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SCHWARZIAN DERIVATIVE RELATED TO MODULES OF DIFFERENTIAL OPERATORS ON A LOCALLY PROJECTIVE MANIFOLD

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Abstract. We introduce a 1-cocycle on the group of diffeomorphisms $\mathrm{Diff}(M)$ of a smooth manifold M endowed with a projective connection. This cocycle represents a nontrivial cohomology class of $\mathrm{Diff}(M)$ related to the $\mathrm{Diff}(M)$-modules of second order linear differential operators on M. In the one-dimensional case, this cocycle coincides with the Schwarzian derivative, while, in the multi-dimensional case, it represents its natural and new generalization. This work is a continuation of [3] where the same problems have been treated in the one-dimensional case.

1. Introduction

1.1. The classical Schwarzian derivative. Consider the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle preserving its orientation. Identifying S^1 with $\mathbb{R}P^1$, fix an affine parameter x on S^1 such that the natural $\mathrm{PSL}(2,\mathbb{R})$-action is given by the linear-fractional transformations:

$$x \to \frac{ax+b}{cx+d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{1.1}$$

The classical Schwarzian derivative is then given by:

$$S(f) = \left(\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right)(dx)^2, \tag{1.2}$$

where $f \in \mathrm{Diff}(S^1)$.

1.2. The Schwarzian derivative as a 1-cocycle. It is well known that the Schwarzian derivative can be intrinsically defined as the *unique 1-cocycle* on $\mathrm{Diff}(S^1)$ with values in the space of quadratic differentials on S^1, *equivariant with respect to the Möbius group*

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$\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}(S^1)$, cf. [2, 6]. That means, the map (1.2) satisfies the following two conditions:

$$S(f \circ g) = g^* S(f) + S(g), \tag{1.3}$$

where f^* is the natural $\mathrm{Diff}(S^1)$-action on the space of quadratic differentials and

$$S(f) = S(g), \qquad g(x) = \frac{af(x)+b}{cf(x)+d}. \tag{1.4}$$

Moreover, the Schwarzian derivative is characterized by (1.3) and (1.4).

1.3. Relation to the module of second order differential operators. The Schwarzian derivative appeared in the classical literature in closed relation with differential operators. More precisely, consider the space of Sturm-Liouville operators: $A_u = -2\left(\frac{d}{dx}\right)^2 + u(x)$, where $u(x) \in C^\infty(S^1)$, the action of $\mathrm{Diff}(S^1)$ on this space is given by $f(A_u) = A_v$ with

$$v = u \circ f^{-1} \cdot (f^{-1\prime})^2 + \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \tag{1.5}$$

(see e.g. [16]).

It, therefore, seems to be clear that the natural approach to understanding of multi-dimensional analogues of the Schwarzian derivative should be based on the relation with modules of differential operators.

1.4. The contents of this paper. In this paper we introduce a multi-dimensional analogue of the Schwarzian derivative related to the $\mathrm{Diff}(M)$-modules of differential operators on M.

Following [4] and [10], the module of differential operators $\mathcal{D}_{\lambda,\mu}$ will be viewed as a *deformation* of the module of symmetric contravariant tensor fields on M. This approach leads to $\mathrm{Diff}(M)$-cohomology first evoked in [4]. The corresponding cohomology of the Lie algebra of vector fields $\mathrm{Vect}(M)$ has been calculated in [10] for a manifold M endowed with a flat projective structure. We use these results to determine the projectively equivariant cohomology of $\mathrm{Diff}(M)$ arising in this context.

Note that multi-dimensional analogues of the Schwarzian derivative is a subject already considered in the literature. We will refer [1, 7, 11, 12, 13, 15, 14] for various versions of multi-dimensional Schwarzians in projective, conformal, symplectic and non-commutative geometry.

2. Projective connections. Let M be a smooth (or complex) manifold of dimension n. There exists a notion of projective connection on M, due to E. Cartan. Let us recall here the simplest (and naive) way to define a projective connection as an equivalence class of standard (affine) connections.

2.1. Symbols of projective connections

Definition. A *projective connection* on M is the class of affine connections corresponding to the same expressions

$$\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1}\left(\delta^k_i \Gamma^l_{jl} + \delta^k_j \Gamma^l_{il}\right), \tag{2.1}$$

where Γ^k_{ij} are the Christoffel symbols and we have assumed a summation over repeated indices.

