EVOLUTION EQUATIONS:
EXISTENCE, REGULARITY AND SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 52
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

NONLOCAL QUADRATIC EVOLUTION PROBLEMS

PIOTR BILER

Mathematical Institute, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: biler@math.uni.wroc.pl

WOJBOR A. WOYCZYŃSKI

Department of Statistics and Center for Stochastic and Chaotic Processes
in Science and Technology, Case Western Reserve University
Cleveland, Ohio 44106, U.S.A.
E-mail: waw@po.cwru.edu

Abstract. Nonlinear nonlocal parabolic equations modeling the evolution of density of mutually interacting particles are considered. The inertial type nonlinearity is quadratic and nonlocal while the diffusive term, also nonlocal, is anomalous and fractal, i.e., represented by a fractional power of the Laplacian. Conditions for global in time existence versus finite time blow-up are studied. Self-similar solutions are constructed for certain homogeneous initial data. Monte Carlo approximation schemes by interacting particle systems are also mentioned.

1. Introduction. This paper is an extended version of the lecture given by the first-named author during the conference "Evolution Equations" at Banach Center in Warsaw, in October 1998. The presentation is partly based on [BW] and [BFW2].

We consider global and exploding solutions for equations of the form

$$u_t = -(-\Delta)^{\alpha/2} u + \nabla \cdot (uB(u)). \tag{1.1}$$

Here $u : \Omega \times (0,T) \subset I\!R^d \times I\!R^+ \to I\!R$, $(-\Delta)^{\alpha/2}$ is a fractional power of the minus

2000 *Mathematics Subject Classification*: Primary 35K60; Secondary 35B40, 35Q, 60H, 82C21.

Key words and phrases: nonlinear nonlocal parabolic equations, fractal anomalous diffusion, asymptotic behavior of solutions, self-similar solutions.

Grant support from KBN (324/P03/97/12) and NSF (INT-97-04231) is gratefully acknowledged.

The paper is in final form and no version of it will be published elsewhere.

Laplacian in $\mathbb{R}^d$, $0 < \alpha \le 2$, and

$$B(u)(x) = \int_{\mathbb{R}^d} b(x,y)u(y)\,dy$$

is a linear $\mathbb{R}^d$-valued integral operator with the kernel $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. The dimension is restricted to the physically interesting cases $d = 1, 2$, or 3.

Equations (1.1) describe various physical phenomena involving diffusion and interaction of pairs of particles when suitable assumptions are made on the possibly singular integral operator B. Since our main interest is in u as a description of the density of particles in $\mathbb{R}^d$, we will consider nonnegative solutions to (1.1), except for the case of self-similar solutions in Section 4.

In the case of *classical Brownian diffusion*, i.e., $\alpha = 2$, a deterministic study of these models in [BN] was initially motivated by the Fokker–Planck type parabolic equations with nonlocal nonlinearity and we studied them mostly in bounded domains of $\mathbb{R}^d$, supplemented with suitable (nonlinear) boundary conditions. For instance, if

$$b(x,y) = c(x-y)|x-y|^{-d}, \tag{1.2}$$

then the equation (1.1) models the diffusion of charge carriers ($c < 0$) in electrolytes, semiconductors or plasmas interacting via Coulomb forces. If $c > 0$, it describes gravitational interaction of particles in a cloud, or galaxies in a nebula.

Related equations and parabolic systems appear in mathematical biology where they are used to model chemotaxis phenomena, see [B3]. There, we have been mainly interested in the possibility of the continuation of local in time solutions of (1.1) up to $T = +\infty$. The answer to this question depends strongly on the type of interaction. For instance, for Newtonian attraction of particles or chemotactic attraction of cells, finite time collapse of solutions is possible, see [B1], [B3], while for the Coulomb forces global in time existence of solutions is guaranteed, cf. [BHN].

Further, for the Biot–Savart kernel

$$b(x,y) = (2\pi)^{-1}(x_2 - y_2, y_1 - x_1)|x-y|^{-2} \tag{1.3}$$

in $\mathbb{R}^2$, the equation (1.1) with $\alpha = 2$ is equivalent to the vorticity formulation of the Navier–Stokes equations. Its solutions are global in time. Also, formally, the singular kernel $b(x,y) = c\delta(x-y)$ leads to the classical Burgers equation

$$u_t = u_{xx} + c(u^2)_x. \tag{1.4}$$

A new important ingredient of a more general class of model problems (1.1) in [BW], studied in the whole space $\mathbb{R}^d$, was the *anomalous Lévy α-stable diffusion* described by a fractional power of the Laplace operator in $\mathbb{R}^d$. In the physical literature such fractal diffusions have been vigorously studied in the context of statistical mechanics, hydrodynamics, acoustics, relaxation phenomena and biology, see e.g. [BPFS] and [SZF]. In probabilistic terms, replacing the Laplacian by its fractional power leads to interesting questions of extension of results for Brownian motion driven stochastic equations to those driven by Lévy α-stable flights; the latter, of course, having discontinuous sample paths.

In fact, the probabilistic theory of interacting particle systems and theory of McKean's diffusions have been our immediate theoretical inspiration for [BW]. McKean's processes and "propagation of chaos" results connect the detailed Liouvillean picture of the evolution of diffusing and interacting particles and the reduced hydrodynamic description. In that context equations like (1.1) appear as "laboratory" models with diffusion and two-particle interactions leading to bilinear nonlinearities. We cite only a few of references that deal with different aspects of this connection in the case of classical Brownian diffusion: [BT], [GK], [KO], [McK], [O], [Sz].

A direct numerical approach to equations like (1.1) is hopelessly difficult because of the nonlocal character of the higher order term $(-\Delta)^{\alpha/2}u$ and/or the nonlinearity $uB(u)$. However, if the "propagation of chaos" property were established for (1.1), an efficient numerical analysis of these equations via Lévy α-stable Monte Carlo simulations would be available, see e.g., [G], [O], [Sz] for this practical aspect of interacting particle systems with Brownian diffusion.

