha f_{,\alpha} - \Gamma^i_{\mu\nu} p^\mu p^\nu f_{,p^i} = Q(f,f), \tag{1.1}$$

$$G_{\mu\nu} = T_{\mu\nu}, \tag{1.2}$$

$$T^{\mu\nu} = \int p^\mu p^\nu f(t,x,p) \frac{|g|^{1/2}}{p_0} d\bar{p}, \tag{1.3}$$

where $Q(f,f)$ is the collision operator and $T^{\mu\nu}$ is the energy-momentum tensor. The first equation (1.1), called the Boltzmann equation, determines the distribution function $f(t,x,p)$ of gas particles. To describe f we need a submanifold $P(M)$ of the tangent bundle TM of the pseudoriemannian manifold M which is defined by the constraint:

$$P_x(p) : g_x(p,p) = g_{\alpha\beta} p^\alpha p^\beta = 1 \qquad (\alpha, \beta = 0,1,2,3), \tag{2}$$

where $g_{\alpha\beta}$ is a metric of M given by the Einstein equations (1.2). Then $f : P(M) \to \mathbf{R}$.

We assume that the spacetime is spatially homogeneous and isotropic. The symmetry implies that the metric simplifies to the form

$$ds^2 = dt^2 - R^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2), \tag{3}$$

where $R(t) > 0$ and the distribution function $f(t,\bar{x},\bar{p})$ does not depend on x and depends only on $p = |\bar{p}|$ $(f(t,\bar{x},\bar{p}) = f(t,p))$.

We consider the initial value problem in such a case. The aim of this paper is to show a global in time mild solution. For the proof we use methods similar to ones applied for the classical spatially homogeneous Boltzmann equation [4].

Local in time results for the Einstein-Boltzmann system in a general case have been considered in [1, 3, 6].

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With the above assumptions, the Einstein-Boltzmann system reads [1, 2]:

$$f_{,t} - 2\frac{\dot{R}}{R} p f_{,p} = \frac{1}{p^0} Q(f,f), \tag{4.1}$$

$$\left(\frac{\dot{R}}{R}\right)^2 = T_{00}, \tag{4.2}$$

where

$$T_{00} = R^3(t) \int f(t,\bar{p}) p^0 d\bar{p}, \tag{4.3}$$

$$Q(f,g) = Q^+(f,g) - Q^-(f,g), \tag{4.4}$$

$$Q^+(f,g) = \int_{R_q^3} \frac{R^3 d\bar{q}}{q_0} \int_{S^2} d\omega f(p') g(q') S(\bar{p},\bar{q},\bar{p}',\bar{q}'), \tag{4.5}$$

$$Q^-(f,g) = \int_{R_q^3} \frac{R^3 d\bar{q}}{q_0} \int_{S^2} d\omega f(p) g(q) S(\bar{p},\bar{q},\bar{p}',\bar{q}'), \tag{4.6}$$

$$\bar{p}' = \bar{p} - (\omega, \bar{p} - \bar{q})\omega, \tag{4.7}$$

$$\bar{q}' = \bar{q} + (\omega, \bar{p} - \bar{q})\omega, \tag{4.8}$$

$$p^0 = \sqrt{1 + R^2 p^2}, \tag{4.9}$$

$$0 \le S(\cdot\,,\cdot\,,\cdot\,,\cdot) \le C_1, \tag{4.10}$$

where $\omega \in S^2$, C_1 is constant and $S(p,q,p',q')$ is the cross section for the collisions.

We examine the initial value problem for system (4) with initial data:

$$R(0) = R_0 > 0, \qquad 0 \le f(0,p) = f_0(p) = f_0(\bar{p}) \in L^1(\mathbf{R}^3). \tag{5}$$

Because of (4.2) the above initial data do not ensure uniqueness. We have to add the extra initial condition $\dot{R}(0) < 0$ or $\dot{R}(0) > 0$.