The symbols (2.1) naturally appear if one considers projective connections as a particular case of so-called Cartan normal connection, see [8].

Remarks. (a) The definition is correct (i.e. does not depend on the choice of local coordinates on M).

(b) The formula (2.1) defines a natural projection to the space of trace-less (2,1)-tensors, one has: $\Pi^k_{ik} = 0$.

2.2. Flat projective connections and projective structures. A manifold M is said to be locally projective (or endowed with a *flat projective structure*) if there exists an atlas on M with linear-fractional coordinate changes:

$$x^i = \frac{a^i_j x^j + b^i}{c_j x^j + d}. \tag{2.2}$$

A projective connection on M is called *flat* if in a neighborhood of each point, there exists a local coordinate system $(x^1, \ldots, x^n)$ such that the symbols Π^k_{ij} are identically zero (see [8] for a geometric definition). Every flat projective connection defines a projective structure on M.

2.3. A projectively invariant 1-cocycle on $\mathrm{Diff}(M)$. A common way of producing nontrivial cocycles on $\mathrm{Diff}(M)$ using affine connections on M is as follows. The map: $(f^*\Gamma)^k_{ij} - \Gamma^k_{ij}$ is a 1-cocycle on $\mathrm{Diff}(M)$ with values in the space of symmetric $(2,1)$-tensor fields. It is, therefore, clear that a projective connection on M leads to the following 1-cocycle on $\mathrm{Diff}(M)$:

$$\ell(f) = \left((f^*\Pi)^k_{ij} - \Pi^k_{ij}\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{2.3}$$

vanishing on (locally) projective diffeomorphisms.

Remarks. (a) The expression (2.3) is well defined (does not depend on the choice of local coordinates). This follows from a well-known fact that the difference of two (projective) connections defines a $(2,1)$-tensor field.

(b) Already the formula (2.3) implies that the map $f \mapsto \ell(f)$ is, indeed, a 1-cocycle, that is, it satisfies the relation $\ell(f \circ g) = g^*\ell(f) + \ell(g)$.

(c) It is clear that the cocycle ℓ is nontrivial (cf. [10]), otherwise it would depend only on the first jet of the diffeomorphism f. Note that the formula (2.3) looks as a coboundary, however, the symbols Π^k_{ij} do not transform as components of a $(2,1)$-tensor field (but as symbols of a projective connection).

Example. In the case of a smooth manifold endowed with a *flat* projective connection, (with symbols (2.1) identically zero) or, equivalently, with a projective structure, the cocycle (2.3) obviously takes the form:

$$\ell(f,x) = \left(\frac{\partial^2 f^l}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial f^l} - \frac{1}{n+1}\left(\delta^k_j \frac{\partial \log J_f}{\partial x^i} + \delta^k_i \frac{\partial \log J_f}{\partial x^j}\right)\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \tag{2.4}$$

where $f(x^1, \ldots, x^n) = \left(f^1(x), \ldots, f^n(x)\right)$ and $J_f = \det\left(\frac{\partial f^i}{\partial x^j}\right)$ is the Jacobian. This expression is globally defined and vanishes if f is given (in the local coordinates of the projective structure) as a linear-fractional transformation (2.2).

The cocycle (2.3,2.4) was introduced in [15, 11] as a multi-dimensional projective analogue of the Schwarzian derivative. However, in contradistinction with the Schwarzian derivative (1.2), this map (2.4) depends on the second-order jets of diffeomorphisms. Moreover, in the one-dimensional case ($n = 1$), the expression (2.3,2.4) is identically zero.

3. Introducing the Schwarzian derivative. Assume that $\dim M \geq 2$. Let $\mathcal{S}^k(M)$ (or $\mathcal{S}^k$ for short) be the space of k-th order symmetric contravariant tensor fields on M.

3.1. Operator symbols of a projective connection. For an arbitrary system of local coordinates fix the following linear differential operator $T : \mathcal{S}^2 \to C^\infty(M)$ given for every $a \in \mathcal{S}^2$ by $T(a) = T_{ij}(a^{ij})$ with

$$T_{ij} = \Pi^k_{ij} \frac{\partial}{\partial x^k} - \frac{2}{n-1} \left(\frac{\partial \Pi^k_{ij}}{\partial x^k} - \frac{n+1}{2} \Pi^k_{il} \Pi^l_{kj} \right), \tag{3.1}$$

where Π^k_{ij} are the symbols of a projective connection (2.1) on M.

