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TIME-PERIODIC SOLUTIONS OF QUASILINEAR PARABOLIC DIFFERENTIAL EQUATIONS II. OBLIQUE DERIVATIVE BOUNDARY CONDITIONS

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Abstract. We study boundary value problems for quasilinear parabolic equations when the initial condition is replaced by periodicity in the time variable. Our approach is to relate the theory of such problems to the classical theory for initial-boundary value problems. In the process, we generalize many previously known results.

1. Introduction. Let P be the quasilinear operator defined by

$$Pu = -u_t + a^{ij}(X, u, Du)D_{ij}u + a(X, u, Du)$$

for a positive definite matrix-valued function a^{ij} and a scalar-valued function a. In the standard theory for such equations [7, 11], the usual problem to study is the initial-boundary value problem

$$Pu = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } S\Omega, \tag{1}$$

$$u = u_0 \text{ on } \omega \tag{2}$$

for some domain $\Omega \subset \mathbf{R}^{n+1}$ with lateral boundary $S\Omega$ and initial surface ω. (The usual example for Ω is $\Omega = \omega \times (0, T)$ in which case $S\Omega = \partial\omega \times (0, T)$.) Alternatively, the Dirichlet condition $u = \varphi$ may be replaced by the nonlinear operator condition $Nu = 0$ with N defined by

$$Nu = b(X, u, Du)$$

and N is assumed to be an oblique derivative condition, that is

$$\frac{\partial b}{\partial p}(X, z, p) \cdot \gamma > 0$$

where γ is the unit inner spatial normal to $S\Omega$. (If $\Omega = \omega \times (0, T)$, then γ is just the inner normal to ω.) Under suitable general conditions on the functions a^{ij}, a, and b,

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it is well-known that (1), (2) has a solution, which is globally smooth under appropriate compatibility conditions on the data. On the other hand, if we modify the initial condition (2), such problems are largely unexplored.

In [13], we discussed the case in which the Dirichlet condition is prescribed on $S\Omega$ and (2) is replaced by the periodic condition $u(x,0) = u(x,T)$ for all $x \in \omega$. In this paper, we consider the corresponding periodic problem with an oblique derivative boundary condition. Of course, this problem has been studied for some time [1, 4, 15, 17] (see also the discussion in Section 3.5 of [20]), but those works are primarily concerned with semilinear problems (so a^{ij} depends only on X and u, and b depends linearly on p). We shall consider the more general structure indicated above, that is, we study the problem

$$Pu = 0 \text{ in } \Omega, \qquad Nu = 0 \text{ on } S\Omega, \qquad u(\cdot,0) = u(\cdot,T) \text{ in } \omega. \tag{3}$$

Moreover, we shall show that, in the case of semilinear problems, the hypotheses in these works can be relaxed. As in [13], we follow the basic idea from [16] of using *a priori* estimates similar to those for initial-boundary value problems.

We present *a priori* estimates of solutions of (3) in Sections 2, 3, and 4 under general structure conditions on the operators P and N. Specifically, we prove L^∞ estimates for the solutions in Section 2 and L^∞ estimates for the gradients of the solutions are given in Sections 3. Section 4 is concerned with Hölder gradients. These estimates form the basis for our existence program, which we also present there. Section 5 gives an application of our results to semilinear problems, in which case some of the estimates can be derived more simply and some hypotheses can be relaxed. Finally, Section 6 discusses the problem in one spatial dimension; again, this special structure makes the estimates and corresponding existence result simpler.

We follow the notation in [11, 13] and we refer the reader to Section 2 of [13] for the definition of periodic domains as well as for the definitions of the various function spaces. In a future paper, we study problem (3) in a conormal form; this means that P can be written in divergence form, that is,

$$Pu = -u_t + \operatorname{div} A(X,u,Du) + B(X,u,Du)$$

for some vector-valued function A and scalar-valued function B, that N has the form

$$Nu = A(X,u,Du)\cdot\gamma + \psi(X,u)$$

for some scalar-valued function ψ (so N is oblique if P is parabolic), and that $\Omega = \omega \times (0,T)$ for some domain $\omega \subset \mathbf{R}^n$. (The significance of this last assumption is discussed in [11, Section 6.10].)

In addition, we write $\lambda(X,z,p)$ and $\Lambda(X,z,p)$ for the minimum and maximum eigenvalues, respectively, of the matrix $(a^{ij}(X,z,p))$. We also write $\Gamma = \Omega \times \mathbf{R}^n \times \mathbf{R}$, $\Gamma' = S\Omega \times \mathbf{R}^n \times \mathbf{R}$, and we write Γ'' for the subset of Γ' on which $p\cdot\gamma = 0$.