First we define the mild solution to system (4). To reach our aim we have to reformulate the problem. For the Boltzmann equation (4.1) we apply the characteristic method. Thus we solve the following system

$$\frac{dp(t,y)}{dt} = -2\frac{\dot{R}}{R} p(t,y), \tag{6.1}$$

$$\frac{df(t,y)}{dt} = \frac{1}{p^0} Q(f,f)(t,y), \tag{6.2}$$

where

$$p^0 = \sqrt{1 + R^2(t) p^2(t,y)}.$$

Equation (6.1) gives the characteristic:

$$p(t,y) = \frac{y R^2(0)}{R^2(t)}. \tag{7}$$

It's easily seen that the jacobian of the transformation $p \to y$ is equal to:

$$\det\left(\frac{\partial \bar{p}}{\partial \bar{y}}\right) = \left(\frac{R^6(0)}{R^6(t)}\right) > 0.$$

In (t, y) coordinates we have

$$p^0 = p_0 = \sqrt{1 + \frac{R^4(0)}{R^2(t)} y^2} \quad \text{and} \quad T_{00} = \frac{R^6(0)}{R^6(t)} \int f p^0 d\bar{y}. \tag{8}$$

From (6.2) and from the property of the collision operator ($\int Q(f,f)d\bar{y} = 0$, see [3]) we get

$$\int \frac{R^6(0)}{R^3(t)} p^0 \frac{df}{dt} d\bar{y}.$$

So by (8) we obtain

$$\frac{d}{dt} T_{00} + 3\frac{\dot{R}}{R} T_{00} + \frac{\dot{R}}{R} \int \frac{R^6(0)}{R^3(t)} f \frac{\frac{R^4(0)}{R^2(t)} y^2}{p^0} d\bar{y} = 0. \tag{9}$$

Setting $L = \frac{\dot{R}}{R}$, from (4.2) and (9) we get

$$\dot{L} + \frac{3}{2} L^2 + \frac{1}{2} \int \frac{R^6(0)}{R^3(t)} f \frac{\frac{R^4(0)}{R^2(t)} y^2}{p^0} d\bar{y} = 0. \tag{10}$$

By (8) and (10) we have two inequalities

$$\dot{L} + \frac{3}{2} L^2 \leq 0 \ \text{ and } \ \dot{L} + 2L^2 \geq 0.$$

Solving them we obtain

$$\frac{1}{\frac{1}{L(0)} + 2t} \leq L(t) \leq \frac{1}{\frac{1}{L(0)} + \frac{3}{2}t}. \tag{11}$$

By the definition of L we get

$$R(t) = R(0) \exp\left\{ \int_0^t L(s) ds \right\}.$$

Notation

$$\bar{x} = (x_1, x_2, x_3), \qquad x = |x| = \sqrt{\sum_{k=1}^{3} x_k^2},$$

$$\frac{\partial}{\partial x} u = u_{,x}, \qquad \frac{\partial}{\partial t} u = u_{,t} = \dot{u},$$

$$X_r = \left\{ f \geq 0 : \int_{R^3} f(x) dx \leq r \right\},$$

$$||u|| = \int |u| d\bar{y}, \qquad ||u||_p = \int |u| d\bar{p},$$

$$d\bar{y} = \frac{R^6(t)}{R^6(0)} d\bar{p}, \qquad ||u||_y = \frac{R^6(t)}{R^6(0)} ||u||_p.$$

The main result is the following theorem:

THEOREM. *Let $f_0(p) \in X_r$ for $r \geq 0$, $R_0 > 0$, $\int p^0 f_0(p) R_0^3 d\bar{p} < \infty$ and $\dot{R}(0) > 0$. Then the Cauchy problem for the system* (4) *has a unique global in time nonnegative mild solution such that*

$$f \in C(0, \infty; L^1(\mathbf{R}^3_y))$$

and $R(\cdot)$ *is an increasing function satisfying the estimate*

$$R(0)\exp\left\{\frac{ds}{\frac{1}{\sqrt{T_{00}(0)}}+2s}\right\}\le R(t)\le R(0)\exp\left\{\frac{ds}{\frac{1}{\sqrt{T_{00}(0)}}+\frac{3}{2}s}\right\}.$$

