EVOLUTION EQUATIONS:
EXISTENCE, REGULARITY AND SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 52
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

BEHAVIOUR OF SOLUTIONS TO $u_t - \Delta u + |\nabla u|^p = 0$ AS $p \to +\infty$

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1. Introduction. We study the behaviour as $p \to +\infty$ of the non-negative and integrable solution u_p to the Cauchy problem

$$u_{p,t} - \Delta u_p + |\nabla u_p|^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1}$$

$$u_p(0) = u_0 \quad \text{in} \quad \mathbb{R}^N, \tag{2}$$

where u_0 is a non-negative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. To explain in a formal way our result let us assume for the moment that the sequence $(u_p)_{p>1}$ converges to some u_∞ as $p \to +\infty$ and try to obtain some information about u_∞. We first observe that the non-negativity and integrability of u_p entail that the sequence $(|\nabla u_p|^p)$ is bounded in $L^1((0,+\infty) \times \mathbb{R}^N)$. The Fatou lemma then implies that

$$|\nabla u_\infty| \leq 1 \quad \text{a.e. in} \quad (0,+\infty) \times \mathbb{R}^N. \tag{3}$$

We will actually prove a stronger assertion, namely that $(|\nabla u_p|^p)$ converges to zero in $L^1_{loc}(0,+\infty; L^1(\mathbb{R}^N))$, from which we deduce that u_∞ is a solution to the linear heat equation. It remains to identify $u_\infty(0)$. Recalling (3) we expect that $|\nabla u_\infty(0)| \leq 1$ a.e. in $\mathbb{R}^N$. Consequently $u_\infty(0)$ does not coincide with $u_0 = \lim_{p\to+\infty} u_p(0)$ if $\|\nabla u_0\|_{L^\infty} > 1$. Also, u_p being a non-negative subsolution to the heat equation we expect that $0 \leq u_\infty(0) \leq u_0$. Summarizing, a possible limit u_∞ of $(u_p)_{p>1}$ as $p \to +\infty$ would be a solution to the linear heat equation with an initial datum $u_\infty(0)$ satisfying

$$0 \leq u_\infty(0) \leq u_0 \quad \text{and} \quad |\nabla u_\infty(0)| \leq 1 \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{4}$$

We will in fact prove that $u_\infty(0)$ is a suitably defined projection of u_0 on the convex subset of $L^1(\mathbb{R}^N)$ defined by the constraints (4).

Before stating precisely our result we first recall the well-posedness of the Cauchy problem (1)-(2) in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$.

2000 *Mathematics Subject Classification*: 35K55, 35B30.
The paper is in final form and no version of it will be published elsewhere.

Proposition 1. *Let $p \in (1,+\infty)$ and u_0 be a non-negative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. There is a unique non-negative function*

$$u \in \mathcal{C}([0,+\infty); L^1(\mathbb{R}^N)) \cap L^\infty(0,+\infty; W^{1,\infty}(\mathbb{R}^N))$$

satisfying

$$u(t) = G(t)u_0 - \int_0^t G(t-s)|\nabla u(s)|^p \; ds, \quad t \in [0,+\infty). \tag{5}$$

Here $G(t)$ denotes the linear heat semigroup. In addition, for each $t \in [0,+\infty)$ we have

$$0 \le u(t) \le G(t)u_0, \tag{6}$$

$$\|u(t)\|_{L^1} \le \|u_0\|_{L^1}, \tag{7}$$

$$\|\nabla u(t)\|_{L^\infty} \le \|\nabla u_0\|_{L^\infty}. \tag{8}$$

We next consider a non-negative function u_0 in $L^1(\mathbb{R}^N)$ and define a convex subset $\mathcal{C}(u_0)$ of $L^1(\mathbb{R}^N)$ by

$$\mathcal{C}(u_0) = \left\{ w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), 0 \le w \le u_0 \;\text{ and }\; |\nabla w| \le 1 \;\text{ a.e. in }\; \mathbb{R}^N \right\}. \tag{9}$$

A crucial step in our analysis is the existence of a projection from $L^1(\mathbb{R}^N)$ on $\mathcal{C}(u_0)$ with suitable properties and is given by [3, Proposition 5.3].