It is clear that the differential operator (3.1) is not intrinsically defined, indeed, already its principal symbol, Π^k_{ij}, is not a tensor field. In the same spirit that the difference of two projective connections $\widetilde{\Pi}^k_{ij} - \Pi^k_{ij}$ is a well-defined tensor field, we have the following

THEOREM 3.1. *Given arbitrary projective connections $\widetilde{\Pi}^k_{ij}$ and Π^k_{ij}, the difference*

$$\mathcal{T} = \widetilde{T} - T \tag{3.2}$$

is a linear differential operator from $\mathcal{S}^2$ to $C^\infty(M)$ well defined (globally) on M (i.e., it does not depend on the choice of local coordinates).

PROOF. To prove that the expression (3.2) is, indeed a well-defined differential operator from $\mathcal{S}^2$ into $C^\infty(M)$, we need an explicit formula of coordinate transformation for such kind of operators.

LEMMA 3.2. *The coefficients of a first-order linear differential operator $A : \mathcal{S}^2 \to C^\infty(M)$ $A(a) = \left(t^k_{ij}\partial_k + u_{ij}\right) a^{ij}$ transform under coordinate changes as follows:*

$$t^k_{ij}(y) \quad = \quad t^c_{ab}(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \frac{\partial y^k}{\partial x^c} \tag{3.3}$$

$$u_{ij}(y) \quad = \quad u_{ab}(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} - 2 t^c_{ab}(x) \frac{\partial^2 y^k}{\partial x^c \partial x^l} \frac{\partial x^a}{\partial y^k} \frac{\partial x^b}{\partial y^{(i}} \frac{\partial x^l}{\partial y^{j)}} \tag{3.4}$$

where round brackets mean symmetrization.

PROOF OF THE LEMMA: straightforward. ∎

Consider the following expression:

$$\mathcal{T}(\alpha, \beta)_{ij} = (\widetilde{\Pi}^k_{ij} - \Pi^k_{ij})\partial_k + \alpha \partial_k (\widetilde{\Pi}^k_{ij} - \Pi^k_{ij}) + \beta (\widetilde{\Pi}^k_{li} \widetilde{\Pi}^l_{jk} - \Pi^k_{li} \Pi^l_{jk})$$

From the definition (3.1,3.2) for

$$\alpha = -\frac{2}{n-1}, \qquad \beta = \frac{n+1}{n-1}, \tag{3.5}$$

one gets $\mathcal{T}(\alpha,\beta) = \mathcal{T}$.

Now, it follows immediately from the fact that $\widetilde{\Pi}^k_{ij} - \Pi^k_{ij}$ is a well-defined $(2,1)$-tensor field on M, that the condition (3.3) for the principal symbol of $\mathcal{T}(\alpha,\beta)$ is satisfied.

The transformation law for the symbols of a projective connection reads:

$$\Pi^k_{ij}(y) = \Pi^c_{ab}(x)\,\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial y^k}{\partial x^c} + \ell(y,x),$$

where $\ell(y,x)$ is given by (2.4). Let $u(\alpha,\beta)_{ij}$ be the zero-order term in $\mathcal{T}(\alpha,\beta)_{ij}$, one readily gets:

$$\begin{aligned} u(\alpha,\beta)(y)_{ij} \;&=\; u(\alpha,\beta)(x)_{ab}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} \\ &\quad -2(\alpha+\beta)\,(\widetilde{\Pi}^c_{ab}(x) - \Pi^c_{ab}(x))\frac{\partial^2 y^k}{\partial x^c \partial x^l}\frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^{(i}}\frac{\partial x^l}{\partial y^{j)}} \\ &\quad +(\alpha + \frac{2\beta}{n+1})(\widetilde{\Pi}^c_{ab}(x) - \Pi^c_{ab}(x))\frac{\partial \log J_y}{\partial x^c}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}. \end{aligned}$$

The transformation law (3.4) for $u(\alpha,\beta)_{ij}$ is satisfied if and only if α and β are given by (3.5). Theorem 3.1 is proven. ■

We call T_{ij} given by (3.1) the *operator symbols* of a projective connection. This notion is the main tool of this paper.