The analogous interacting particle system approximation questions for the "fractal" Burgers equation with α-stable processes

$$u_t = -(-\Delta)^{\alpha/2}u + a \cdot \nabla(u^r) \tag{1.5}$$

have been dealt with in [FW] for $d = 1$ and $r = 2$. Based on various estimates of solutions to the deterministic Burgers equation with fractal diffusion in [BFW1], theorems in the "propagation of chaos" spirit have been recently proved in [FW]. The paper [BFW2] relates to [BW] as [FW] to [BFW1]. For a survey of related more classical issues, see also [W].

Notation. $|u|_p$ stands for the Lebesgue $L^p(\mathbb{R}^d)$-norm of the function u, and $\|u\|_k$ is the Sobolev $H^k \equiv W^{k,2}$-norm. Inessential constants will be denoted generically by C, even if they vary from line to line.

2. Local and global existence of solutions. In this section we provide existence results for the local and global in time (weak) solutions of the initial value problem for (1.1). We consider in the sequel only the simplest case of $\Omega = \mathbb{R}^d$, although most of results in this section extends to u defined on an open subset Ω of $\mathbb{R}^d$ and satisfying suitable boundary conditions on $\partial\Omega$.

We restrict ourselves to the most important in the aplications case of convolution operators B in (1.1), so that from now on $b(x, y) = b(x - y)$. Moreover, we assume that b satisfies potential estimates like either

$$|b(x)| \le C|x|^{\beta-d} \tag{2.1}$$

or

$$|Db(x)| \le C|x|^{\gamma-d} \tag{2.2}$$

for some $0 < \beta < d$, $0 < \gamma < d$, which is motivated by the examples (1.2), (1.3). Formally, (1.4) corresponds to the limit case $\beta = 0$ but, of course, the operator $B(u) = cu$, $0 \ne c \in \mathbb{R}^d$, is not an integral one. In fact, assumptions (2.1), (2.2) can be weakened as, e.g., in [BW, Sec. 2] but we prefer to keep the potential character and smoothing properties of B clear. By the fractional power of the minus Laplacian in $\mathbb{R}^d$ we mean the

Fourier multiplier

$$(-\Delta)^{\alpha/2}v(x) \equiv D^\alpha v(x) = \mathcal{F}^{-1}\left(|\xi|^\alpha \hat{v}(\xi)\right)(x).$$

Now, we recall and extend results from [BW] on the local in time existence of solutions to (1.1) with the initial condition

$$u(x,0) = u_0(x), \tag{2.3}$$

under assumptions (2.1) or (2.2) specified to the case $d \le 3$, in order to use the framework of Hilbertian Sobolev spaces $H^k(I\!\!R^d)$. By a solution we mean a *weak* one, i.e. a function $u \in L^2((0,T); H^{\alpha/2}(I\!\!R^d))$ such that the integral identity

$$\int_{I\!\!R^d} u(x,t)\eta(x,t)\,dx - \int_0^t ds \int_{I\!\!R^d} u\eta_s\,dx + \int_0^t ds\int_{I\!\!R^d}\left(D^{\alpha/2}uD^{\alpha/2}\eta + uB(u)\cdot\nabla\eta\right)dx$$
$$= \int_{I\!\!R^d} u_0(x)\eta(x,0)\,dx$$

holds for every test function $\eta \in H^1(I\!\!R^d \times (0,T))$, cf. [BW, Sec. 2].

THEOREM 2.1. *Suppose that $\alpha+\beta > d/2+1$ in (2.1), $\alpha \in (0,2]$, $\beta \in (0,d)$, $d = 1, 2, 3$, and the initial condition is $0 \le u_0 \in L^2(I\!\!R^d) \cap L^1(I\!\!R^d)$. Then there exist $T > 0$ and a weak solution $u \ge 0$ of the Cauchy problem (1.1), (2.3). Moreover, $|u(t)|_1 = |u_0|_1$ for all $t \in (0,T)$.*

The above theorem is related to Theorems 2.1 and 2.2 in [BW], and improves over those results for some $0 < \beta < 1$ and for $d = 1$ not considered there.

PROOF. We give only a crucial *a priori* estimate of $u(t)$ in L^2 referring to [BW] for a description of the construction of u. Observe that

$$\frac{d}{dt}|u|_2^2 + 2|D^{\alpha/2}u|_2^2 = -2\int_{I\!\!R^d} uB(u)\cdot\nabla u\,dx \tag{2.4}$$

and the right hand side of (2.4) can be transformed into

$$-\int_{I\!\!R^d}\nabla(u^2)\cdot B(u)\,dx = \int_{I\!\!R^d} u^2\nabla\cdot\ B(u)\,dx.$$

Then we estimate, from the Schwarz inequality and the condition (2.1) which assure smoothing properties of B,

$$\left|\int_{I\!\!R^d} u^2\nabla\cdot B(u)\,dx\right| \le |u^2|_2\|B(u)\|_1 \le C|u|_4^2\|u\|_{1-\beta}. \tag{2.5}$$

Note that the assumptions (2.1)–(2.2) on the potential nature of the kernel b are stricter than those imposed in [BW], thus permitting stronger estimates than L^p estimates in that paper.

Next, by interpolation we get

$$\begin{aligned}\left|\int_{I\!\!R^d} u^2\nabla\cdot B(u)\,dx\right| &\le C\|u\|_{\alpha/2}^{d/\alpha+2(1-\beta)/\alpha}|u|_2^{3-d/\alpha-2(1-\beta)/\alpha}\\ &\le \|u\|_{\alpha/2}^2 + C|u|_2^m\end{aligned} \tag{2.6}$$

for some $m > 0$ if $1 - \beta \le \alpha/2$ and $d/\alpha + 2(1-\beta)/\alpha < 2$ which is the assumption in Theorem 2.1. Now, (2.4) and (2.6) lead to the differential inequality

$$\frac{d}{dt}|u|_2^2 + |D^{\alpha/2}u|_2^2 \le C(|u|_2^2 + |u|_2^m) \tag{2.7}$$

which implies a local bound $|u(t)|_2 \le C(T) < \infty$ for some $T = T(|u_0|_2) > 0$ and all $t \in (0,T)$. Note that for $\beta \ge 1$ the proof of Th. 2.2 in [BW] involved another reasoning based on the Hardy–Littlewood–Sobolev inequality.