Proposition 2. *Let u_0 be a non-negative function in $L^1(\mathbb{R}^N)$. For each $v \in L^1(\mathbb{R}^N)$ there is a unique $\mathbb{P}v \in \mathcal{C}(u_0)$ such that*

$$\int j\left(w - \mathbb{P}v\right) \; dx \le \int j\left(w - \mathbb{P}v + \lambda \left(\mathbb{P}v - v\right)\right) \; dx \tag{10}$$

for every $w \in \mathcal{C}(u_0)$, $\lambda \in [0,+\infty)$ and $j \in \mathcal{J}_0$, where

$$\mathcal{J}_0 = \{\text{convex lower semicontinuous maps } \; j : \mathbb{R} \to [0,+\infty] \;\text{ satisfying }\; j(0) = 0\}.$$

The mapping $\mathbb{P} : L^1(\mathbb{R}^N) \to \mathcal{C}(u_0)$ satisfies $\mathbb{P}v = v$ if $v \in \mathcal{C}(u_0)$ and

$$\int j\left(\mathbb{P}v - \mathbb{P}\hat{v}\right) \; dx \le \int j\left(v - \hat{v}\right) \; dx \tag{11}$$

for every $v \in L^1(\mathbb{R}^N)$, $\hat{v} \in L^1(\mathbb{R}^N)$ and $j \in \mathcal{J}_0$.

We may now state our main result.

Theorem 3. *Let u_0 be a non-negative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ and denote by $\mathbb{P}u_0$ the projection of u_0 on the convex set $\mathcal{C}(u_0)$ defined by (9). For $p \in (1,+\infty)$ we denote by u_p the unique solution to (1)-(2) given by Proposition 1. For every $t_1 \in (0,+\infty)$ and $t_2 \in (t_1,+\infty)$ we have*

$$\lim_{p \to +\infty} \sup_{t \in [t_1,t_2]} \left\|u_p(t) - G(t)\mathbb{P}u_0\right\|_{L^1} = 0. \tag{12}$$

Let us mention at this point that related results have been obtained for the solutions to the equation

$$u_t - \Delta u + u^p = 0$$

and its generalisations in [4] and [5].

An interesting consequence of Theorem 3 is that it allows to identify the behaviour of

$$\lim_{t\to+\infty} \|u_p(t)\|_{L^1}$$

as $p \to +\infty$. More precisely we have the following result.

COROLLARY 4. *With the assumptions and notations of Theorem 3 we define*

$$I_p(u_0) = \lim_{t\to+\infty} \|u_p(t)\|_{L^1}, \tag{13}$$

(which exists as $t \mapsto \|u_p(t)\|_{L^1}$ is non-increasing and bounded from below). We have

$$\lim_{p\to+\infty} I_p(u_0) = \|\mathbb{P}u_0\|_{L^1}. \tag{14}$$

We now briefly describe the remainder of the paper: in the next section we sketch the proof of Proposition 1 and derive some bounds on the gradients of solutions to (1)-(2). In Section 3 we check that the convex set $\mathcal{C}(u_0)$ defined by (9) enjoys the desired properties which allow us to apply [3, Proposition 5.3]. The proof of Theorem 3 is done in Section 4 while the last section is devoted to the proof of Corollary 4.

2. Preliminaries. We first briefly recall some arguments towards the proof of Proposition 1. If u_0 is a non-negative function in $\mathcal{D}(\mathbb{R}^N)$ there exists a unique non-negative classical solution to (1)-(2) (see, e.g., [7] or [1]) and (6), (8) follow from the comparison principle while (7) follows from (1) and the non-negativity of u. For a non-negative function u_0 in $L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$ the existence part of Proposition 1 and (6)-(8) are obtained by approximation and weak compactness arguments while the uniqueness is a consequence of the Gronwall lemma and the Duhamel formula.

We now fix a non-negative function u_0 in $L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$ and for $p\in(1,+\infty)$ we denote by u_p the unique non-negative solution to (1)-(2) given by Proposition 1. The next result is a consequence of [2, Theorem 1].

LEMMA 5. *For $p\in(1,+\infty)$ we have*

$$|\nabla u_p(t,x)|^p \le \frac{u_p(t,x)}{(p-1)t}, \quad (t,x)\in(0,+\infty)\times\mathbb{R}^N. \tag{15}$$

PROOF. It follows from [2, Theorem 1] that

$$\|\nabla u_p^{(p-1)/p}(t)\|_{L^\infty} \le (p-1)^{(p-1)/p}\ p^{-1}\ t^{-1/p}, \quad t\in(0,+\infty).$$

As

$$\nabla u_p(t,x) = \frac{p}{p-1}\ u_p(t,x)^{1/p}\ \nabla u_p^{(p-1)/p}(t,x),$$

we easily deduce (15). ∎

We next study the behaviour of u_p for large values of $x\in\mathbb{R}^N$.