Remark. The scalar term of (3.1) looks similar to the symbols $\Pi_{ij} = -\partial \Pi^k_{ij}/\partial x^k + \Pi^k_{il}\Pi^l_{kj}$, which together with Π^k_{ij} characterize the normal Cartan projective connection (see [8]). We will show that the operator symbols T_{ij}, and not the symbols of the normal projective connection, lead to a natural notion of multi-dimensional Schwarzian derivative.

3.2. The main definition. Consider a manifold M endowed with a projective connection. The expression

$$S(f) = f^*(T) - T, \tag{3.6}$$

where T is the (locally defined) operator (3.1), is a linear differential operator well defined (globally) on M.

Proposition 3.3. *The map $f \mapsto S(f)$ is a nontrivial 1-cocycle on* $\mathrm{Diff}(M)$ *with values in* $\mathrm{Hom}(\mathcal{S}^2, C^\infty(M))$.

Proof. The cocycle property for $S(f)$ follows directly from the definition (3.6). This cocycle is not a coboundary. Indeed, every coboundary $\mathrm{d}B$ on $\mathrm{Diff}(M)$ with values in the space $\mathrm{Hom}(\mathcal{S}^2, C^\infty(M))$ is of the form $B(f)(a) = f^*(B) - B$, where $B \in \mathrm{Hom}(\mathcal{S}^2, C^\infty(M))$. Since $S(f)$ is a first-order differential operator, the coboundary condition $S = \mathrm{d}B$ would imply that B is also a first-order differential operator and so, $\mathrm{d}B$ depends at most on the second jet of f. But, $S(f)$ depends on the third jet of f. This contradiction proves that the cocycle (3.6) is nontrivial. ■

The cocycle (3.6) will be called the *projectively equivariant Schwarzian derivative.* It is clear that the kernel of S is precisely the subgroup of $\mathrm{Diff}(M)$ preserving the projective connection.

EXAMPLE. In the projectively flat case, $\Pi^k_{ij} \equiv 0$, the cocycle (3.6) takes the form:

$$S(f)_{ij} = \ell(f)^k_{ij} \frac{\partial}{\partial x^k} - \frac{2}{n-1} \frac{\partial}{\partial x^k} \left(\ell(f)^k_{ij}\right) + \frac{n+1}{n-1} \ell(f)^k_{im} \ell(f)^m_{kj}, \tag{3.7}$$

where $\ell(f)^k_{ij}$ are the components of the cocycle (2.3) with values in symmetric (2,1)-tensor fields. The cocycle (3.7) vanishes if and only if f is a linear-fractional transformation.

It is easy to compute this expression in local coordinates:

$$S(f)_{ij} = \ell(f)^k_{ij} \frac{\partial}{\partial x^k} + \frac{\partial^3 f^k}{\partial x^i \partial x^j \partial x^l} \frac{\partial x^l}{\partial f^k} - \frac{n+3}{n+1} \frac{\partial^2 J_f}{\partial x^i \partial x^j} J_f^{-1} + \frac{n+2}{n+1} \frac{\partial J_f}{\partial x^i} \frac{\partial J_f}{\partial x^j} J_f^{-2}. \tag{3.8}$$

To obtain this formula from (3.7), one uses the relation:

$$\frac{\partial^3 f^k}{\partial x^i \partial x^j \partial x^l} \frac{\partial x^l}{\partial f^k} - \frac{\partial^2 f^k}{\partial x^i \partial x^m} \frac{\partial^2 f^l}{\partial x^j \partial x^s} \frac{\partial x^m}{\partial f^l} \frac{\partial x^s}{\partial f^k} = \frac{\partial^2 J_f}{\partial x^i \partial x^j} J_f^{-1} + \frac{\partial J_f}{\partial x^i} \frac{\partial J_f}{\partial x^j} J_f^{-2}.$$

We observe that, in the one-dimensional case ($n = 1$), the expression (3.8) is precisely $-S(f)$, where S is the classical Schwarzian derivative. (Recall that in this case $\ell(f) \equiv 0$.)

REMARKS. (a) The infinitesimal analogue of the cocycle (3.7) has been introduced in [10].

(b) We will show in Section 4.3, that the analogue of the operator (3.6) in the one-dimensional case, is, in fact, the operator of multiplication by the Schwarzian derivative.