The positivity and total mass preserving properties of (1.1) are the consequences of those properties of Lévy and Gauss semigroups $\exp(-t(-\Delta)^{\alpha/2}) = \mathcal{F}^{-1}(\exp(-t|\xi|^\alpha)\mathcal{F})$ of *probability* measures corresponding to the cases $0 < \alpha < 2$ and $\alpha = 2$, respectively. Moreover, weak solutions to (1.1) enjoy some supplementary regularity properties, due to parabolic smoothing by $(-\Delta)^{\alpha/2}$, see [BN, Sec. 2], [BHN, Sec. 2, 3], [BW]. ∎

REMARK 2.1. Although the calculations above are not directly applicable to the Burgers equation (1.5), the assumption $\alpha + \beta > d/2 + 1$ gives a correct result. This guarantees even the global existence of solutions if $d = 1$, $\beta = 0$, $u_0 \in H^1(\mathbb{R})$, so that $\alpha > 3/2$, see [BFW1, Th. 2.1]. Concerning the higher dimensional quadratic Burgers equation (1.5) with $r = 2$, the condition $\alpha + \beta > d/2 + 1$ may suggest that no weak solutions exist for $d \ge 2$ and $\alpha \in (0, 2]$. This can serve as an heuristic motivation for the study of another kind of solutions, namely *mild* ones in [BFW1, Sec. 6].

The theorem below recalls sufficient conditions for the global in time existence of solutions, see [BW, Sec. 3].

THEOREM 2.2. *Suppose that $\alpha + \beta > d + 1$ in (2.1), $\alpha \in (0, 2]$, $\beta \in (0, d)$, $d = 1, 2, 3$. Then any local solution to the Cauchy problem (1.1), (2.3) with $u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ can be continued to the whole half-line $(0,\infty)$.*

PROOF. The right hand side of the energy identity (2.4) can be estimated as in (2.5) for $0 < \beta \le 1$. After interpolation of norms this quantity is bounded by

$$C\|u\|_{\alpha/2}^{3d/(\alpha+d)+(d+2-2\beta)/(\alpha+d)}|u|_1^m$$

with some $m > 0$. Our assumption shows that the exponent of $\|u\|_{\alpha/2}$ above is strictly less than 2. Hence, (2.4) implies that

$$\frac{d}{dt}|u|_2^2 + |D^{\alpha/2}u|_2^2 \le C(|u|_2^2 + |u|_1^M),$$

so a locally uniform estimate of $|u(t)|_2$ follows, and by the results of Th. 2.1 $u(t)$ has a continuation to $(0,\infty)$.

For $\beta > 1$ we apply to the second factor on the right hand side of (2.7) the Hardy–Littlewood–Sobolev inequality, and then the interpolation to obtain

$$\|B(u)\|_1 \le C|u|_q \le C\|u\|_{\alpha/2}^k |u|_1^{1-k}$$

with $1/2 = 1/q - (\beta-1)/d$ and $k = (d + 2\beta - 2)/(\alpha + d)$. The conclusion follows now as before when $0 < \beta \le 1$. ∎

For $\alpha = 2$ we recover Theorem 3.1 in [BW] where $\beta > d - 1$.

Note that under essentially the same growth assumption $\gamma > d - \alpha$ in (2.2) (since $\beta = \gamma - 1$ for smooth kernels b satisfying (2.2)) Theorem 2.2 for $0 < \alpha < 2$ has been proved in [BW, Th. 3.2].

EXAMPLE 2.1. The Biot–Savart kernel (1.3). In this example (borrowed from [Sz, Ch. I.1, c)] and studied in, e.g., [GMO], [G], [O]) global solutions exist under weaker growth assumptions on b than those in Theorem 2.2, but even local in time existence needs a more delicate argument than that in Theorem 2.1 because of a specific structure of b. Here, of course, $\nabla \cdot B(u) = 0$, so the continuation argument would follow from $\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 = 0$, if the local existence were proved for (2.1) with $u_0 \in L^2(\mathbb{R}^2)$ (which is much subtler than the arguments in Theorem 2.1).

3. Finite time blow-up of solutions. The existence result for classical Brownian diffusion in Theorem 2.2 is sharp in the sense that there exists a two-dimensional kernel b satisfying (2.1) with $\beta = d - 1 = 1$ and such that certain solutions of (1.1) with $\alpha = 2$ blow up in finite time. This is the content of

PROPOSITION 3.1. *Suppose that $d = 2$, $\alpha = 2$, and that the linear operator B is defined by the potential kernel*

$$b(x,y) = +(x-y)|x-y|^{-2} = +\nabla(\log|x-y|) \tag{3.1}$$

corresponding to the gravitational interaction of the particles. If a compactly supported initial condition (density) $u_0 \geq 0$ has a sufficiently big integral (total mass) $M = \int u_0$, then the solution to the Cauchy problem (1.1), (2.3) cannot be global in time.

