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LINEARIZATION AND STAR PRODUCTS

VERONIQUE CHLOUP

Département de mathématiques, Université de Metz
Ile du Saulcy, F-57045 Metz Cedex, France
E-mail: chloup@poncelet.univ-metz.fr

Abstract. The aim of this paper is to give an overview concerning the problem of linearization of Poisson structures, more precisely we give results concerning Poisson-Lie groups and we apply those cohomological techniques to star products.

1. Introduction. The notion of Poisson structure appears in the study of Hamiltonien systems in mechanics. It generalizes the notion of symplectic structure. If the rank of a Poisson tensor Λ on a manifold M, is constant and equals the dimension of M then Λ is invertible and its inverse defines a symplectic structure on M. Darboux's theorem claims that two same dimensional symplectic manifolds are isomorphic. But there is no similar result for general Poisson structures.

First of all to study locally a Poisson structure we apply Weinstein's decomposition theorem [15]: locally the Poisson manifold (M, Λ) is isomorphic to a Poisson product of a symplectic manifold and a Poisson manifold which vanishes at one point. So the local study of Poisson structures reduces to the study of Poisson structures which vanish at one point.

A *Poisson structure* on a manifold M is a Lie bracket on $C^\infty(M)$, denoted by $\{\ ,\ \}$: $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ and satisfying the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}\ \forall f, g, h \in C^\infty(M).$$

In the local coordinates $(x^1, \ldots, x^n)$, we obtain:

$$\{f, g\} = \sum_{1 \le i,j \le n} \Lambda^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

where Λ is a skew symmetric contravariant two tensor field satisfying $[\Lambda, \Lambda] = 0$ ($[\ ,\ ]$ is

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the Schouten bracket). If Λ vanishes at e, we can write

$$\Lambda^{ij}(x) = \sum_{1\leq k\leq n} C_k^{ij} x^k + R^{ij}(x)$$

where R^{ij} are the non-linear terms in the Taylor series expansion of Λ^{ij} and where C_k^{ij} are the structure constants of a Lie algebra $\mathfrak{h}$ called the *linear approximation of the structure* (Weinstein [15]).

We may ask whether the Poisson structure Λ is isomorphic to its linear approximation in a neighborhood of e. If so we shall say that the Poisson structure Λ is *formally linearizable in a neighborhood of* e, i.e. there are new local coordinates $(y^1, \dots, y^n)$ in a neighborhood of e in which the expression of Λ is

$$\Lambda^{ij}(y) = \sum_{1\leq k\leq n} C_k^{ij} y^k.$$

We talk about *smooth* resp. *analytic* linearization if the change of coordinates is smooth resp. analytic.

V. Arnold was the first to prove that any Poisson structure whose linear approximation is the non-trivial two-dimensional Lie algebra is linearizable. In the formal case, A. Weinstein [15] proved that if the linear approximation is semi-simple then the structure is linearizable. Furthermore J. Conn [7] showed that if Λ is analytic, the linearization is analytic. Dufour [10] gave a counterexample of a three-dimensional solvable Lie algebra. In the case of a smooth Poisson structure, J. Conn [8] proved that if the linear approximation is semi-simple and of compact type then the linearization is smooth. A. Weinstein [16] exhibited examples of smooth and smoothly linearizable Poisson structures where the linear approximation is a semi-simple Lie algebra of non-compact type and of real rank at least two.

In the first part of this paper we explain how the problem of local linearization of a Poisson structure can be reformulated in cohomological terms. In particular this gives the proof of the result of A. Weinstein in the formal case. Then we will consider the particular case of Poisson-Lie groups this means that the tensor is multiplicative. In $\mathbb{R}^n$, any Poisson-Lie structure is linear and there exist some Poisson structures which are not linearizable. We give some results of linearization of Poisson-Lie structures which admit counterexamples in the general Poisson case. Finally we explain how those cohomological techniques can be used to find results concerning star products on the algebra of polynomials on the dual of a semi-simple Lie algebra.

