POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

SINGULARITIES OF POISSON AND NAMBU STRUCTURES

JEAN-PAUL DUFOUR

Département de mathématiques, Université Montpellier II
Pl. E. Bataillon, 34095 Montpellier Cedex 05, France
E-mail: dufourj@math.univ-montp2.fr

To the memory of Stanisław Zakrzewski

1. Introduction. Poisson tensors in dimension 2 have the local form $f(x,y)\partial/\partial x \wedge \partial/\partial y$. So the study of their singularities reduces to the study of singularities of the function of two variables f. In that direction the essential of the work was done by V.I. Arnol'd ([A]).

The task becomes more difficult in dimension 3, where Jacobi's identity begins to give non trivial constraints. Fortunately, in that dimension, we can use what I will call the Ω *trick*: choose a volume form Ω (in this paper we always work locally); then we consider the isomorphism

$$\flat : A \mapsto i_A\Omega, \quad V_p \to \Lambda_{n-p},$$

where V_p is the set of p-vectors (contravariant skew-symmetric tensors of order p) and Λ_{n-p} is the set of $(n-p)$-forms on our manifold. When the dimension n of the manifold is 3, it is well known that $\flat$ exchanges Poisson vectors and integrable 1-forms, i.e. 1-forms ω such that $\omega \wedge d\omega = 0$. Moreover the symplectic foliation of the Poisson tensor Π is exactly the foliation associated with $\flat(\Pi)$. So, in that case, the study of singularities of Poisson structures is equivalent to the study of singularities of integrable 1-forms. For the latter a considerable litterature is available: see, for example, [W-R], [Ku], [C-LN], [C-C]...

For higher dimensions the Ω trick doesn't work any more for general Poisson tensors. However it still works if we replace Poisson tensor by their generalizations: Nambu tensors.

In the following paragraph we will recall the Nambu formalism. In [D-N] we had shown that, by the Ω trick, Nambu tensors correspond to what was called by different authors *integrable p-forms*. We had also given a classification of linear Nambu tensors which are

2000 *Mathematics Subject Classification*: 53D17.
The paper is in final form and no version of it will be published elsewhere.

not Poisson and some linearization results. In this paper we will extend these results to Poisson structures with maximal rank 2. We will also give a more general "normal form" result for Nambu tensors with "type 2" linear part.

2. Nambu structures. Following an idea of Y. Nambu ([Na]), L. Takhtajan gave in 1994 ([T]) a formalism which generalizes the Poisson bracket: let M be a manifold and A the algebra of smooth functions on M; he defines a *Nambu structure of order* r on M as a r-linear skew-symetric map

$$A \times \cdots \times A \to A, \quad (f_1, \ldots, f_r) \mapsto \{f_1, \cdots, f_r\},$$

which satisfies properties

$$\{f_1, \ldots, f_{r-1}, gh\} = \{f_1, \ldots, f_{r-1}, g\}h + g\{f_1, \ldots, f_{r-1}, h\} \tag{L}$$

$$\{f_1, \ldots, f_{r-1}, \{g_1, \ldots, g_r\}\} = \sum_{i=1}^{r} \{g_1, \ldots, g_{i-1}, \{f_1, \ldots, f_{r-1}, g_i\}, g_{i+1}, \ldots, g_r\} \tag{FI}$$

for every $f_1, \ldots, f_{r-1}$, g, h, $g_1, \ldots, g_r$ in A.

In the above definition (L) stands for *Leibniz property*, (FI) stands for *fundamental identity* or for *Filippov identity* (see [M-V-V]). For $r = 2$, (FI) is nothing but Jacobi identity, so a Nambu structure of order 2 is a Poisson structure.

Identity (L) implies that $X_{f_1 \cdots f_{r-1}} : g \mapsto \{f_1, \ldots, f_{r-1}, g\}$ is a dervivation of A, hence a vector field on M: it is, by definition, the *Hamiltonian vector field* associated to $f_1 \cdots f_{r-1}$.