PROOF. We repeat essentially the reasoning in [B1], where the method of moments has been employed. The assumption on u_0, used in the construction of solution with good decay properties, can be relaxed to $u_0 \in L^1(\mathbb{R}^2, (1+|x|^2)\,dx)$. Indeed, a weak solution $u(t)$ constructed as the limit of solutions with compactly supported initial conditions, satisfies $u(t) \in L^1(\mathbb{R}^2, (1+|x|^2)\,dx)$ whenever $u(x,t)$ is finite a.e. Consider now the function

$$w(t) = \int |x|^2 u(x,t)\,dx \geq 0. \tag{3.2}$$

Using the equation (1.1), after some integrations by parts we arrive at

$$\frac{dw}{dt} = 2dM - 2\iint b(x,y)\cdot x\, u(y,t)u(x,t)\,dy\,dx \tag{3.3}$$

(remember that $\int |x|^2 \Delta u = 2d\int u$). After the symmetrization of the double integral we have

$$\begin{aligned}\frac{dw}{dt} &= 2dM - \iint (b(x,y)\cdot x + b(y,x)\cdot y)u(x,t)u(y,t)\,dx\,dy \\ &= 2dM - \iint ((x-y)\cdot x + (y-x)\cdot y)|x-y|^{-2}u(x,t)u(y,t)\,dx\,dy \\ &= 2dM - M^2\end{aligned}$$

for our particular kernel b. It is now obvious that for $M = \int u_0 = \int u(x,t)\,dx > 2d = 4$ the moment function $w(t)$ becomes negative in a finite time, a contradiction. ∎

REMARK 3.1. Notice that in the case of electrostatic repulsion ($d = 2$, $-$ sign in (3.1) instead of $+$) the local solutions can be continued to the global ones for arbitrarily large initial data $u_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. The proof can be obtained by a suitable modification of that in [BHN, Th. 3], using some supplementary *a priori* estimates. For the Biot–Savart kernel in Example 2.1 the global in time existence of solutions for $u_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ holds, but the reasons for this are quite different (here $\nabla \cdot B(u) = 0$, while for the above two examples $\nabla \cdot B(u) \not\equiv 0$).

The conclusion of Proposition 3.1 extends to more general situations.

PROPOSITION 3.2. *Suppose that* $d \geq 2$, $\alpha = 2$, *and that the linear operator* B *is defined by the potential kernel* $b(x, y) = (x - y)|x - y|^{-d+\beta-1}$ *with* $0 < \beta \leq d - 1$. *Then there exists* $u_0 \geq 0$ *such that the solution to (1.1), (2.3) blows up in a finite time.*

The above indirect proof of the nonexistence of global solutions gives no quantitative information about the character of the blow-up in the neighborhood of the critical point $t = T < \infty$. Actually, it can be proved that $\lim_{t\to T^-} |u(t)|_p = \infty$ for each $p > 1$ and even $\lim_{t\to T^-} \int u(x, t) \log u(x, t)\, dx = \infty$ for some u_0's. For related parabolic systems of chemotaxis (arising in mathematical biology) the solutions can blow up either with u converging to a point mass ($d \geq 2$), or without the concentration of mass ($d = 3$). Particular radially symmetric solutions that display this chemotactic collapse behavior have been constructed in [HMV]. Propositions 3.1 and 3.2 give rather general conditions on the initial data that guarantee blow-up of solutions to (1.1), (2.3). We do not know how to solve an analogous problem of blowing up solutions to the fractal ($0 < \alpha < 2$) nonlinear equation (1.1).

4. Self-similar solutions. In this section we consider self-similar (automorphic) solutions of the equation (1.1) with $1 < \alpha \leq 2$ and

$$b(x, y) = \pm(x - y)|x - y|^{-d+\beta-1}, \ \beta > 0. \tag{4.1}$$

Due to the homogeneity of the kernel b, if a function u (not necessarily positive in this section) solves (1.1), then the rescaled function $u_\lambda(x, t) = \lambda^\gamma u(\lambda x, \lambda^\alpha t)$ with $\gamma = \alpha+\beta-1$ also verifies (1.1) for each $\lambda > 0$. The *self-similar* solutions are those satisfying $u_\lambda \equiv u$ for each $\lambda > 0$.

These solutions also determine the generic asymptotic behavior as t tends to $+\infty$ of other globally defined solutions. They are expected to be the leading terms in asymptotic (as $t \to +\infty$) expansions of global solutions because if $\lim_{\lambda\to\infty} \lambda^\gamma u(\lambda x, \lambda^\alpha t) = U(x, t)$ exists in a suitable sense, then $t^{\gamma/\alpha} u(xt^{1/\alpha}, t) \to U(x, 1)$ as $t \to \infty$, and $U_\lambda \equiv U$. So U is a self-similar solution and

$$U(x, t) = t^{-\gamma/\alpha} U(xt^{-1/\alpha}, 1) \tag{4.2}$$

is determined by a function of d variables $U(y) \equiv U(y, 1)$. Moreover, if

$$u_0(x) = \lim_{t\to 0} t^{-\gamma/\alpha} U(xt^{-1/\alpha})$$

is the initial value corresponding to a self-similar solution, then u_0 is necessarily homogeneous of degree $-\gamma$, where $\gamma = \alpha + \beta - 1$.

Solutions (4.2) are called *forward* self-similar, while solutions of the form $v(x,t) = (T-t)^{-\gamma/\alpha}V(xt^{-1/\alpha})$ for some $T>0$ are *backward* self-similar ones. They would be the simplest solutions that blow up at $t=T$. However, the problem of their existence seem to be open. Note that blowing up solutions to chemotactic systems constructed in [HMV] are not self-similar but only close to them.

We formulate below two results on the existence of self-similar solutions which extend those in [B2, Sec. 3] and [BFW1, Sec. 7]. Their proofs are obtained by suitable modifications of demonstrations in these references, where the cases $\alpha=2$, $\beta=1$, and $1<\alpha<2$, $\beta=0$ have been considered. Note that we cannot pose directly $\beta=0$ in (4.1), as this leads to divergent integrals in (1.1), but formally this corresponds to $b(x,y)=c\delta(x-y)$, $c\in I\!R^d$. In such a case the nonlinear term in (1.1) has the differential (not integro-differential) form $u\nabla u\cdot a$ with some $a\in I\!R^d$ like in (1.5), see also Example (1.9) b) in [Sz]. Thus, Theorems 4.1, 4.2 below will be obtained via "interpolation" and "extrapolation" of the range of β's and α's, and employing Banach spaces that we denote generically by $\mathcal{B}$, which will be specified for each theorem separately.