2. Pointwise bounds. Our first step in the existence program is to prove a bound in L^∞ for solutions of (3). We begin with two cases that are immediate consequences of known results.

LEMMA 2.1. *Let u be a solution of* (3)*, and suppose that there are nonnegative constants μ_1 and μ_2 and an increasing function M_1 such that*

$$(\operatorname{sgn} z)a(X,z,p) \le \lambda(X,z,p)[\mu_1|p| + \mu_2] \tag{4}$$

for all $(X,z,p) \in \Gamma$, and

$$(\operatorname{sgn} z)b(X,z,p) < 0 \tag{5}$$

for all $(X,z,p) \in \Gamma'$ with $|z| \ge M_1(|p|)$. If $S\Omega \in H_2$, then $\sup|u| \le C(\mu_1,\mu_2,M_1,\Omega)$.

PROOF. We follow the proof of [8, Lemma 3.1] using the strong maximum principle as in [13, Lemma 3.3]. ∎

For our next lemma, we note (by combining the discussions from p. 51 of [8] and Section 10.3 of [11]) that if $S\Omega \in H_2$, then there is a function $\rho \in C^{2,1}(\Omega) \cap H_2(\Omega)$ which vanishes on $S\Omega$ and is positive in Ω. We write ρ_0 for the maximum of ρ, ρ_1 for the maximum of $|\rho_t|$, and ρ_2 for $\sup|D^2\rho|$.

LEMMA 2.2. *Let u be a solution of* (3) *with $S\Omega \in H_2$, and suppose that there are nonnegative constants μ_3 and M_2 such that*

$$b(X,z,-(\operatorname{sgn} z)\mu_3\gamma)(\operatorname{sgn} z) < 0 \tag{6}$$

for all $(X,z) \in S\Omega \times \mathbf{R}$ with $|z| \ge M_2$ and

$$[\rho_1 + \rho_2\Lambda(X,z,p)]\mu_3 + (\operatorname{sgn} z)a(X,z,p) < 0 \tag{7}$$

for all $(X,z,p) \in \Gamma$ with $|z| \ge M_2$ and $|p| \le \mu_3$. Then $\sup|u| \le M_2 + \rho_0\mu_3$.

PROOF. Now we follow the proof of [8, Lemma 3.2] and use the strong maximum principle. ∎

For our general existence theory, we shall use a modification of Lemma 2.2 when a has a special form and the operator P is uniformly parabolic.

LEMMA 2.3. *Let $S\Omega \in H_{1+\alpha}$ for some constant $\alpha \in (0,1)$, let P have the form*

$$Pu = a^{ij}(X,u,Du)D_{ij}u - u + f(X), \tag{8}$$

and suppose there are constants a_0, M_3, λ_0, and Λ_0, such that

$$\Lambda(X,z,p) \le \Lambda_0,\ \ \lambda(X,z,p) \ge \lambda_0 \tag{9}$$

for all $(X,z,p) \in \Gamma$ with $|z| \ge M_3$,

$$|f| \le a_0\lambda_0 d^{\alpha-1}. \tag{10}$$

Suppose also that there is a constant μ_3 such that (6) *holds for $|z| \ge M_3$. Then there is a constant C determined only by a_0, M_3, λ_0, Λ_0, μ_3, and Ω such that $|u| \le C$ in Ω.*

PROOF. We first recall from [11, Section 4.5] that there is a proper regularized distance ρ. This means that ρ/d is bounded above and below in Ω and that there is a positive constant ε_1 such that $|D\rho| \ge 1$ wherever $d < \varepsilon_1$. Moreover

$$|D\rho| \le 2 \text{ and } |\rho_t| + |D^2\rho| \le C(\Omega)d^{\alpha-1} \text{ in } \Omega.$$