Identity (L) implies also that there is a $r-$vector Λ such that

$$\{f_1, \ldots, f_r\} = \Lambda(df_1, \ldots, df_r).$$

This Λ is called a *Nambu tensor.*

3. Foliation. Identity (FI) implies that Hamiltonian vector fields give an integrable distribution, like in the Poisson's case. So we have on M a singular foliation which generalizes symplectic foliations of Poisson manifolds. This work is essentially devoted to the study of the singularities of such foliations.

In 1996 appeared three (independent) proofs of the following surprising result.

LOCAL TRIVIALITY THEOREM ([G], [A-G], [N]). *Let* Λ *be a Nambu tensor of order* $r > 2$. *Near every point where* Λ *doesn't vanish there are local coordinates* $x_1, \ldots, x_n$ *such that*

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_r}.$$

In particular this theorem shows that there are only two types of leaf for the foliation associated to Λ: either it reduces to a point (zero of Λ) or it is r-dimensional.

This leads to the idea that *Nambu structures of order* $r > 2$ *are, in fact, generalization of Poisson structures* **of maximum rank 2.**

VOCABULARY. In the following a *Nambu structure (*resp. *tensor)* of order r will be simply a Nambu structure (resp. tensor) for $r > 2$ and a Poisson structure (resp. tensor) of maximal rank 2 for $r = 2$.

4. Ω trick: covariant characterization of Nambu structures. Here we suppose that we have a volume form Ω on our manifold M. We put $\omega := i_\Lambda \Omega$ and we have the following result ([D-N]).

THEOREM 2. *Suppose Λ is a r-vector on M such that either $r > 2$ or $r = 2$ but, in this case, maximal rank of Λ is 2. If r is equal to the dimension n of M, then Λ is always a Nambu tensor. When $r < n$, Λ is a Nambu tensor if and only if we have*

$$i_A\omega \wedge \omega = 0, \qquad i_A\omega \wedge d\omega = 0$$

for every $(n-r-1)$-vector A.

The first relation in this theorem says that ω is decomposable at each point, the second is an "integrability" condition. In the case $r = n-1$, ω is nothing but an integrable 1-form. In the case $r < n-1$, ω is what was called an *integrable $(n-r)$-form* by different authors (see e.g. [Me]). Roughly speaking, this theorem 2 says that a Nambu structure (or a Poisson structure of maximal rank 2) is exactly the "dual" of an integrable $p-$form.

The idea of the proof of the last theorem is as follows. First we remark that Λ is Nambu if, and only if, it is Nambu when restricted to the open set U where it is different from zero. Then apply local triviality theorem near each point of U to obtain the "only if" part. For the "if" part, use the relations to obtain $\omega = \theta_1 \wedge \cdots \wedge \theta_r$ where $\theta_1, \ldots, \theta_r$ is an integrable system of 1-forms near any point of U. Then apply Frobenius theorem.

As a corollary all classical results concerning these integrable forms pass to Nambu structures.

5. Normal form problem. Suppose m is a zero of the Nambu tensor Λ. In a system of local coordinates (vanishing at m) we have a Taylor expansion

$$\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \cdots$$

where $\Lambda^{(k)}$ consists of the terms of order k. Using the covariant characterization of Nambu tensors, we can show that $\Lambda^{(1)}$ is a *linear* Nambu tensor (it gives a Nambu structure such that bracket of linear functions is a linear function). It is uniquely determined up to linear isomorphism and is called the *linear part of* Λ at m.

The linearization problem is: is it possible to find local coordinates which kill higher order terms $\Lambda^{(2)}$, $\Lambda^{(3)}$,...? More generally, when we cannot linearize, we can try to find *normal forms* like in the theory of vector fields (which can be considered as Nambu tensors of order 1).

Before giving results for that problem we must understand better what is a linear Nambu structure.

6. Linear Nambu structures. For $r = 2$ there is a 1-1 correspondance between linear Poisson structures and Lie algebras: each linear Poisson structure $\{\ ,\ \}$ on a vector space V corresponds to the Lie algebra structure $[\ ,\ ]$ on V^* given simply by $[\alpha, \beta] := \{\alpha, \beta\}$.