Our first result is expressed in terms of the *homogeneous Besov space*

$$\mathcal{B}=\dot{B}^s_{2\,\infty}(I\!R^d)\equiv\Big\{v:\ \|v;\dot{B}^s_{2\,\infty}\|\equiv\sup_{k\in Z\!\!\!Z}2^{ks}|\phi_k*v|_2<\infty\Big\}. \tag{4.3}$$

with $s=d/2-\gamma$. The family of functions $(\phi_k)_{k\in Z\!\!\!Z}$ generates a smooth decomposition into dyadic annuli on the space $I\!R^d_\xi$ of Fourier variables, i.e., for ψ with $\hat\psi\in C_0^\infty(I\!R^d)$, $0\le\hat\psi\le1$, $\hat\psi(\xi)=1$ if $|\xi|\le1$, $\hat\psi(\xi)=0$ for $|\xi|\ge2$, ϕ_k are defined by $\hat\phi_k(\xi)=\hat\psi(2^{-k}\xi)-\hat\psi(2^{-(k+1)}\xi)$, $k\in Z\!\!\!Z$. Thus, $\operatorname{supp}\hat\phi_k\subset A_k\equiv\{\xi:2^{k-1}\le|\xi|\le2^{k+1}\}$, $\sum_k\hat\phi_k(\xi)=1$ for each $\xi\ne0$, with at most two nonzero terms in the series. $\mathcal{B}$ defined as above is a nonseparable Banach space, with $s=(d/2)-\gamma$ the space $\mathcal{B}\subset\mathcal{S}'(I\!R^d)$ is contained in the space of tempered distributions. It is fairly easy to check that, for $0<\gamma<d$, $|x|^{-\gamma}\in\mathcal{B}$, because $\mathcal{F}(|x|^{-\gamma})=c_{\gamma,d}|\xi|^{\gamma-d}$. Moreover, if a function u_0 homogeneous of degree $-\gamma$ is sufficiently smooth on the unit sphere in $I\!R^d$, then u_0 belongs to $\mathcal{B}$.

THEOREM 4.1. *Let $\alpha\in(1,2]$, $\beta>0$, and $d>2\gamma=2(\alpha+\beta-1)$. Suppose that the linear operator B is defined by the potential kernel (4.1). If $u_0\in\mathcal{B}=\dot{B}^{d/2-\gamma}_{2\,\infty}(I\!R^d)$ is homogeneous of degree $-\gamma$ and its norm $\|u_0\|_{\mathcal{B}}$ is sufficiently small, then there exists a self-similar mild solution U of the equation (1.1) with u_0 as the initial data.*

Our second result is expressed in terms of the functional space

$$\mathcal{B}=E^{\gamma,m}(I\!R^d)\equiv\Big\{v\in C^m(I\!R^d):\ |D^\zeta v(x)|\le C(1+|x|)^{-\gamma-|\zeta|},\ |\zeta|\le m\Big\}, \tag{4.4}$$

m being a nonnegative integer, which consists of smooth functions of polynomial decay at infinity, and its homogeneous version

$$\dot{E}^{\gamma,m}(I\!R^d)\equiv\Big\{v\in C^m(I\!R^d\setminus\{0\}):\ |D^\zeta v(x)|\le C|x|^{-\gamma-|\zeta|},\ |\zeta|\le m\Big\},$$

whose elements admit some singularity at the origin. The norms of v are the least constants satisfying the aforementioned conditions.

THEOREM 4.2. *Let $\alpha \in (1,2]$, $\beta > 0$, $\gamma = \alpha+\beta-1$, $m+\gamma < d$. Suppose that the linear operator B is defined by the potential kernel (4.1). If $u_0 \in \dot{E}^{\gamma,m}(\mathbb{R}^d)$ is homogeneous of degree $-\gamma$ and has a sufficiently small norm, then there exists a self-similar mild solution $t^{-\gamma/\alpha}U(xt^{-1/\alpha})$ of the equation (1.1) with $U \in \mathcal{B} = E^{\gamma,m}(\mathbb{R}^d)$.*

The concept of a *mild solution* replaces the original formulation (1.1) by an integral equation

$$U(t) = e^{-t(-\Delta)^{\alpha/2}}u_0 + \int_0^t \left(\nabla e^{-(t-s)(-\Delta)^{\alpha/2}}\right) \cdot (U(s)B(U(s)))\, ds, \tag{4.5}$$

a consequence of the variation of parameters formula. Here $\exp(-t(-\Delta)^{\alpha/2})$ is the Lévy semigroup of linear operators generated by $-(-\Delta)^{\alpha/2}$, so

$$\frac{\partial}{\partial t}e^{-t(-\Delta)^{\alpha/2}} = -(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}},$$

and $\exp(-t(-\Delta)^{\alpha/2})$ commutes with ∇. A solution U of (4.5) is looked for in the space of (weakly) continuous functions $\mathcal{C}([0,\infty);\mathcal{B})$ with values in $\mathcal{B}$.

The difference between the two theorems is the range of parameters for which they apply and, more importantly, the dramatically different nature of the solution spaces, (4.3) $\dot{B}^s_{2\,\infty}$ and (4.4) $E^{\gamma,m}$. The former is the space of distributions enjoying, in general, no significant pointwise regularity, while the latter consists of smooth functions.

The observation crucial for the proofs is that if u_0 is homogeneous of degree $-\gamma$, then $\exp(-t(-\Delta)^{\alpha/2})u_0$ is of the self-similar form (4.2). Moreover, if U of the form (4.2) is put into the nonlinear integral operator on the right-hand side of (4.5), then the result is again of the form (4.2). Hence, the equation (4.5) has a reproducing property of the self-similar form (4.2). Thus, it can be studied for $t = 1$ only, which reduces all the considerations to the spaces $\mathcal{B}$. Moreover, it is suitable to be solved by iterative methods and the contraction theorem. These arguments can be applied when the integral in (4.5) defines a quadratic form continuous on $\mathcal{B}$. And this is indeed the case under the assumptions of Theorems 4.1, 4.2. These ideas have been originally applied to the Navier–Stokes system by M. Cannone, Y. Meyer and F. Planchon, cf. [C]. We skip the details of rather technical proofs, referring to [BW].