LEMMA 6. *For each $t\in[0,+\infty)$ we have*

$$\lim_{R\to+\infty} \sup_{p\in(1,+\infty)} \sup_{s\in[0,t]} \int_{\{|x|\ge R\}} u_p(s,x)\ dx = 0, \tag{16}$$

$$\lim_{R\to+\infty} \sup_{p\in(1,+\infty)} \int_0^t \int_{\{|x|\ge R\}} |\nabla u_p(s,x)|^p\ dxds = 0. \tag{17}$$

Proof. Let ϑ be a $\mathcal{C}^\infty$-smooth function such that $0 \le \vartheta \le 1$ and
$$\vartheta(x) = 0 \ \text{ if } \ |x| \le 1/2 \ \text{ and } \ \vartheta(x) = 1 \ \text{ if } \ |x| \ge 1.$$
For $R \ge 1$ we put $\vartheta_R(x) = \vartheta(x/R)$, $x \in \mathbb{R}^N$. It follows from (1) and (7) that
$$\begin{aligned} &\int u_p(t,x)\ \vartheta_R(x)\ dx + \int_0^t \int |\nabla u_p(s,x)|^p\ \vartheta_R(x)\ dxds \\ &\le \int u_0(x)\ \vartheta_R(x)\ dx + \frac{\|\vartheta\|_{W^{2,\infty}}}{R^2} \int_0^t \int u_p(s,x)\ dxds \\ &\le \int u_0(x)\ \vartheta_R(x)\ dx + \frac{t\ \|\vartheta\|_{W^{2,\infty}}\ \|u_0\|_{L^1}}{R^2}. \end{aligned}$$
The function u_0 being integrable, (16) and (17) follow from the above inequality by letting $R \to +\infty$. ∎

3. The convex set $\mathcal{C}(u_0)$. Consider a non-negative function u_0 in $L^1(\mathbb{R}^N)$. The set $\mathcal{C}(u_0)$ defined by
$$\mathcal{C}(u_0) = \{w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), 0 \le w \le u_0 \ \text{ and } \ |\nabla w| \le 1 \ \text{ a.e. in } \ \mathbb{R}^N\} \qquad (18)$$
is a closed and convex subset of $L^1(\mathbb{R}^N)$ which is non-empty (as $0 \in \mathcal{C}(u_0)$). The following additional property is enjoyed by the set $\mathcal{C}(u_0)$.

Lemma 7. *Consider $w \in \mathcal{C}(u_0)$, $\hat{w} \in \mathcal{C}(u_0)$ and a function $\xi \in W^{1,\infty}(\mathbb{R})$ satisfying*
$$\xi(0) = 0 \ \text{ and } \ 0 \le \xi' \le 1 \ \text{ a.e. in } \ \mathbb{R}. \qquad (19)$$
Then
$$w + \xi\,(\hat{w} - w) \in \mathcal{C}(u_0). \qquad (20)$$

Proof. We first notice that (19) ensures that
$$\min\,(w,\hat{w}) \le w + \xi\,(\hat{w} - w) \le \max\,(w,\hat{w}),$$
from which we deduce that
$$0 \le w + \xi\,(\hat{w} - w) \le u_0 \ \text{ a.e.}$$

We next have
$$\nabla\,(w + \xi\,(\hat{w} - w)) = \xi'\,(\hat{w} - w)\ \nabla\hat{w} + (1 - \xi'\,(\hat{w} - w))\ \nabla w.$$
We then infer from (19) and the convexity of the euclidean norm in $\mathbb{R}^N$ that
$$|\nabla\,(w + \xi\,(\hat{w} - w))| \le 1.$$
We have thus proved that $w + \xi\,(\hat{w} - w)$ belongs to $\mathcal{C}(u_0)$. ∎

Thanks to Lemma 7, Proposition 2 is a straightforward consequence of [3, Proposition 5.3].

4. Convergence. In this section we prove Theorem 3. Let u_0 be a non-negative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ and, for $p \in (1,+\infty)$, denote by u_p the unique non-negative solution to (1)-(2) given by Proposition 1. We next define the set $\mathcal{C}(u_0)$ by (9) and denote by $\mathbb{P}$ the projection on $\mathcal{C}(u_0)$ given by Proposition 2. We first notice the following consequence of (8) and (15).