2. Cohomology. The problem of formal linearization can be written in terms of cohomology. Denote by

$$\Lambda^{ij}(x) = \sum_{1\leq p\leq n} C_p^{ij} x^p + \Lambda_{(k)}^{ij}(x) + \Lambda_{(k+1)}^{ij}(x) + \dots$$

the formal series expansion of Λ^{ij} where $\Lambda_{(k)}^{ij}(x)$ is the homogeneous term of degree k in $x^1, \dots, x^n$ ($k \geq 2$). The idea for formal linearization is to eliminate the terms of degree k, i.e. to transform them into higher order terms by a change of coordinates of degree k. Let $y^i = x^i + f^i(x)$ where f^i are homogeneous polynomials of degree k. Consider

$\Lambda_{(k)} : \mathfrak{h} \wedge \mathfrak{h} \to S^k(\mathfrak{h})$ $(x^i \wedge x^j) \mapsto \Lambda^{ij}_{(k)}$. At order k, the relation $[\Lambda, \Lambda] = 0$ can be written $\partial_2 \Lambda_{(k)} = 0$ where ∂_2 is the Chevalley cohomological operator of $\mathfrak{h}$ associated to the adjoint representation of $\mathfrak{h}$ on $S(\mathfrak{h})$:

$$(\partial_2(f))(x \wedge y \wedge z) = \underset{x,y,z}{S} \Big(ad(x) f(y \wedge z) + f(x, [y, z]) \Big).$$

Thus $\Lambda_{(k)}$ is a 2-cocycle. Consider $f_{(k)} : \mathfrak{h} \to S^k(\mathfrak{h})$ $x^i \mapsto f^i_{(k)}$. Then $(\{y^i, y^j\} - \sum_k C^{ij}_k y^k)$ is of order strictly higher then k if and only if $(\Lambda_{(k)} + \partial_1 f_{(k)})(x^i, x^j) = 0$, i.e. Λ is a coboundary. So, if the 2-cocycle $\Lambda_{(k)}$ is a coboundary, we transform the terms of degree k in terms of degree $\geq k+1$. By iteration, we obtain a sequence $\{x^i_\nu\}$ which in the limit determines a system of local formal coordinates $x^i_\infty = x^i + g^i_\infty(x)$ in which Λ is linear. In particular, Λ is formally linearizable if $H^2(\mathfrak{h}, S(\mathfrak{h})) = 0$. For example if $\mathfrak{h}$ is semi-simple or if $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathbb{R}$ where $\mathfrak{g}_1$ is semi-simple (in this case the linearization is analytic; see Molinier [13]).

3. Poisson-Lie groups. Now we consider the case of Poisson-Lie structures. All the results mentioned here are in V. Chloup-Arnould [3].

Let G be a Lie group and P a Poisson tensor on G. (G, P) is a *Poisson-Lie group* (Drinfeld [9]) if P is multiplicative: $P(xy) = L_{x*}P(y) + R_{y*}P(x)$. In particular $P(e) = 0$.

The problem of the linearization of Poisson-Lie structure is a particular case of the problem of the linearization of Poisson structures. In the Poisson-Lie case the structure is analytic and is totally determined by its linear part without always being linearizable; see M. Cahen, S. Gutt and J. Rawnsley [6].

Let $\mathfrak{g}$ be a Lie algebra. $(\mathfrak{g}, \mathfrak{g}^*)$ (also denoted $(\mathfrak{g}, p)$) is a *Lie bialgebra* if there exists a Lie algebra structure on $\mathfrak{g}^*$ such that its dual map, denoted by p, is a cocycle on $\mathfrak{g}$, i.e. $p : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ satisfies: $p([X, Y]) = [p(X), Y] - [p(Y), X]$.

Let (G, P) be a Poisson-Lie group, then $P_r(x) = R_x^{-1}{}_* P(x)$ is a cocycle for the representation Ad, i.e. $P_r(xy) = P_r(x) + Adx P_r(y)$. And $p = P_{r*e}$ defines a Lie bialgebra structure on $\mathfrak{g}$ the Lie algebra of the Lie group G. Reciprocally if $(\mathfrak{g}, p)$ is a Lie bialgebra, then (G, P) is a Poisson-Lie group, where G is the connected and simply connected Lie group of Lie algebra $\mathfrak{g}$ and where P is given in a neighborhood of e by:

$$P(expX) = R_{expX*} \frac{e^{adX} - 1}{adX} p(X).$$

Thus the structure is determined by its linear approximation $\mathfrak{h} = \mathfrak{g}^*$.