Similarly we can define a *Nambu-gebra structure* (or a *Filippov algebra structure*) of order r on the vector space E as an r-linear skew-symmetric map

$$(v_1, \dots, v_r) \mapsto [v_1, \dots, v_r], \qquad E \times \dots \times E \to E$$

which satisfies

$$[v_1, \dots, v_{r-1}, [u_1, \dots, u_r]] = \sum_{i=1}^{r} [u_1, \dots, u_{i-1}, [v_1, \dots, v_{r-1}, u_i], u_{i+1}, \dots, u_r]$$

for every $v_1, \dots, v_{r-1}$, $u_1, \dots, u_r$ in E.

A linear Nambu structure is a Nambu structure on a vector space V such that the bracket of linear functions is always linear. So, exactly as in the Poisson's case, *a linear Nambu structure on V determines a Nambu-gebra structure on V^*.*

The strange thing is that *the converse is false for $r > 2$.*

For example, given two Nambu-gebra structures of order r, $[\ ,\dots,\ ]'$ on V' and $[\ ,\dots,\ ]''$ on V'', we can construct their direct product:

$$[(v_1', v_1''), \dots, (v_r', v_r'')] := ([v_1', \dots, v_r']', [v_1'', \dots, v_r'']'').$$

But the corresponding operation is forbidden for linear Nambu structures because, as a result of the local triviality theorem, a Nambu tensor must be decomposable (the sum of two decomposable tensors is, in general, not decomposable).

There is another interesting construction of Nambu-gebras: suppose $[\dots]_0$ is a Nambu-gebra structure of order r on the linear space V; we construct a Nambu-gebra structure $[\dots]$ of order $r+1$ on $V \oplus K e_0$ (K is the scalar field) if we impose

$$[v_1, \dots, v_r, e_0] := [v_1, \dots, v_r]_0, \quad [v_1, \dots, v_{r+1}] := 0,$$

the v_i being in V. Using this construction, we can start with a Lie algebra, for example, to obtain Nambu-gebras of order 3. If this Lie algebra corresponds to a non-decomposable Poisson tensor, the linear tensor corresponding to this Nambu-gebra is also non-decomposable: This gives examples of Nambu-gebras (in dimension 5, 6...) which don't correspond to linear Nambu structures.

We have the following classification of linear Nambu structures.

Theorem 3. *Let Λ be a linear Nambu tensor of order r (we suppose either $r > 2$ or $r = 2$ but with 2 as maximal rank). Then, up to a linear isomorphism, Λ has one of the following two types:*

Type 1: $\Lambda = \pm x_1 \partial/\partial x_2 \wedge \dots \wedge \partial/\partial x_{r+1} + \dots \pm x_u \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_{u-1} \wedge \partial/\partial x_{u+1} \wedge \dots \wedge \partial/\partial x_{r+1} + x_{r+2} \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_u \wedge \partial/\partial x_{u+2} \wedge \dots \wedge \partial/\partial x_{r+1} + \dots + x_{r+s+1} \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_{u+s-1} \wedge \partial/\partial x_{u+s+1} \wedge \dots \wedge \partial/\partial x_{r+1}$ *with* $0 \le u + s \le r + 1$.

Type 2: $\Lambda = \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_{r-1} \wedge X$ *where X is a linear vector field which is independent of $x_1, \dots, x_{r-1}$ (and can be put in Jordan normal form).*

To understand better this theorem we can rewrite type 1 in the form

$$\Lambda = \sum_{i=1}^{r+1} l_i \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \dots \wedge \partial/\partial x_{r+1}$$

where l_i are (precise) linear functions. Choose $\Omega = dx_1 \wedge \cdots \wedge dx_n$: by the Ω trick Λ becomes

$$dx_{r+2} \wedge \ldots \wedge dx_n \wedge \alpha$$

with $\alpha = \sum_i \pm l_i dx_i$ (the precise form of Λ corresponds to the fact that we can choose for α the precise form $d(\pm x_1^2/2 + \cdots \pm x_u^2/2 \pm x_{u+1}x_{r+2} + \cdots \pm x_{u+s}x_{r+s+1}))$.