Now we set $v = -\mu_3\rho - K\rho^{1+\alpha} + v_0$ with
$$K = \frac{(C(\Omega)\Lambda_0 + 1)\mu_3}{\lambda_0\alpha} + \frac{a_0}{2\alpha} + 1$$
and v_0 a constant chosen so that $\sup_\Omega(u-v) = 0$. To prove an upper bound for u we need only bound v_0 from above, so we may assume without loss of generality that $v_0 \ge M_3$. As in [8], it follows that the maximum cannot occur on $S\Omega$. To study the interior maximum, we define the operator L by $Lw = -w_t + a^{ij}D_{ij}w - w$. If $d < \varepsilon_1$, then we have
$$Lv \le -a_0\lambda_0\rho^{\alpha-1} + C(\Omega)(\Lambda+1)\rho^{2\alpha-1} + \mu_3\rho + K\rho^{1+\alpha} - \alpha(1+\alpha)\lambda\rho^{\alpha-1},$$
so $Lv \le -f$ wherever $d < \varepsilon$ for a suitably small constant ε. Once $d \ge \varepsilon$, we have $Lv \le C_1 - v_0$ for some $C_1(a_0, \alpha, \varepsilon, \lambda_0, \Lambda_0, \mu_3)$. It follows that $u-v$ cannot have an interior maximum if $v_0 \ge C_1$ and hence $v_0 \le C_1$. A lower bound for u is proved similarly. ■

3. Gradient estimates. The usual gradient bound is a local one (see e.g. [11, Section 13.3], [2], [19]), so these results can be applied to the periodic case. First, we apply [11, Theorem 13.13] to the periodic case (although we point out that condition (13.49) there should be replaced by the more general condition (12) below).

THEOREM 3.1. *Suppose that there are positive constants* b_0, b_1, M, β_0, λ, Λ, Λ_1, Λ_2, *and* Λ_3 *such that* a^{ij} *and* a *satisfy*

$$a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2, \quad |a^{ij}| \le \Lambda, \quad |a| \le \Lambda_1(1+|p|^2) \tag{11}$$

for all $(X,z,p) \in \Gamma$,

$$\pm b(X,z,p' \pm \beta_0(1+|p'|)\gamma) > 0 \tag{12}$$

for all $(X,z,p') \in \Gamma''$,

$$|p|^2|a^{ij}_p| + |p||a^{ij}_z| + |a^{ij}_x| \le \Lambda_2|p|, \tag{13}$$

$$|p|^2|a_p| + |a_x| \le \Lambda_3|p|^3, \tag{14}$$

$$|p|a_z \le \Lambda_3|p|^2 \tag{15}$$

for all $(X,z,p) \in \Gamma$ *with* $|p| \ge M$, *and*

$$|b_p| \le b_0 b_p\cdot\gamma, \tag{16}$$

$$|p|^2|b_z| + |p||b_x| + |b_t| \le b_1|p|^3 b_p\cdot\gamma, \tag{17}$$

for all $(X,z,p) \in \Gamma'$ *with* $|p| \ge M$. *If* $u \in C^{2,1}(\overline{\Omega})$ *is a solution of* (3) *and if* $S\Omega \in H_3$, *then there is a positive constant* C *determined only by* b_0, b_1, n, β_0, λ, Λ, Λ_1, Λ_2, Λ_3, Ω, *and* $\operatorname{osc} u$ *such that* $\sup_\Omega|Du| \le C$.

In fact, the regularity of $S\Omega$ can be relaxed to $S\Omega \in H_{2+\alpha}$ with $\alpha > 0$ by using the ideas at the end of [14, Section 3].

If we follow the approach in [19], we can remove the differentiability assumptions on a and we can relax the smoothness of $S\Omega$ provided we assume more regularity for b. To state the additional structure conditions, we introduce the vector differential operator δ defined by $\delta f(X,z,p) = f_z(X,z,p)p + f_x(X,z,p)$.

THEOREM 3.2. *Suppose there are positive constants* b_0, b_1, Λ_1, Λ_2, *and* Λ_3 *such that* a^{ij} *and* a *satisfy* (11) *for all* $(X,z,p) \in \Gamma$ *and* (13) *for all* $(X,z,p) \in \Gamma$ *with* $|p| \ge M$.

Suppose also that b *satisfies* (12) *for all* $(X, z, p') \in \Gamma''$ *and* (16), (17) *for all* $(X, z, p) \in \Gamma'$ *with* $|p| \geq M$. *Suppose also that*

$$|p'||b_{pp}| \leq b_2 b_p \cdot \gamma, \tag{18}$$

$$|p|^2|\delta b_p| + |p||\delta b_z| + |\delta b_x| \leq b_2|p|^3 b_p \cdot \gamma \tag{19}$$

for all $(X, z, p) \in \Gamma'$ *with* $|p| \geq M$. *If* $u \in C^{2,1}(\overline{\Omega})$ *is a solution of (3) and if* $S\Omega \in H_2$, *then there is a positive constant* C *determined only by* b_0, b_1, b_2, n, λ, Λ, Λ_1, Λ_2, Ω, *and* $\operatorname{osc} u$ *such that* $\sup_\Omega |Du| \leq C$.