REMARK. This paper can be treated as an erratum to [7]. In eq. (4.1) in [7] there should have been factor 2 in the second term of l.h.s. Therefore we cannot compute explicitly $R(t)$ ((10) and (11) in [7] are wrong). But since we have (11), all proofs of lemmas and *Proof of the Theorem* from [7] are correct.

To prove the theorem we need some lemmas. Since R^{-1} is a decreasing function, all constants below are independent of time.

LEMMA 1. *If* $f,g\in X_r$ *then*

$$\left|\left|\frac{1}{p^0}Q^+(f,f)-\frac{1}{p^0}Q^+(g,g)\right|\right|\le N(r)||f-g||, \tag{12}$$

$$\left|\left|\frac{1}{p^0}Q^-(f,f)-\frac{1}{p^0}Q^-(g,g)\right|\right|\le N(r)||f-g|| \tag{13}$$

and $N(r)=C_1r$.

LEMMA 2. *For any* $r>0$ *there exists* $n(r)>0$ *such that the equation:*

$$nu-\frac{1}{p^0}Q(u,u)=v \tag{14}$$

with $v\in X_r$ *has a unique nonnegative solution* u *belonging to* $L^1(\mathbf{R}^3)$ *for any* $n\ge n(r)$.

DEFINITION. Let $R(n,Q)=(n-\frac{1}{p^0}Q)^{-1}:X_r\to X_r$ for $n\ge n(r)$.

For $R(n,Q)$ we prove the following estimates:

LEMMA 3. *If* $g,h\in X_r$ *and* $n\ge\max\{8N(r),1\}$ *then*

$$||R(n,Q)g||\le\frac{1+\varepsilon}{n}||g||, \tag{15}$$

where $\varepsilon=\frac{4N(r)}{n^2}$ *and*

$$||R(n,Q)nu-R(n,Q)nv||\le N_1(r)||u-v||, \tag{16}$$

where $N_1(r)<2$.

DEFINITION. We define the Yosida approximation of the operator $\frac{1}{p^0}Q$ by

$$Q_n=nR(n,Q)n-n=\frac{1}{p^0}QR(n,Q)n. \tag{17}$$

LEMMA 4. *For* Q_n *defined by (17) we have* $\lim_{n\to\infty}Q_n=\frac{1}{p^0}Q$ *in* $L^1(\mathbf{R}^3)$.

The solution of the Boltzmann equation (6.2) will be approximated by the solution of the equation on the intervals $[t_0,t_0+t]$:

$$f_n(t_0+t,y)=f_n(t_0,y)+\int_{t_0}^{t_0+t}Q_n(f_n,f_n)(s,y)ds. \tag{18}$$

LEMMA 5. *There exists a unique solution* $f_n(t,y)$ *of equation* (18) *in* $C(t_0,t_0+\frac{1}{3n};L^1(\mathbf{R}^3))$ *such that* $||f_n(t_0+t)||\le\frac{1}{1-\delta}||f_n(t_0)||$ *for* $0<t\le\frac{1}{3n}$ *and* $\delta=\frac{4N(r)}{n^2}$.

Proof of the Theorem. We can construct an approximation of the solution of the Boltzmann equation (6.2) on the interval $[0, T]$ for any $T > 0$ and small $\delta > 0$. We take $n_1 = [4N(r)] + l = k_0 + 1$, where l is such that

$$\exp \frac{4N(2r)}{k_0} < \frac{1}{1-\delta}. \tag{19}$$

Then from Lemma 5 we get a unique solution of (18) on the interval $[0, T_1]$, where $T_1 = \frac{1}{3n_1} = \frac{1}{3(k_0+1)}$; we denote this solution by f_{n_1}. By Lemma 5 we get for $0 \le t \le T_1$:

$$||f_{n_1}||(t) \le \frac{1}{1 - \frac{4N(r)}{n_1^2}} ||f_0|| \le \frac{1}{1 - \frac{4N(2r)}{(k_0+1)^2}} ||f_0||.$$

Solving (18) with $t_0 = T_1$ and greater n we obtain T_2 etc.