3.3. A remark on the projectively equivariant cohomology. Consider the standard $\mathrm{sl}(n+1,\mathbb{R})$-action on $\mathbb{R}^n$ (by infinitesimal projective transformations). The first group of differential cohomology of $\mathrm{Vect}(\mathbb{R}^n)$, vanishing on the subalgebra $\mathrm{sl}(n+1,\mathbb{R})$, with coefficients in the space $\mathcal{D}(\mathcal{S}^k,\mathcal{S}^\ell)$ of linear differential operators from $\mathcal{S}^k$ to $\mathcal{S}^\ell$, was calculated in [10]. For $n \geq 2$ the result is as follows:

$$H^1(\mathrm{Vect}(\mathbb{R}^n), \mathrm{sl}(n+1,\mathbb{R}); \mathcal{D}(\mathcal{S}^k,\mathcal{S}^\ell)) = \begin{cases} \mathbb{R}, & k-\ell = 2, \\ \mathbb{R}, & k-\ell = 1, \ell \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

The cocycle (3.7) is, in fact, corresponds to the nontrivial cohomology class in the case $k = 2, \ell = 0$ integrated to the group $\mathrm{Diff}(\mathbb{R}^n)$, while the nontrivial cohomology class in the case $k - \ell = 1$ is given by the operator of contraction with the tensor field (2.4).

For any locally projective manifold M it follows that the cocycle (3.6) generates the unique nontrivial class of the cohomology of $\mathrm{Diff}(M)$ with coefficients in $\mathcal{D}(\mathcal{S}^2, C^\infty(M))$, vanishing on the (pseudo)group of (locally defined) projective transformations. The same fact is true for the cocycle (2.3).

4. Relation to the modules of differential operators. Consider, for simplicity, a smooth oriented manifold M. Denote $\mathcal{D}(M)$ the space of scalar linear differential operators $A : C^\infty(M) \to C^\infty(M)$. There exists a two-parameter family of $\mathrm{Diff}(M)$-module structures on $\mathcal{D}(M)$. To define it, one identifies the arguments of differential operators with tensor densities on M of degree λ and their values with tensor densities on M of degree μ.

4.1. Differential operators acting on tensor densities. Consider the the space $\mathcal{F}_\lambda$ of *tensor densities* on M, that mean, of sections of the line bundle $(\Lambda^n T^*M)^\lambda$. It is clear that $\mathcal{F}_\lambda$ is naturally a $\mathrm{Diff}(M)$-module. Since M is oriented, $\mathcal{F}_\lambda$ can be identified with $C^\infty(M)$ as a vector space. The $\mathrm{Diff}(M)$-module structures are, however, different.

Definition. We consider the differential operators acting on tensor densities, namely,

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu. \tag{4.1}$$

The $\mathrm{Diff}(M)$-action on $\mathcal{D}(M)$, depending on two parameters λ and μ, is defined by the usual formula:

$$f_{\lambda,\mu}(A) = {f^*}^{-1} \circ A \circ f^*, \tag{4.2}$$

where f^* is the natural $\mathrm{Diff}(M)$-action on $\mathcal{F}_\lambda$.

Notation. The $\mathrm{Diff}(M)$-module of differential operators on M with the action (4.2) is denoted $\mathcal{D}_{\lambda,\mu}$. For every k, the space of differential operators of order $\leq k$ is a $\mathrm{Diff}(M)$-submodule of $\mathcal{D}_{\lambda,\mu}$, denoted $\mathcal{D}^k_{\lambda,\mu}$.

In this paper we will only deal with the special case $\lambda = \mu$ and use the notation $\mathcal{D}_\lambda$ for $\mathcal{D}_{\lambda,\lambda}$ and f_λ for $f_{\lambda,\lambda}$.

The modules $\mathcal{D}_{\lambda,\mu}$ have already been considered in classical works (see [16]) and systematically studied in a series of recent papers (see [4, 9, 10, 3, 5] and references therein).

4.2. Projectively equivariant symbol map. From now on, we suppose that the manifold M is endowed with a projective structure. It was shown in [10] that there exists a (unique up to normalization) *projectively equivariant symbol map*, that is, a linear bijection σ_λ identifying the space $\mathcal{D}(M)$ with the space of symmetric contravariant tensor fields on M.