5. Nonlinear Markov processes and approximating particle systems. We sketch in this section the construction of a nonlinear Markov process for which the equation (1.1) serves as the Fokker–Planck–Kolmogorov equation. The assumption $\alpha \in (1,2)$ permits us to freely use the expectations of the α-stable processes involved in the construction.

Let $u \geq 0$ be a (local in time) solution of (1.1) with u_0 regular enough. Without loss of generality we can assume that u is bounded, i.e.

$$\sup_{x\in\mathbb{R}^d, t\in[0,T]} |u(x,t)| < \infty, \tag{5.1}$$

and, since we are working with $(L^1 \cap L^\infty)$-solutions, $\sup_{x\in\mathbb{R}^d,t\in[0,T]} |B(u(t))(x)| < \infty$, which follows from the potential estimate (2.1), Sobolev embedding theorem and (5.1).

This is a property similar to that in Th. 2.1(iii) in [BW] where the case $\alpha = 2$ was considered. Whenever a local solution u can be defined, the parabolic regularization property of $(-\Delta)^{\alpha/2}$, $\alpha \in (1,2]$, leads to an instantaneous smoothing of u to a locally bounded function, see [BFW2].

Consider a solution $X(t)$ of the stochastic differential equation

$$dX(t) = dS(t) - B(u(t))(X(t))\,dt, \tag{5.2}$$

where u is a given (bounded) solution of (1.1), $X(0) \sim u(x,0)\,dx$ in law, and $S(t)$ is a standard α-stable spherically symmetric process with its values in $\mathbb{R}^d$. Since the coefficient $B(u)$ in (5.2) is bounded, we infer that the stochastic differential equation (5.2) has a unique solution X. The measure-valued function $v(dx,t) \equiv P(X(t) \in dx)$ satisfies the weak forward equation

$$\frac{d}{dt}\langle v(t), \eta\rangle = \langle v(t), \mathcal{L}_{u(t)}\eta\rangle, \tag{5.3}$$

for all $\eta \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class of functions on $\mathbb{R}^d$, with the initial condition $v(0) = u(x,0)\,dx$ and the operator $\mathcal{L}_u = -(-\Delta)^{\alpha/2} - B(u)\cdot\nabla$, $\quad u = u(x,t)$.

PROPOSITION 5.1. *Let $1 < \alpha < 2$ and u be a solution of (1.1) satisfying (5.1). The process $X(t)$ in (5.2) is the McKean process (nonlinear Markov process) corresponding to (1.1), that is, it satisfies the relation $P(X(t) \in dx) = u(x,t)\,dx$.*

PROOF. The following two statements are equivalent:

- The martingale problem for the operator $\mathcal{L}_{u(t)}$ is well posed, and
- The existence and uniqueness theorem holds for the corresponding *linear* weak forward equation (5.3).

Here, the martingale problem associated with (5.2) is well posed. However, $u(dx,t) \equiv u(x,t)\,dx$ is also a solution of (5.3) since $\frac{d}{dt}\langle u(t),\eta\rangle = \langle -(-\Delta)^{\alpha/2}u + \nabla\cdot(uB(u)), \eta\rangle = \langle u, (-(-\Delta)^{\alpha/2} - B(u)\cdot\nabla)\eta\rangle$. Since the coefficients of the *linear* equation (5.3) are regular ($B(u) \in L^\infty$), the problem $w_t = -(-\Delta)^{\alpha/2}w - B(u)\cdot\nabla w$, $w(0) = 0$, has the unique solution $w \equiv 0$. This can be easily seen from the energy estimates as in the proof of Theorem 2.1. Now, the uniqueness for (5.3) implies that $v(dx,t) = u(dx,t)$, which yields Proposition 5.1. ∎

The classical *"propagation of chaos"* result for the partial differential equation (1.1) would show that the empirical distribution

$$\tilde{X}^n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(X^{i,n}(t)) \tag{5.4}$$

of n interacting particles with positions $\{X^{i,n}(t)\}_{i=1,\dots,n}$, whose dynamics is described by the system of stochastic differential equations

$$dX^{i,n}(t) = dS^i(t) - \frac{1}{n}\sum_{j\neq i} b(X^{i,n}(t), X^{j,n}(t))\,dt, \tag{5.5}$$

is close to the distribution of the McKean process $X(t)$ in the sense that

$$\tilde{X}^n(t) \Rightarrow u(x,t)\,dx, \text{ in probability, as } n \to \infty, \tag{5.6}$$

where $\Rightarrow$ denotes the weak convergence of measures. In our situation $\{S^i(t)\}_{i=1,\dots,n}$ are independent copies of symmetric Lévy α-stable processes with the common infinitesimal generator $-(-\Delta)^{\alpha/2}$.

Results in this spirit, when S is replaced by a more familiar Wiener process, have been proved in various situations after the pioneering work [McK]. We have chosen some classical as well as new references containing reformulations, extensions and generalizations of the above scheme for various evolution problems of physical origin: [GK], [KO], [O], [G], [Sz], [BT]. Besides a purely mathematical interest, they give also reasonably well working tools for the numerical approximation of solutions, especially when convergence rates can be found.

The recent paper [FW] deals with the first, to the best of our knowledge, interacting particle system approximation result for Lévy α-stable processes driven stochastic differential equations associated with the fractal Burgers equation (1.5). Because of rather weak parabolic regularization effect of $(-\Delta)^{\alpha/2}$, a preliminary step involving the replacement of $X^{i,n}$ by solutions of regularized stochastic differential equations ((5.7) below) seems to be necessary in order to have an analogue of (5.6).