For type 2, we can write $X = \sum_{i=r}^n k_i \partial/\partial x_i$, where the k_i are linear functions. Then by the Ω trick, Λ corresponds to the $(n-r)$-form

$$\sum_{i=r}^{n} \pm k_i dx_r \wedge \cdots \wedge dx_{i-1} \wedge dx_{i-1} \cdots \wedge dx_n.$$

So we observe an interesting duality between type 1 and type 2: type 2 are decomposable exactly as the duals of type 1, type 1 are in general non-decomposable and have an analogous form as the duals of type 2. For type 1 the covariant form is simpler: it reduces to the 1-form α. For type 2 the contravariant form is simpler: it reduces to the vector field X.

In [D-N] there is a proof of this theorem for the case $r > 2$. The same proof works also for the case $r = 2$. Its principle is as follows. Use the covariant characterization of paragraph 4. Write $\omega(= i_\Lambda \Omega)$ in the form $\sum_i x_i \omega_i$ where ω_i are constant $(n-r)$-forms. The decomposability of ω leads to the decomposability of each ω_i and also to the decomposability of $\omega_i + \omega_j$. If we denote by V_i the span of ω_i when it is non trivial, we obtain that each V_i has dimension r and that $V_i \cap V_j$ has at least dimension $r-1$. So a linear algebra exercise gives that there are two possibilities: either the intersection of the V_i has dimension $r-1$ or they are all included in a $(r+1)$-dimensional subspace. The first case leads to $\omega = dx_{r+2} \wedge \ldots \wedge dx_n \wedge \alpha$ where α is a linear 1-form in a good coordinates system; the second case leads to $\omega = \sum_i k_i dx_r \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$ where the k_i are linear functions. Then the integrability condition permits to give the precise forms of α and the k_i which leads respectively to type 1 and type 2.

Foliations corresponding to linear Nambu structures of type 2 are simply "suspension of phase portraits" of linear vector fields (X). In the type 1 case the foliation is given by x_i=constant, for $i = r+2, \ldots, n$ and $\alpha = 0$.

Remark. This last theorem gives, as a corollary, the classification of Lie algebras with coadjoint orbits of dimension at most 2 ([A-C-L]) and Nambu linear tensors of order $n-1$ (which appears, for example, in [M-V-V]).

7. Normal forms. A type 1 linear Nambu structure is called *regular* if we have $u = r$, $s = 0$; this means that it is isomorphic to $\Lambda = \sum_{j=1}^{r+1} \pm x_j \partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_{j-1} \wedge \partial/\partial x_{j+1} \wedge \ldots \wedge \partial/\partial x_{r+1}$. We will say that it is *elliptic* if the $\pm$ in that formula are all equal.

A regular type 1 linear Nambu structure with $r = 2$ corresponds to a Lie algebra which is a direct product of $sl(2)$ or $so(3)$ ($so(3)$ in the elliptic case) with a commutative one.

Theorem 4. *Let Λ be a Nambu tensor which vanishes at a point m (if it is a Poisson tensor we impose that its maximal rank is 2). If it admits a linear part at m which is* *type 1 and regular we can formally linearize it (we can linearize up to flat terms). If it is elliptic we can linearize up to a non-zero factor. If all the data are analytic we can linearize analytically up to a non-zero factor.*

The proof is a slight extension of the one we gave in [D-N] for the case of Nambu structures of order $r > 2$; it uses classical technics from the study of integrable forms.

A type 2 linear Nambu structure $\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_{r-1} \wedge X$ is called *regular* if the trace of X is different from 0. Remark that if $\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_{r-1} \wedge X$ and $\partial/\partial y_1 \wedge \ldots \wedge \partial/\partial y_{r-1} \wedge Y$ are equivalent, they are simutaneously regular or non-regular.