We note here that some of the assumptions in Theorem 3.2 can be further weakened, and we refer the interested reader to [19] for details.

4. Hölder gradient estimates and existence theorems. Since the usual Hölder gradient estimates for quasilinear equations are local in nature, they immediately apply to the periodic case. We shall quote the results in [11] because they are in a particularly convenient form for our purposes. To state these results, we note that the equation $b(X, u, Du) = 0$ can be rewritten as $Du \cdot \gamma = g(X, u, D'u)$ for a function g under suitable hypotheses, in particular, if there is an increasing function h such that

$$\pm b(X, z, p' \pm h(|z| + |p'|)\gamma) > 0 \tag{20}$$

for all $(X, z, p') \in \Gamma''$. This condition is satisfied, for example, if (12) holds. Moreover g is uniformly Lipschitz with respect to p' if $|b_p| \leq C b_p \cdot \gamma$.

Theorem 4.1. *Suppose there are positive constants* a_0, K, R, α, λ, λ_0, Λ, μ, μ_1 *such that* $S\Omega \in H_{1+\alpha}$,

$$a^{ij}\xi_i\xi_j \geq \lambda |\xi|^2, \ \left|a^{ij}\right| \leq \Lambda, \tag{21}$$

$$\left|a_p^{ij}\right| \leq \lambda_0, \tag{22}$$

$$|a| \leq a_0 d^{\alpha-1} \tag{23}$$

for all $(X, z, p) \in \Gamma$ *with* $|z| + |p| \leq K$, *and*

$$|g_{p'}(X, z, p')| \leq \mu, \tag{24}$$

$$|g(X, z, p') - g(Y, w, p')| \leq \mu_1(|X - Y|^\alpha + |z - w|^\alpha) \tag{25}$$

for all (X, z, p') *and* (Y, w, p') *in* Γ'' *with* $\max\{|z|, |w|\} + |p'| \leq K$. *Suppose also that there is a continuous, nonnegative increasing function* ζ *with* $\zeta(0) = 0$ *such that*

$$\left|a^{ij}((x,t), z, p) - a^{ij}((y,t), w, p)\right| \leq \zeta(|x - y| + |z - w|) \tag{26}$$

for all $((x,t), z, p)$ *and* $((y,t), w, p)$ *in* Γ *with* $\max\{|z|, |w|\} + |p| \leq K$. *If* $u \in C^{2,1}(\Omega)$ *with* $Du \in C(\overline{\Omega})$ *is a periodic solution of*

$$-u_t + a^{ij}(X, Du)D_{ij}u + a(X) = 0 \text{ in } \Omega \tag{27}$$

$$Du \cdot \gamma - g(X, D'u) = 0 \text{ on } S\Omega, \tag{28}$$

such that $|u| + |Du| \leq K$, *then there is a positive constant* θ *determined only by* K, n, α, λ, λ_0, Λ, μ, μ_1 *and* ζ *such that*

$$|Du|_\theta \leq C(K, n, \alpha, \lambda, \lambda_0, \Lambda, \zeta, \Omega)(1 + a_0 + \mu_1). \tag{29}$$

Proof. By a standard localization argument and known results for interior estimates, it suffices to prove (29) in a neighborhood N of a point in $S\Omega$. Since $S\Omega \in H_{1+\alpha}$, our hypotheses are invariant under a suitable change of coordinates which transforms $N \cap \Omega$ to the half-cylinder $Q^+ = \{X : |x| \le 1, -1 < t < 0, x^n > 0\}$ and $N \cap S\Omega$ to $Q^0 = \{X : |x| \le 1, -1 < t < 0, x^n = 0\}$. The Hölder estimate in this case is just [11, Lemma 13.22] once we note that we may allow A^{ij} in [11, Lemma 13.21] to depend on t without changing the proof or result of that lemma. ∎

For our existence theory, it will be useful to prove a Hölder gradient estimate under slightly different hypotheses, which is a simple modification of Theorem 4.1.