Let us consider a standard smoothing kernel $\delta_\epsilon(x) = (2\pi\epsilon)^{-d/2}\exp(-|x|^2/(2\epsilon))$, $\epsilon > 0$, and the system of regularized equations (5.5)

$$dX^{i,n,\epsilon}(t) = dS^i(t) - \frac{1}{n}\sum_{j\neq i} b_\epsilon\left(X^{i,n,\epsilon}(t) - X^{j,n,\epsilon}(t)\right)\,dt, \tag{5.7}$$

where $b(x,y) = b(x-y)$, $b_\epsilon = b * \delta_\epsilon$. Then define random empirical measures

$$Y^{n,\epsilon}(t) = \frac{1}{n}\sum_{i=1}^{n}\delta(X^{i,n,\epsilon}(t)),$$

instead of previously considered $\tilde{X}^n$ in (5.4).

First, we prove propagation of chaos property for a regularized version ((5.9) below) of equation (1.1), including an error estimate. Then we prove a weaker property "propagation of chaos in a wide sense" for the original equation (1.1), which is, however, a satisfactory basis for an approximation scheme for numerical solving of that equation. The extension will rely on purely analytic estimates of solutions u^ϵ of (5.9).

THEOREM 5.1. *Let the conditions of Theorem 2.1 ensuring the local in time existence of solutions to (1.1) on $\mathbb{R}^d\times(0,T)$ be satisfied. Moreover, assume that $|\hat{b}(\xi)| \le C(1+|\xi|^{-\beta})$ (which is, of course, compatible with the potential estimate (2.1)) and that the initial conditions $\{X^{i,n,\epsilon}(0)\}_{i=1,\dots,n}$ satisfy*

$$\sup_n \sup_{\lambda\in\mathbb{R}^d} n^{1-1/\alpha}(1+|\lambda|^a)^{-1} E\left[\langle Y^{n,\epsilon}(0) - u^\epsilon(x,0), \chi_\lambda\rangle\right] < \infty$$

for some $a \ge 0$ and all the characters $\chi_\lambda(x) = e^{i\lambda x}$. Then:

(i) For each $\epsilon > 0$ the empirical process is weakly convergent

$$Y^{n,\epsilon}(t) \Rightarrow u^\epsilon(x,t)\,dx, \text{ in probability, as } n\to\infty. \tag{5.8}$$

The limit density $u^\epsilon = u^\epsilon(x,t)$, $x \in I\!R^d$, $t \in (0,T)$, solves the regularized equation (1.1)

$$u_t^\epsilon = -(-\Delta)^{\alpha/2} u^\epsilon + \nabla\cdot(u^\epsilon B_\epsilon(u^\epsilon)) \tag{5.9}$$

with $B_\epsilon = \delta_\epsilon * B$ defined by the kernel $b_\epsilon = \delta_\epsilon * b$.

(ii) For each $\epsilon > 0$, there exists a constant C_ϵ such that for any $\phi \in \mathcal{S}(I\!R^d)$

$$E\,|\langle Y^{n,\epsilon}(t) - u^\epsilon(t),\phi\rangle| \le C_\epsilon n^{1/\alpha-1}\int_{I\!R^d}(1+|\lambda|^a)|\hat\phi(\lambda)|\,d\lambda. \tag{5.10}$$

(iii) Under the assumptions of Theorem 2.2 guaranteeing the global in time existence of solutions to (1.1), the conclusions (i), (ii) are valid for all $t \in (0,\infty)$.

The proof of above theorem involves some delicate probabilistic methods, see [BFW2].

By "propagation of chaos in a wide sense" property for equation (1.1) we mean that given any sequence of regularizations (5.9) with $\epsilon \to 0$, the family of empirical distributions $\{Y^{n,\epsilon}(t)\}$ contains a subsequence weakly convergent to a solution $u(t)$ of (1.1).

THEOREM 5.2. *Let the general conditions of Theorem 5.1 be satisfied. Assume that $u^\epsilon(t)$ are solutions of the regularized equation (5.9) such that their initial conditions satisfy $|u^\epsilon(0) - u(0)|_2 \to 0$ as $\epsilon \to 0$ for some $u(0) \in L^2(I\!R^d)$. Then given any sequence $\epsilon_k \to 0$ as $k \to \infty$, there exists a sequence $n_k \to \infty$ and a weak solution $u(t)$ of (1.1) such that for each $\phi \in C_0^\infty(I\!R^d)$*

$$E\,|\langle Y^{n_k,\epsilon_k}(t) - u(t),\phi\rangle| \to 0. \tag{5.11}$$

Moreover, under the assumptions of Theorem 5.1 (iii), (5.11) can be strengthened to the global in time convergence for all $t \in (0,\infty)$.

The proof requires only the following purely analytic weak convergence result

$$|\langle u^{\epsilon_k}(t) - u(t),\phi\rangle| \to 0, \tag{5.12}$$

as $\epsilon_k \to 0$ for each ϕ in a suitable function class containing $C_0^\infty(I\!R^d)$. Indeed, (5.10) combined with (5.12) shows that $E\,[|\langle Y^{n_k,\epsilon_k}(t) - u(t),\phi\rangle|] \to 0$ for some sequence $\epsilon_k \to 0$ and suitably large $n_k \to \infty$ as $k \to \infty$, see [BFW2].

REMARK 5.1. Under fairly general assumptions of Th. 5.1 (i)–(ii), when only local in time solutions exist (and it may actually happen that they blow up in a finite time), (5.12) is a rather weak result. When stronger assumptions in Th. 5.1 (iii) guarantee the global in time existence of solutions, convergence of solutions of regularized equations (5.9) to those of the original one (1.1) will be, of course, stronger. To obtain those convergence properties, we show compactness of the family of approximating solutions using either the Aubin–Lions or the Ascoli–Arzelà criteria for vector-valued functions.