Precisely, we construct F_{k_0} - the approximation of the solution on $[0, T]$:

1. $F_{k_0}|_{[0,T_1]} = f_{n_1}$, where $n_1 = k_0 + 1$, $T_1 = \frac{1}{3(k_0+1)}$, $f_{n_1}(0, y) = f_0(y)$ and

$$\sup_{0 \le t \le T_1} ||f_{n_1}(t)|| \le \frac{1}{1 - \frac{4N(2r)}{(k_0+1)^2}} ||f_0||.$$

2. $F_{k_0}|_{[T_1,T_2]} = f_{n_2}$, where $n_2 = k_0 + 2$, $T_2 = T_1 + \frac{1}{3(k_0+2)}$, $f_{n_2}(T_1, y) = f_{n_1}(T_1, y)$ and

$$\sup_{T_1 \le t \le T_2} ||f_{n_2}(t)|| \le \prod_{j=1}^{2} \frac{1}{1 - \frac{4N(2r)}{(k_0+j)^2}} ||f_0||.$$

$$\vdots$$

i. $F_{k_0}|_{[T_{i-1},T_i]} = f_{n_i}$, where $n_i = k_0 + i$, $T_i = T_{i-1} + \frac{1}{3(k_0+i)}$, $f_{n_i}(T_{i-1}, y) = f_{n_{i-1}}(T_{i-1}, y)$ and

$$\sup_{T_{i-1} \le t \le T_i} ||f_{n_i}(t)|| \le \prod_{j=1}^{i} \frac{1}{1 - \frac{4N(2r)}{(k_0+j)^2}} ||f_0||.$$

$$\vdots$$

n_K. $F_{k_0}|_{[T_{K-1},T_K]} = f_{n_K}$, where $n_K = k_0 + K$, $T_K = T_{K-1} + \frac{1}{3(k_0+K)}$, $f_{n_K}(T_{n_{K-1}}, y) = f_{n_{K-1}}(T_{n_{K-1}}, y)$ and

$$\sup_{T_{K-1} \le t \le T_K} ||f_{n_K}(t)|| \le \prod_{j=1}^{K} \frac{1}{1 - \frac{4N(2r)}{(k_0+j)^2}} ||f_0||,$$

where K is so large that $T_K \ge T$ or $\sum_{j=1}^{K} \frac{1}{3(k_0+j)} > T$ (it is always possible) and from (19) $\prod_{j=1}^{\infty} \frac{1}{1 - \frac{4N(2r)}{(k_0+j)^2}} < \frac{1}{1-\delta}$ ($\sum \frac{1}{n} = \infty$ and $\sum \frac{1}{n^2} < \infty$). And this implies that

$$\sup_{t \in [0,T]} ||F_{k_0}|| \le \frac{1}{1-\delta} ||f_0||. \tag{20}$$

Thus we have constructed F_{k_0}. By Lemmas 1 and 4 we can show that for small fixed T

$$\lim_{k_0 \to \infty} F_{k_0} = f \quad \text{in} \quad C(0, T; L^1(\mathbf{R}^3)),$$

hence we have obtained the solution of (4). Since $\delta > 0$ can be arbitrarily chosen we get

$$\sup_{t\in[0,T]} ||f(t)||_y \le ||f(0)||_y. \tag{21}$$

By (21) and (11) we can continue the solution in intervals $[T,2T],[2T,3T],\ldots$, etc. Thus we constructed the solution of (4) for any T.

Proofs of the lemmas one can find in [7].

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