Lemma 4.2. *Let $u \in C^{2,1}(\Omega)$ be periodic with $Du \in C(\overline{\Omega})$ for some Ω with $S\Omega \in H_{1+\alpha}$. Suppose that u solves*

$$-u_t + a^{ij}(X)D_{ij}u + a(X) = 0 \text{ in } \Omega,\ \gamma \cdot Du = g(X, D'u) \text{ on } S\Omega \tag{30}$$

for functions a^{ij}, a, and g satisfying (21), (23), (24),

$$|g(X,p') - g(Y,p')| \le \mu_1[1 + |p'|]^{1+\alpha}|X - Y|^\alpha, \tag{31}$$

$$|a^{ij}(x,t) - a^{ij}(y,t)| \le \zeta(|x - y|) \tag{32}$$

for positive constants a_0, λ, Λ, μ, μ_1 and a continuous, nonnegative, increasing function ζ with $\zeta(0) = 0$. Then there is a constant $\theta \in (0, \alpha]$ determined only by n, Ω, α, λ, and Λ such that $Du \in H_\theta$ and

$$|Du|_\theta \le C(n, \Omega, \lambda, \Lambda, \alpha, a_0, \mu_1, |u|_0). \tag{33}$$

Proof. We introduce the notation

$$Q[R](X_0) = \{X \in \Omega : |X - X_0| < R,\ t < t_0\}$$

for R a positive number and $X_0 \in \overline{\Omega}$. We also recall the weighted Hölder seminorm

$$[w]^*_{\theta;Q[R](X_0)} = \sup_{r<R}(R - r)^\theta [w]_{\theta;Q[r](X_0)}.$$

¿From the explicit form of the Hölder estimate in [11, Lemma 13.18] (with μ_1 there replaced by $\mu_1[1 + \sup_{\Sigma^+(R)} |Dv|]^\alpha$), it is easy to see that

$$[Dv]^*_{\theta;Q[R](X_0)} \le C_1(1 + M_1 R)(1 + M_1) \tag{34}$$

where $M_1 = \sup_{Q[R]} |Dv|$ and C_1 is determined by the same quantities as C in (33). (In particular, this estimate guarantees that $Dv \in H_\theta$.) From [11, Corollary 7.36], we have also a Hölder estimate for u and then the obvious parabolic analog of Trudinger's interpolation inequality [18, Lemma 1] gives a uniform bound for Dv. Another application of (34) gives the desired result. ∎

From these estimates we can prove several existence theorems, depending on the specifics of the equation and boundary condition. Our first step is an existence theorem for linear equations with nonlinear boundary conditions. To state this result, we recall the definition of the "starred" seminorms and norms from [10]:

$$[u]^{*(b)}_\alpha = \sup_{(x,t),(y,t) \text{ in } \Omega} (\min\{d(x,t), d(y,t)\})^{\alpha+b}|u(x,t) - u(y,t)|/|x - y|^\alpha,$$

$$|u|^{*(b)}_\alpha = [u]^{*(b)}_\alpha + |u|^{\alpha+b}_0, \qquad |u|^*_\alpha = |u|^{*(-\alpha)}_\alpha.$$

PROPOSITION 4.3. *Let* $S\Omega \in H_{1+\alpha}$, *and suppose* $a^{ij} \in H_\beta^{*(0)}(\Omega)$ *and* $a \in H_\beta^{*(1-\alpha)}$ *for some constants* α *and* β *in* $(0,1)$. *Suppose also that* a^{ij} *satisfies* (21) *and* (32). *If, finally,* g *satisfies conditions* (24) *and* (31), *then there is a unique periodic solution of*

$$-u_t + a^{ij} D_{ij} u + a - u = 0 \text{ in } \Omega,\ Du \cdot \gamma = g(X, D'u) \text{ on } S\Omega. \tag{35}$$

PROOF. Suppose first that a^{ij} and a are in H_β^*, that $g \in H_3$, and that $S\Omega \in H_{2+\beta}$. The argument of, for example, [9, Theorem 1] (in conjunction with the linear theory in [12]) shows that (35) has a solution in $H_{2+\beta}$ provided we can show that any solution of (35) satisfies an estimate of the form

$$|u|_0 + |Du|_\theta \le C \tag{36}$$

for some $\theta \in (0,1)$ and the constant C depends on a only through the norm $|a|_\beta^*$. A bound on $|u|_0$ is immediate from Lemma 2.1, and then $|Du|_\theta \le C$ from Lemma 4.2.