REMARK 5.2. Note that so far the issue of uniqueness of solutions to (1.1) was not addressed in this paper. For $\alpha = 2$ the uniqueness of weak solutions holds true, see [BW]. For $1 < \alpha < 2$, we can only prove the uniqueness of more regular solutions in $L^\infty((0,T);H^1(I\!R^d))$. However, we do not develop this issue here because, although the convergence in (5.12) would then be improved to all $\epsilon \to 0$, in (5.11) we still would need to select a subsequence $n_k \to \infty$. We suspect that solutions to (1.1) with sufficiently regular initial data are unique, but they are not necessarily unique in general. In such a case our interacting particle approximation selects a solution of (1.1) similarly to the way the viscosity method selects a unique, so-called *viscosity*, solution of conservation laws.

REMARK 5.3. Unlike the case of the one-dimensional fractal Burgers equation in [FW], the estimates of u_ϵ leading to (5.12) (gaining extra information from the degree of approximation of $\delta_\epsilon * u^\epsilon - u^\epsilon$) will be similar to those of u in the existence Theorems 2.1 and 2.2. It seems that in the higher dimensional case, $d \geq 2$, we cannot obtain results in the same spirit for the fractal Burgers equation (1.5) with $r > 1$, because the diffusion operator $(-\Delta)^{\alpha/2}$, $\alpha < 2$, is too weak compared to the nonlinear term.

REMARK 5.4. The case $\alpha = 2$ is substantially different (and easier to treat) than that of $\alpha < 2$. Namely, the global in time solutions to (1.1) are expected (by e.g. [BW]) to satisfy a Gaussian bound in the space variable. However, we cannot expect such an exponential decay of solutions to (1.1) if $\alpha < 2$. Even for linear equations, in particular for the Lévy semigroup, the best we can obtain is an algebraic decay rate $|x|^{-d-\alpha}$. This is an heuristic explanation of seemingly very weak convergence properties obtained in Theorem 5.1.

References

[BPFS] C. BARDOS, P. PENEL, U. FRISCH and P. L. SULEM, *Modified dissipativity for a nonlinear evolution equation arising in turbulence*, Arch. Rat. Mech. Anal. 71 (1979), 237–256.

[B1] P. BILER, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, III*, Colloq. Math. 68 (1995), 229–239.

[B2] P. BILER, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, Studia Math. 114 (1995), 181–205.

[B3] P. BILER, *Local and global solvability of parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. 8 (1998), 715–743.

[BFW1] P. BILER, T. FUNAKI and W. A. WOYCZYŃSKI, *Fractal Burgers equations*, J. Diff. Eq. 148 (1998), 9–46.

[BFW2] P. BILER, T. FUNAKI and W. A. WOYCZYŃSKI, *Interacting particle approximation for nonlocal quadratic evolution problems*, Probab. Math. Statist. 19 (1999), 321–340.

[BHN] P. BILER, W. HEBISCH and T. NADZIEJA, *The Debye system: existence and large time behavior of solutions*, Nonlinear Analysis T.M.A. 23 (1994), 1189–1209.

[BN] P. BILER and T. NADZIEJA, *A class of nonlocal parabolic problems occurring in statistical mechanics*, Colloq. Math. 66 (1993), 131–145.

[BW] P. BILER and W. A. WOYCZYŃSKI, *Global and exploding solutions for nonlocal quadratic evolution problems*, SIAM J. Appl. Math. 59 (1999), 845–869.

[BT] M. BOSSY and D. TALAY, *Convergence rate for the approximation of the limit law of weakly interacting particles: application to the Burgers equation*, Ann. Appl. Prob. 6 (1996), 818–861.

[C] M. CANNONE, *Ondelettes, paraproduits et Navier–Stokes*, Diderot Editeur; Arts et Sciences, Paris, 1995.

[FW] T. Funaki and W. A. Woyczyński, *Interacting particle approximation for fractal Burgers equation*, in: Stochastic Processes and related topics, A volume in memory of S. Cambanis, I. Karatzas, B. Rajput and M. Taqqu, Eds., Birkhäuser, Boston 1998, 141–166.

[GMO] Y. Giga, T. Miyakawa and H. Osada, *Two-dimensional Navier–Stokes flow with measures as initial vorticity*, Arch. Rat. Mech. Anal. 104 (1988), 223–250.

[G] J. Goodman, *Convergence of the random vortex method*, Comm. Pure Appl. Math. 40 (1987), 189–220.

[GK] E. Gutkin and M. Kac, *Propagation of chaos and the Burgers equation*, SIAM J. Appl. Math. 43 (1983), 971–980.

[HMV] M. A. Herrero, E. Medina and J. J. L. Velázquez, *Finite-time aggregation into a single point in a reaction-diffusion system*, Nonlinearity 10 (1997), 1739–1754.

[KO] S. Kotani and H. Osada, *Propagation of chaos for Burgers' equation*, J. Math. Soc. Japan 37 (1985), 275–294.

[McK] H. P. McKean, *Propagation of chaos for a class of non-linear parabolic equations*, in: Stochastic differential equations VII, Lecture Series in Differential Equations, Catholic University, Washington D.C., 1967, 177–194.

[O] H. Osada, *Propagation of chaos for two dimensional Navier–Stokes equation*, Proc. Japan Ac. 62A (1986), 8–11, and Taniguchi Symp. PMMP, Katata, 1985, 303–334.

[SZF] M. F. Shlesinger, G. M. Zaslavsky and U. Frisch, Eds., *Lévy Flights and Related Topics in Physics*, Lecture Notes in Phys. 450, Springer-Verlag, Berlin, 1995.

[Sz] A. S. Sznitman, *Topics in propagation of chaos*, in: École d'été de St. Flour, XIX - 1989, Lecture Notes in Math. 1464, Springer-Verlag, Berlin, 1991, 166–251, .

[W] W. A. Woyczyński, *Burgers-KPZ Turbulence, Göttingen Lectures*, Lecture Notes in Math. 1700, Springer-Verlag, Berlin, 1998.