LEMMA 8. *For every $t_1 \in (0,+\infty)$ and $t_2 \in (t_1,+\infty)$ we have*

$$\lim_{p\to+\infty} \int_{t_1}^{t_2} \int |\nabla u_p(s,x)|^p \, dxds = 0. \tag{21}$$

Assume further that

$$\|\nabla u_0\|_{L^\infty} < 1. \tag{22}$$

Then for every $T \in (0,+\infty)$ we have

$$\lim_{p\to+\infty} \int_0^T \int |\nabla u_p(s,x)|^p \, dxds = 0. \tag{23}$$

PROOF. By (15) and (7) we have

$$\begin{aligned}\int_{t_1}^{t_2} \int |\nabla u_p(s,x)|^p \, dxds &\le \frac{1}{p-1} \int_{t_1}^{t_2} s^{-1} \int u_p(s,x) \, dxds \\ &\le \frac{\|u_0\|_{L^1}}{(p-1)} \ln\left(\frac{t_2}{t_1}\right),\end{aligned}$$

hence (21).

Assuming now that u_0 fulfils (22) we deduce from (8) and (22) that

$$\lim_{p\to+\infty} |\nabla u_p(t,x)|^p = 0 \quad \text{for a.e.} \quad (t,x) \in (0,+\infty)\times \mathbb{R}^N. \tag{24}$$

Owing to (8), (17), (22) and (24) we may apply the Vitali convergence theorem (see, e.g., [6, p. 13]) and obtain (23). ■

The next result is a straightforward consequence of Lemma 8 and states that Theorem 3 is valid when u_0 fulfils (22). Note that in that case $\mathbb{P}u_0 = u_0$.

PROPOSITION 9. *If u_0 satisfies the additional assumption (22) we have for each $T \in (0,+\infty)$*

$$\lim_{p\to+\infty} \sup_{t\in[0,T]} \|u_p(t) - G(t)u_0\|_{L^1} = 0.$$

PROOF. By the Duhamel formula (5) we have for $t \in [0,T]$

$$\begin{aligned}\|u_p(t) - G(t)u_0\|_{L^1} &\le \int_0^t \|G(t-s)|\nabla u_p(s)|^p\|_{L^1} \, ds, \\ &\le \int_0^T \int |\nabla u_p(s,x)|^p \, dxds,\end{aligned}$$

and Proposition 9 follows at once from (23) and the above inequality. ■

We now turn to the general case. Consider $t_1 \in (0,+\infty)$ and $t_2 \in (t_1,+\infty)$. By (6)-(8) and (15) the sequence $(u_p)_{p>1}$ is bounded in $L^\infty(t_1,t_2;L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N))$. We may then proceed as in [2, Section 3] to prove that the sequence $(u_p)_{p>1}$ is relatively compact in $\mathcal{C}(\mathcal{K})$ for every compact subset $\mathcal{K}$ of $(0,+\infty)\times\mathbb{R}^N$. Therefore there is a sequence $(p_j)_{j\ge1}$, $p_j \to +\infty$ and a function $u_\infty \in \mathcal{C}((0,+\infty)\times\mathbb{R}^N)$ such that

$$\lim_{j\to+\infty} \|u_{p_j} - u_\infty\|_{\mathcal{C}(\mathcal{K})} = 0 \tag{25}$$

for every compact subset $\mathcal{K}$ of $(0,+\infty)\times\mathbb{R}^N$. Clearly u_∞ is a non-negative function as the limit of non-negative functions. Observe next that, owing to (25) and (16) we may apply the Vitali convergence theorem and obtain

$$\lim_{j\to+\infty} \|u_{p_j}(t) - u_\infty(t)\|_{L^1} = 0, \quad t \in (0,+\infty). \tag{26}$$

Recalling (21) we may then let $p_j \to +\infty$ in the Duhamel formula (5) and obtain

$$u_\infty(t_2) = G(t_2 - t_1)u_\infty(t_1), \quad 0 < t_1 < t_2. \tag{27}$$

Thanks to (5), (21), (26) and (27) we may improve (26) to

$$\lim_{j\to+\infty} \sup_{t\in[t_1,t_2]} \|u_{p_j}(t) - u_\infty(t)\|_{L^1} = 0, \quad 0 < t_1 < t_2. \tag{28}$$