We generalize the result of J. Conn [7] to the case where the Lie algebra $\mathfrak{h} = \mathfrak{g}^*$ is reductive:

THEOREM 1. *Let G be a connected and simply connected Lie group, of Lie algebra $\mathfrak{g}$. Let P a Poisson-Lie tensor on G such that $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, has the form $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \mathfrak{R}$ direct product where $\mathfrak{R}$ is an abelian ideal and where $\mathfrak{g}_1$ is a semi-simple Lie algebra. Then P is analytically linearizable.*

This theorem is not satisfied for some general Poisson (not Poisson-Lie) structures which have the same linear approximation $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{R}$. Let P be a Poisson tensor on

$\mathbb{R}^{n+r}$ (where $n = dim\mathfrak{g}_1$ and $r = dim\mathfrak{R}$, $r > 2$), vanishing at 0 and such that:

$$P^{ij}(x) = \sum_{1\le k\le n} C_k^{ij} x^k + B^{ij}$$

where C_k^{ij} are the constants of structure of $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{R}$ in the basis $\langle e_1, \ldots, e_{n+r}\rangle$ where $e_i \in \mathfrak{g}_1$, $\forall i \le n$ et $e_j \in \mathfrak{R}$, $\forall j > n$; and where B^{ij} are the non-linear terms of P^{ij}, and we assume that $B^{ij} = 0$ for i or $j \le n$.

Then if B is non-zero, the maximal dimension of the symplectic leaves for the linear structure is different from the maximal dimension of the symplectic leaves for P, so P is not linearizable.

THEOREM 2. *Let G be a connected and simply connected Lie group, of Lie algebra $\mathfrak{g}$. Let P be a Poisson-Lie tensor on G such that $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, has the form $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \mathfrak{R}$ (direct product) where $\mathfrak{R}$ is an r-dimensional ideal and where $\mathfrak{g}_1$ is an n-dimensional semi-simple Lie algebra. Then $P = L \oplus T$ where L is the linear Poisson-Lie tensor given by constants of structure of $\mathfrak{g}_1$ and*

$$T(x^1, \ldots, x^{n+r}) = \sum_{n+1\le i,j\le n+r} T^{ij}(x^{n+1}, \ldots, x^{n+r}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

This theorem is not satisfied for some general Poisson (not Poisson-Lie) structures which have the same linear approximation $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{R}$. We have the following result:

PROPOSITION 3. *Let Λ be a Poisson structure on a manifold M, vanishing at a point z and such that its linear approximation at z is $\mathfrak{h} = \mathfrak{g}_1 \ltimes \mathfrak{R}$ the semi-direct product of an n-dimensional semi-simple Lie algebra and of an r-dimensional ideal. Then there exist some formal coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^r)$ in a neighborhood of z such that:*

$$\begin{aligned}
\{x^i, x^j\} &= \sum_{1\le k\le n} C_k^{ij} x^k, \\
\{x^i, y^\alpha\} &= \sum_{1\le \gamma\le r} C_\gamma^{i\alpha} y^\gamma, \\
\{y^\alpha, y^\gamma\} &= \sum_{1\le \nu\le r} C_\nu^{\alpha\gamma} y^\nu + R^{\alpha\gamma}(x, y),
\end{aligned}$$

where C_k^{ij} are the constants of structure of $\mathfrak{h}$ and where $R^{\alpha\gamma}$ are series in x and y of order at least two.

4. Star products. We apply the same cohomological techniques to obtain results concerning star products on the space of polynomials on the dual of a semi-simple Lie algebra (Chloup-Arnould [5]).