To complete the proof, we note that we can approximate the coefficients in (35) along with $S\Omega$ so that the hypotheses of this proposition are satisfied uniformly and such that the approximating coefficients and domain are as smooth as we wish. It suffices to show that the solutions to these approximating problems satisfy (36) with a uniform constant C. By virtue of Lemma 4.2, we are reduced to proving a uniform L^∞ estimate, and this estimate follows from Lemma 2.3. ∎

From this existence result, we can infer a conditional existence theorem for oblique derivative problems along the lines suggested by [14, Theorem 7.6].

THEOREM 4.4. *Let* $S\Omega \in H_{1+\alpha}$, *and suppose* a^{ij} *and* a *are functions in* $H_\alpha^*(K)$ *for any compact subset* K *of* Γ. *Suppose* $g \in H_\alpha(K')$ *for any compact subset* K' *of* Γ''. *Suppose also that* g *is (globally) Lipschitz with respect to* p *and that there is an increasing function* μ_1 *such that*

$$|g(X,z,p) - g(Y,w,p)| \le \mu_1(|z|+|w|)(1+|p|)[(1+|p|)^\alpha |X-Y|^\alpha + |z-w|^\alpha] \tag{37}$$

for any (X,z,p) *and* (Y,w,p) *in* Γ''. *Suppose finally that there are functions* $a_\tau^{ij}(X,z,p)$, $a_\tau(X,z,p)$, *and* $g_\tau(X,z,p)$ *for* $0 \le \tau \le 1$ *such that*

- (i) $a_1^{ij}(X,z,p) = a^{ij}(X,z,p)$, $a_1(X,z,p) = a(X,z,p)$, $g_1(X,z,p) = g(X,z,p)$;
- (ii) *The maps* T_1 *and* T_2 *defined on* $[0,1]$ *by* $T_1(\tau) = a_\tau^{ij}$, $T_2(\tau) = a_\tau$ *are continuous into* $H_\alpha^*(K)$ *for any compact subset* K *of* Γ *and the map* T_3 *defined on* $[0,1]$ *by* $T_3(\tau) = g_\tau$ *is continuous into* $H^\alpha(K')$ *for any compact subset* K' *of* Γ'';
- (iii) $(a_\tau^{ij}(X,z,p))$ *is positive definite for all* $\tau \in [0,1]$;
- (iv) $a_0(X,z,p) = -z$ *and* $g_0(X,z,p') = 0$;
- (v) g_τ *satisfies condition (37).*

If there are constants C *and* θ *with* $\theta \in (0,\alpha]$ *such that any periodic solution of the problem*

$$-u_t + a_\tau^{ij}(X,u,Du) D_{ij} u + a_\tau(X,u,Du) = 0 \quad \text{in } \Omega, \tag{38}$$

$$Du \cdot \gamma = g_\tau(X,u,D'u) \quad \text{on } S\Omega, \tag{39}$$

with $\tau \in [0,1]$, *satisfies the estimate* $|u|_0 + |Du|_\theta \le C$, *then there is a solution of* (3).

PROOF. Let B be the Banach space of all functions u with $|u|_0 + |Du|_{1+\theta/2}$ finite, and define the map $J : B \times [0,1] \to B$ by letting $u = J(v,\tau)$ be the unique solution of

$$-u_t + a_\tau^{ij}(X,v,Dv)D_{ij}u + a_\tau(X,v,Dv) + v - u = 0 \text{ in } \Omega,$$
$$Du \cdot \gamma = g_\tau(X,v,D'u) \text{ on } S\Omega$$

given by Proposition 4.3. It is easy to check that J is a compact mapping and that $J(v,0) = 0$ for all $v \in B$. It follows from [3, Theorem 11.6] that there is an element $u \in B$ such that $u = J(u,1)$, and this u is our desired solution. ∎

In general, the choice for the homotopy in Theorem 4.4 will be made to take into account the specific structure of the operators in question. For the uniformly parabolic problems studied here, we make a simple choice. First we take $a_\tau^{ij} = a^{ij}$. Next, we define a_τ by

$$a_\tau(X,z,p) = \tau a(X,z,p) + (1-\tau)z.$$

Finally, we assume that b satisfies (20) for an increasing function h, and then we define $g_\tau = \tau g$. For notational convenience, we also define b_τ in terms of g as

$$b_\tau(X,z,p) = p \cdot \gamma - \tau g(X,z,p').$$

We can now state our basic existence results.