We next derive some further properties of u_∞. First notice that (6) and (28) entail

$$0 \le u_\infty(t) \le G(t)u_0, \quad t \in (0,+\infty). \tag{29}$$

In addition it follows from (6) and (15) that, for $t \in (0,+\infty)$

$$\limsup_{p\to+\infty} \|\nabla u_p(t)\|_{L^\infty} \le 1,$$

and (26) and a weak compactness argument yield

$$\|\nabla u_\infty(t)\|_{L^\infty} \le 1, \quad t \in (0,+\infty). \tag{30}$$

In particular we infer from (29) and (30) that $(u_\infty(t))_{t\in(0,1)}$ is bounded in $L^1(\mathbb{R}^N)$ and in $W^{1,\infty}(\mathbb{R}^N)$. This fact, (27) and (29) allow us to conclude that there is a non-negative function

$$\bar{u}_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$$

such that

$$u_\infty(t) = G(t)\bar{u}_0, \quad t \in (0,+\infty). \tag{31}$$

Moreover (29) and (30) yield

$$0 \le \bar{u}_0 \le u_0 \ \text{ and } \ |\nabla\bar{u}_0| \le 1 \ \text{ a.e. in } \ \mathbb{R}^N.$$

In other words

$$\bar{u}_0 \in \mathcal{C}(u_0). \tag{32}$$

We now proceed to show that in fact $\bar{u}_0 = \mathbb{P}u_0$. For that purpose we first notice that (32) ensures that $\mathbb{P}\bar{u}_0 = \bar{u}_0$ while (11) with $j(r) = r^+ = \max(r,0)$ yields

$$\int (\mathbb{P}\bar{u}_0 - \mathbb{P}u_0)^+ \ dx \le \int (\bar{u}_0 - u_0)^+ \ dx = 0,$$

as $\bar{u}_0 \le u_0$ by (32). We thus conclude that

$$\bar{u}_0 = \mathbb{P}\bar{u}_0 \le \mathbb{P}u_0 \ \text{ a.e. in } \ \mathbb{R}^N. \tag{33}$$

We next consider $\varepsilon \in (0,1)$ and put

$$u_0^\varepsilon = \frac{\mathbb{P}u_0}{1+\varepsilon} = \frac{1}{1+\varepsilon}\ .\ \mathbb{P}u_0 + \left(1 - \frac{1}{1+\varepsilon}\right)\ .\ 0 \in \mathcal{C}(u_0).$$

Note that, since $\mathbb{P}u_0 \in \mathcal{C}(u_0)$,

$$\|\nabla u_0^\varepsilon\|_{L^\infty} \le \frac{1}{1+\varepsilon} < 1. \tag{34}$$

For $p \in (1, +\infty)$ we denote by u_p^ε the solution to (1) with initial datum u_0^ε. On the one hand we infer from (34) and Proposition 9 that

$$\lim_{p \to +\infty} \left\| u_p^\varepsilon(t) - G(t) u_0^\varepsilon \right\|_{L^1} = 0, \quad t \in [0, +\infty). \tag{35}$$

On the other hand we have $u_0^\varepsilon \le u_0$ as $u_0^\varepsilon \in \mathcal{C}(u_0)$ and the comparison principle entails

$$u_{p_j}^\varepsilon(t,x) \le u_{p_j}(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{36}$$

Combining (28), (35) and (36) yields

$$G(t) u_0^\varepsilon \le u_\infty(t), \quad t \in (0, +\infty).$$

Letting $t \to 0$ in the above inequality we deduce from (31) that

$$u_0^\varepsilon \le \bar{u}_0 \ \text{ a.e. in } \ \mathbb{R}^N.$$

This inequality being valid for each $\varepsilon \in (0,1)$ we finally obtain

$$\mathbb{P} u_0 \le \bar{u}_0.$$

Recalling (33) we have proved that $\bar{u}_0 = \mathbb{P} u_0$. Thus (31) reads

$$u_\infty(t) = G(t) \mathbb{P} u_0, \quad t \in [0, +\infty),$$

while (28) becomes

$$\lim_{j \to +\infty} \sup_{t \in [t_1, t_2]} \| u_{p_j}(t) - G(t) \mathbb{P} u_0 \|_{L^1} = 0, \quad 0 < t_1 < t_2. \tag{37}$$

Finally, owing to the uniqueness of the cluster points of the sequence $(u_p)_{p>1}$ a standard argument ensures that the convergence (37) actually holds for the whole sequence $(u_p)_{p>1}$, which completes the proof of Theorem 3.

Remark 1. Since the sequence $(u_p)_{p>1}$ is bounded in $L^\infty(0, +\infty; W^{1,\infty}(\mathbb{R}^N))$ it is easy to deduce from Theorem 3 that the convergence (12) also holds in $L^q(\mathbb{R}^N)$ for $q \in (1, \infty]$.