The notion of star product on the space N of functions on a Poisson manifold (M, P) has been introduced by Bayen et al. [1]. More generally, N can be a subspace of smooth functions on M which is stable under pointwise multiplication and under the Poisson bracket. Here we shall consider the case where M is the dual $\mathfrak{g}^*$ of a semi-simple Lie algebra $\mathfrak{g}$ with its Lie-Poisson tensor P and where N is the space of polynomials on $\mathfrak{g}^*$ which can be identified with $S(\mathfrak{g})$.

A *formal star product* on the space N is a bilinear map

$$N \times N \to N[[\nu]], \qquad (u,v) \mapsto u * v = uv + \nu P(du,dv) + \sum_{r\geq 2} \nu^r C_r(u,v),$$

which is associative when extended ν-linearly to $N[[\nu]] \times N[[\nu]]$, where $N[[\nu]]$ is the space of formal power series in the parameter ν with coefficients in N and where the C_r are bidifferential operators vanishing on constants.

Two formal star products $*_1$ and $*_2$ defined on N are said to be *formally equivalent* if there is a series $T = Id + \sum_{r=1}^{\infty} \nu^r T_r$ where the T_r are linear operators on N, such that $u *_2 v = T^{-1}(Tu *_1 Tv)$.

Let P be a Poisson tensor on M. A contravariant skew symmetric 2-tensor B on M is said to be P-*closed* if $[P,B] = 0$ where $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket.

It is said to be P-*exact* if there exists a contravariant 1-tensor E such that $B = [P,E]$.

We consider skew symmetric contravariant p-tensors on $\mathfrak{g}^*$ whose coefficients are polynomials, they are p-forms on $\mathfrak{g}$ with values in $S(\mathfrak{g})$. We call them *polynomial* contravariant p-tensors. Using cohomology, we obtain:

PROPOSITION 4. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $M = \mathfrak{g}^*$ the manifold with the Lie-Poisson bracket, denoted by P. Then any P-closed polynomial contravariant skew symmetric 2-tensor on M is P-exact and is the coboundary of a polynomial contravariant 1-tensor on $\mathfrak{g}^*$.*

This result with the two following propositions gives Theorem 7.

PROPOSITION 5 ([11], [14]). *Any 2-cocycle C for the Hochschild cohomology on the space $N = S(\mathfrak{g})$ can be written as*

$$C(u,v) = \delta F(u,v) + B(u,v),$$

where δ is the Hochschild coboundary operator, F is a 1-cochain and B is a polynomial skew symmetric contravariant 2-tensor.

PROPOSITION 6 ([2]). *If $u *_\nu v$ and $u *'_\nu v$ are two star products which coincide up to order k, the skew symmetric part of their difference at order $k+1$ yields a P-closed skew symmetric contravariant 2-tensor on M. If that closed 2-tensor is P-exact and is the coboundary of a polynomial contravariant 1-tensor, then there is a star product $u *''_\nu v$, equivalent to $u *'_\nu v$, such that $u *''_\nu v$ and $u *_\nu v$ coincide up to order $k+1$.*

We get:

THEOREM 7. *Any star product on the space $S(\mathfrak{g})$ of polynomials on the dual $\mathfrak{g}^*$ of a semi-simple Lie algebra $\mathfrak{g}$ is equivalent to the standard one:*

$$R_r *_\nu Q_q = \sum_{s=0}^{r+q} (2\nu)^s (\sigma(R) \circ \sigma(Q))_{r+q-s}$$

where

$$\sigma : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}), \qquad X_1 \ldots X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} X_{\sigma(1)} \circ \ldots \circ X_{\sigma(r)},$$

and where an element $u \in U(\mathfrak{g})$ *is written* $u = \sum_n u_n$ *relative to the decomposition* $U(\mathfrak{g}) = \oplus\sigma(S^n(\mathfrak{g}))$.

This result can be found in M. Kontsevich [12] in a more general framework.

We can also prove:

THEOREM 8. *Let* $*$ *be a star product on the space* $S(\mathfrak{g})$ *of polynomials on the dual* $\mathfrak{g}^*$ *of a semi-simple Lie algebra* $\mathfrak{g}$. *Then any* ν*-linear derivation of* $*$ *is essentially inner.*

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