THEOREM 5. *Let Λ be a Nambu tensor which vanishes at a point m (if it is a Poisson tensor we impose that its maximum rank is 2). If it admits a linear part at m which is type 2 and regular, there are local coordinates which give to it the form $\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_{r-1} \wedge X$ where X is a vector field independent of $x_1, \ldots, x_{r-1}$.*

PROOF. First we will recall the BV-structure of the set of p-vectors ([K], [KS],...). It is an instance of the Ω trick which works as follows: choose a volume form Ω and consider the operator

$$D_\Omega : V_p \to V_{p-1}$$

defined by $D_\Omega := \sharp \circ d \circ \flat$ where $\sharp$ is the inverse of $\flat$ (defined in the introduction). Then ([K]) the Schouten bracket of the p-vector A and the q-vector B satisfies

$$[A, B] = (-1)^p (D_\Omega(A \wedge B) - (D_\Omega A) \wedge B - (-1)^p A \wedge (D_\Omega B)).$$

Now, if Λ is a Nambu tensor, $D_\Omega(\Lambda)$ will be called the *curl of* Λ (relatively to Ω). If Π is a Poisson structure, the curl $(D_\Omega(\Pi))$ of Π plays a fundamental role in the study of its singularities. To my knowledge, it was first introduced by C. Camacho and A. Lins Neto ([C-LN]) for the study of integrable 1-forms in dimension 3 (they use the converse of the Ω trick). Later we remark in [D-H] that, although there is no evident analog in the context of integrable 1-forms for higher dimensions, this curl can be defined for any Poisson structure, whatever the dimension is. It gives, in general, a non trivial infinitesimal isomorphism of the Poisson structure (unique up to addition of a Hamiltonian vector field). We used it in [D-H] and [D] to classify quadratic Poisson structures and in [D-Z] to obtain normal forms for 3-dimensional Poisson structures. In [W] A. Weinstein shows that this curl plays an important role even when there are no singularities; he also rebaptises it *modular vector field.*

The curl of a Nambu tensor of order r is a $(r-1)$-tensor. Using the local triviality theorem we prove the following lemma.

LEMMA. *Let Λ be a Nambu tensor and C its curl relatively to a fixed volume form. Then we have*

(i)- C is a Nambu tensor (a vector field if $r = 2$)
(ii)- $i_\gamma C \wedge \Lambda = 0$
(iii)- $[i_\gamma C, \Lambda] = 0$
for any $(r-2)$-form γ.

Note that we have only (iii), i.e. $[C, \Lambda] = 0$, in the Poisson case; (ii), i.e. $C \wedge \Lambda = 0$, works only if the maximal rank is 2.

To finish the proof of the theorem we proceed now as follows. We choose coordinates $y_1, \ldots, y_n$ such that $\Lambda^{(1)} = \partial/\partial y_1 \wedge \ldots \wedge \partial/\partial y_{r-1} \wedge Y$, ($Y$ linear) and choose volume form $\Omega = dy_1 \wedge \ldots \wedge dy_n$. Then the curl of Λ relatively to it has the form $C = C^{(0)} + C^{(1)} + \cdots$, where $C^{(k)}$ is the curl of $\Lambda^{(k+1)}$. Moreover the fact that Y has a non zero trace (regularity hypothesis) implies $C^{(0)} \neq 0$. So we obtain that C is different from zero at m and we can apply the local triviality theorem to get

$$C = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{r-1}}$$

in a new system of local coordinates. Then we can apply (ii) of the lemma to get $\Lambda = \partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_{r-1} \wedge X$ where X is a vector field. Finally we apply (iii) of the lemma to get that X is independent of $x_1, \ldots, x_{r-1}$. ■

This last theorem implies that the normal forms of Nambu tensors with regular type 2 linear parts reduce to normal forms for vector fields (see [Ma], [A-I]). For example, if the eigenvalues $\lambda_1, \ldots, \lambda_{n-r}$ of the linear part of the vector field X don't satisfy any "resonance relation"

$$\lambda_i = \sum_{j=1}^{n-r} n_j \lambda_j,$$

$i = 1, \ldots, n-r$, where n_j are non-negative integers then the Nambu tensor can be linearized.

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