THEOREM 4.5. *Let $S\Omega \in H_3$, and suppose that either conditions* (4) *and* (5) *or conditions* (6) *and* (7) *are satisfied. Suppose also that conditions* (11)–(17) *are satisfied and that*

$$\Lambda(X,z,p) = o(|z|) \tag{40}$$

as $|z| \to \infty$ uniformly for $X \in \Omega$ and p in any bounded subset of $\mathbf{R}^n$. Then there is a solution u of (3)*. Moreover, $u \in H_{2+\alpha}$ for any $\alpha \in (0,1)$.*

PROOF. If b satisfies (6), then $\operatorname{sgn} zg(X,z,0) < \mu_3$ for $|z| \ge M_2$, so (6) holds for b_τ. In addition, if a^{ij} and a satisfy (7) whenever $|z| \ge M_2$ and $|p| \le \mu_3$ and if (40) holds only for $|p| \le \mu_3$, then a_τ^{ij} and a_τ satisfy (7) with M_2 replaced by a sufficiently large constant (specifically, by M_3 such that $M_3 \ge M_2 + 2\rho_1$ and

$$\Lambda(X,z,p) \le |z|/(2\rho_2\mu_3)$$

if $|z| \ge M_3$.). On the other hand, if conditions (4), (5), and (40) are satisfied, we consider separately the cases $\tau \le 1/2$ and $\tau > 1/2$. In the first case, we see that (6) holds (with b_τ in place of b) for $M_2 \ge M_1(0)$ and $\mu_3 = 0$ and that (7) holds (with a_τ in place of a) for $M_2 \ge 2\rho_1$. In the second case we see that (4) holds for all $(X,z,p) \in \Gamma$ and that (5) holds if $|z| \ge M_1(2|p|)$. In either case, we note that if a^{ij}, a and b satisfy (11)–(17) and (20), then so do a_τ^{ij}, a_τ, and b_τ. We then obtain the uniform estimate as required in Theorem 4.4. ∎

THEOREM 4.6. *Let $S\Omega \in H_2$, and suppose that either conditions* (4) *and* (5) *or conditions* (6) *and* (7) *are satisfied. Suppose also that conditions* (11), (18), (19), *and* (40) *are satisfied. If $a \in H_\alpha(K)$ for any bounded subset K of Γ, then there is a solution u of* (3)*. Moreover, $u \in H_{2+\alpha}^{(-1-\beta)}$ for any $\beta \in (0,1)$.*

5. Semilinear equations with nonlinear boundary conditions. Now we consider problem (3) when a^{ij} does not depend on p. We suppose that $S\Omega \in H_{1+\alpha}$ for some $\alpha \in (0,1)$ and that there are positive constants M_0 and μ such that a^{ij}, a and b satisfy the conditions

$$[\rho_1 + \rho_2 \Lambda(X,z)]\mu + (\operatorname{sgn} z) a(X,z,p) < 0$$

for all $(X,z,p) \in \Gamma$ with $|z| \geq M_0$ and $|p| \leq \mu$ (with ρ_1 and ρ_2 as in Lemma 2.2),

$$b(X,z,-(\operatorname{sgn} z)\mu\gamma)(\operatorname{sgn} z) < 0$$

for all $(X,z) \in S\Omega$ with $|z| \geq M_0$ and (12) holds. Suppose also that there is an increasing function θ_1 such that

$$\Lambda(X,z) \leq \theta_1(|z|),\ \lambda(X,z) \geq 1/\theta_1(|z|)$$

for all $(X,z) \in \Omega \times \mathbf{R}$,

$$|a(X,z,p)| \leq \theta_1(|z|)[|p|^2 + d^{\alpha-1}]$$

for all $(X,z,p) \in \Gamma$, and

$$|b_p(X,z,p)|[1+|p|^3] + |b_z(X,z,p)|[1+|p|^2] + |b_x(X,z,p)|[1+|p|]$$
$$+|b_t(X,z,p)| \leq \theta_1(|z|) b_p(X,z,p) \cdot \gamma[1+|p|^3]$$

for all $(X,z,p) \in \Gamma'$ with $b(X,z,p) = 0$. Finally suppose that for any $K > 0$, there is a continuous, increasing function ζ_K with $\zeta_K(0) = 0$ such that

$$|a^{ij}((x,t),z) - a^{ij}((y,t),w)| \leq \zeta_K(|x-y| + |z-w|)$$

for all (x,t) and (y,t) in Ω and all z and w in $[-K,K]$. We note first that the hypotheses of Lemma 2.1 are satisfied so we obtain a pointwise bound for u. Next, from condition (12), we infer that the boundary condition can be written in the form $Du \cdot \gamma = g(X,u,D'u)$ and that g satisfies the condition