5. Behaviour of $I_p(u_0)$ as $p \to +\infty$. We now prove Corollary 4. Recall that the assumptions and the notations used are those of Theorem 3 and Corollary 4. For technical reasons we further assume that $p > (N+2)/(N+1)$. Integrating (1) on $(t, +\infty) \times \mathbb{R}^N$ yields

$$I_p(u_0) + \int_t^\infty \int |\nabla u_p(s,x)|^p \, dx ds = \|u_p(t)\|_{L^1}, \quad t \in [0, +\infty). \tag{38}$$

We define

$$\tau_p = \frac{1}{\pi} \ \left(\pi \ \|u_0\|_{L^1}^2 \ (p-1) \right)^{p/(p(N+1)-(N+2))},$$

and introduce

$$J_{1,p}(t) = \int_t^{\tau_p} \int |\nabla u_p(s,x)|^p \, dx ds, \quad t \in (0, \tau_p),$$

$$J_{2,p} = \int_{\tau_p}^\infty \int |\nabla u_p(s,x)|^p \, dx ds.$$

With these notations (38) reads

$$I_p(u_0) = \|u_p(t)\|_{L^1} - J_{1,p}(t) - J_{2,p}, \quad t \in (0, \tau_p). \tag{39}$$

We first estimate $J_{2,p}$. For that purpose we use an upper bound on $\|\nabla u_p^{(p-1)/p}\|_{L^\infty}$ obtained in [2, Theorem 1], namely

$$\|\nabla u_p^{(p-1)/p}(s)\|_{L^\infty} \le (p-1)^{1/2}\ \pi^{1/2}\ \|u_0\|_{L^1}^{(p-1)/p}\ (\pi s)^{-(p(N+1)-N)/2p}, \quad s \in (0,+\infty).$$

Therefore for $(s,x) \in (0,+\infty) \times \mathbb{R}^N$ we have

$$|\nabla u_p(s,x)| \le p\ (p-1)^{-1/2}\ \pi^{1/2}\ \|u_0\|_{L^1}^{(p-1)/p}\ (\pi s)^{-(p(N+1)-N)/2p}\ u_p(s,x)^{1/p}.$$

Plugging this estimate in $J_{2,p}$ and using (7) yield

$$\begin{aligned} J_{2,p} &\le \left(\frac{p}{p-1}\right)^p\ (\pi(p-1))^{p/2}\ \|u_0\|_{L^1}^{p-1} \int_{\tau_p}^{\infty} (\pi s)^{-(p(N+1)-N)/2} \int u_p(s,x)\ dx\ ds \\ &\le \left(\frac{p}{p-1}\right)^p\ \left(\pi(p-1)\|u_0\|_{L^1}^2\right)^{p/2}\ \frac{2\ (\pi\tau_p)^{-(p(N+1)-(N+2))/2}}{\pi\ (p(N+1)-(N+2))} \\ J_{2,p} &\le \left(\frac{p}{p-1}\right)^p\ \frac{2}{\pi\ (p(N+1)-(N+2))} \end{aligned}$$

(recall that $p > (N+2)/(N+1)$). Consequently

$$\lim_{p\to+\infty} J_{2,p} = 0. \tag{40}$$

Next, by (7) and (15) we have

$$\begin{aligned} J_{1,p}(t) &\le \frac{\|u_0\|_{L^1}}{p-1} \int_t^{\tau_p} s^{-1}\ ds \\ &\le \frac{\|u_0\|_{L^1}}{p-1} \left(\ln\left(\frac{1}{\pi t}\right) + \frac{p}{p(N+1)-(N+2)}\ \ln\left(\pi\ \|u_0\|_{L^1}^2\ (p-1)\right)\right), \end{aligned}$$

hence

$$\lim_{p\to+\infty} J_{1,p}(t) = 0, \quad t \in (0,\tau_p). \tag{41}$$

As $\tau_p \to +\infty$ we may choose $t > 0$ such that $t \in (0,\tau_p)$ for p large enough and we may let $p \to +\infty$ in (39) and use (40), (41) and Theorem 3 to obtain

$$\lim_{p\to+\infty} I_p(u_0) = \|G(t)\mathbb{P}u_0\|_{L^1} = \|\mathbb{P}u_0\|_{L^1}.$$

Acknowledgments. I thank S. Benachour and Ph. Bénilan for helpful discussions during the preparation of this work.

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