$$|g(X,z,p')| \leq \beta_0(1+|p'|),$$

so [11, Corollary 7.36] gives a Hölder estimate for u. Next, it is easy to check that

$$|g_z|(1+|p|)^2 + |g_x|(1+|p|) + |g_t| \leq \theta_1(|z|)(1+|p|)^3$$

and hence that (37) holds. An argument like that in Lemma 4.2 then gives the Hölder gradient bound without the use of the gradient bound from Section 3. In addition, these estimates hold uniformly for solutions of (38), (39). If also $a^{ij} \in H^*_\beta(K_0)$ for any bounded subset K_0 of $\Omega \times \mathbf{R}$, and $a \in H^*_\beta(K)$ for any bounded subset K of Γ, then Theorem 4.4 gives the existence of a solution to (3). To remove these conditions, we approximate a^{ij} and a by smooth functions satisfying the given conditions and note that the solutions of the approximating problems are uniformly bounded in $H_{1+\theta}$ for some $\theta \in (0,1)$ and also in $W^{2,1}_p(\Omega')$ for any $p \in (1,\infty)$ and any domain Ω' whose closure is in Ω. By taking a convergent subsequence, it follows that the (periodic) limit function satisfies the differential equation almost everywhere in Ω and the boundary condition everywhere on $S\Omega$. In other words, the limit function solves (3).

A similar result was proved by different methods by Nkashama [15], who assumes in addition that Ω is cylindrical and that b is linear with respect to p. Further, Nkashama made stronger regularity hypotheses on $S\Omega$, a^{ij}, a, and b. To complete our comparison of results to that paper, we note that the use of subsolutions and supersolutions there in place of our results in Section 2 is easily modified to the current case as in [5] (see also [13, Section 7]). Moreover, when b is linear with respect to p, the regularity of b can be further relaxed. Specifically, suppose $b(X,z,p) = \beta(X,z)\cdot p + \beta^0(X,z)$ for functions β and β^0 in $H_\alpha(K'')$ for any bounded subset K'' of $S\Omega \times \mathbf{R}$ with $\beta(X,z)\cdot(X) > 0$. From Lemma 4.1, we infer the Hölder gradient estimate with some exponent $\theta \in (0,1)$. Next, we note that, when g in [11, Lemma 13.17] is a linear function, then the exponent θ in that lemma can be chosen arbitrarily in the range $(0,1)$. It follows that θ in [11, Lemma 13.18] can be taken equal to α, so our solution is in $H_{1+\alpha}$.

Because our results improve those of Nkashama, they also improve those of Amann [1]. Note also that our method relies on a simple linear existence theory [12] which only uses the Poincaré map in the space of continuous functions $C(\overline{\Omega})$ unlike Amann's, which uses the Poincaré map on a more complicated Banach space. In addition (like Nkashama but unlike Amann), we may consider time-dependent boundary conditions.

6. One space dimension. When problem (3) is presented with only one space dimension, the gradient estimates can be streamlined considerably; the ideas are very similar to those in [6] so we only mention the results. First we note that the boundary condition can be solved in the form $u_x = g(X,u)$. As discussed on page 351 of [11], a gradient bound follows from a pointwise bound under the conditions

$$|a(X,z,p)| \le \theta_1(|z|)a^{11}(X,z,p)[1+|p|^2],\ |g(X,z)| \le \theta_1(|z|)$$

for some increasing function θ_1. If we also assume that g is Hölder with respect to X and z, then, since $w = u_x$ is a weak solution of the equation

$$-w_t + (a^{11}(X,u,Du)w_x + a(X,u,u_x))_x = 0,$$

we obtain a Hölder gradient estimate. This estimate depends on

$$\inf_K a^{11}(X,z,p),\ \sup_K a^{11}(X,z,p),\ \sup_K |a(X,z,p)|,$$

where $K = \{(X,z,p) \in \Gamma : |z| \le |u|_0,\ |p| \le |Du|_0\}$, and on

$$[g]_{\alpha;\Sigma\times[-|u|_0,|u|_0]}.$$

A pointwise bound follows from, say, conditions (4), (5); in this context, they can be rewritten as $\operatorname{sgn} z g(X,z) \le \mu$ for $|z| \ge M$ and

$$\rho_1 + \rho_2 a^{11}(X,z,p)\mu + (\operatorname{sgn} z)a(X,z,p) < 0$$

for $|z| \ge M$ and $|p| \le \mu$.

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