Let us give here the explicit formula of σ_λ in the case of second order differential operators. In coordinates of the projective structure, σ_λ associates to a differential operator

$$A = a_2^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + a_1^i \frac{\partial}{\partial x^i} + a_0, \tag{4.3}$$

where $a_\ell^{i_1 \dots i_\ell} \in C^\infty(M)$ with $\ell = 0, 1, 2$, the tensor field:

$$\sigma_\lambda(A) = \bar{a}_2^{ij} \partial_i \otimes \partial_j + \bar{a}_1^i \partial_i + \bar{a}_0, \tag{4.4}$$

and is given by

$$\begin{aligned} \bar{a}_2^{ij} &= a_2^{ij} \\ \bar{a}_1^i &= a_1^i - 2 \frac{(n+1)\lambda + 1}{n+3} \frac{\partial a_2^{ij}}{\partial x^j} \\ \bar{a}_0 &= a_0 - \lambda \frac{\partial a_1^i}{\partial x^i} + \lambda \frac{(n+1)\lambda + 1}{n+2} \frac{\partial^2 a_2^{ij}}{\partial x^i \partial x^j} \end{aligned} \tag{4.5}$$

The main property of the symbol map σ_λ is that it commutes with (locally defined) $\mathrm{SL}(n+1, \mathbb{R})$-action. In other words, the formula (4.5) does not change under linear-fractional coordinate changes (2.2).

4.3. Diff(M)*-module of second order differential operators.* In this section we will compute the Diff(M)-action f_λ given by (4.2) with $\lambda = \mu$ on the space $\mathcal{D}^2_\lambda$ (of second order differential operators (4.3) acting on λ-densities).

Let us give here the explicit formula of Diff(M)-action in terms of the projectively invariant symbol σ^λ. Namely, we are looking for the operator $\bar{f}_\lambda = \sigma_\lambda \circ f_\lambda \circ (\sigma_\lambda)^{-1}$ (such that the diagram below is commutative):

$$
\begin{tikzcd}
\mathcal{D}^2_\lambda \arrow[r, "f_\lambda"] \arrow[d, "\sigma_\lambda"'] & \mathcal{D}^2_\lambda \arrow[d, "\sigma_\lambda"] \\
\mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0 \arrow[r, "\bar{f}_\lambda"] & \mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0
\end{tikzcd}
\tag{4.6}
$$

where $\mathcal{S}^2 \oplus \mathcal{S}^1 \oplus \mathcal{S}^0$ is the space of second-order contravariant tensor fields (4.4) on M.

The following statement, whose proof is straightforward, shows how the cocycles (2.3) and (3.6) are related to the module of second-order differential operators.

PROPOSITION 4.1. *If* $\dim M \geq 2$*, the action of* Diff(M) *on the space of the space* $\mathcal{D}^2_\lambda$ *of second-order differential operators, defined by (4.2,4.6) is as follows:*

$$
\begin{array}{rcl}
\left(\bar{f}_\lambda \bar{a}_2\right)^{ij} & = & (f^*\bar{a}_2)^{ij} \\
\left(\bar{f}_\lambda \bar{a}_1\right)^{i} & = & (f^*\bar{a}_1)^{i} \; + \; (2\lambda - 1)\, \dfrac{n+1}{n+3}\, \ell^i_{kl}(f^{-1})(f^*\bar{a}_2)^{kl} \\
\bar{f}_\lambda \bar{a}_0 & = & f^*\bar{a}_0 \; - \; \dfrac{2\lambda(\lambda-1)}{n+2}\, S_{kl}(f^{-1})(f^*\bar{a}_2)^{kl}
\end{array}
\tag{4.7}
$$

where f^* is the natural action of f on the symmetric contravariant tensor fields.

REMARK. In the one-dimensional case, the formula (4.7) holds true, recall that $\ell(f) \equiv 0$ and $S_{kl}(f^{-1})(f^*\bar{a}_2)^{kl} = S(f^{-1})f^*\bar{a}_2$ with the operator of multiplication by the classical Schwarzian derivative in the right hand side (cf. [3]). This shows that the cocycle (3.6) is, indeed, its natural generalization.

Note also that the formula (1.5) is a particular case of (4.7).

4.4. Module of differential operators as a deformation. The space of differential operators $\mathcal{D}^2_\lambda$ as a module over the Lie algebra of vector fields Vect(M) was first studied in [4], it was shown that this module can be naturally considered as a deformation of the module of tensor fields on M. Proposition 4.1 extends this result to the level of the diffeomorphism group Diff(M). The formula (4.7) shows that the Diff(M)-module of second order differential operators on M $\mathcal{D}^2_\lambda$ is a *nontrivial deformation* of the module of tensor fields $\mathcal{T}^2$ generated by the cocycles (2.3) and (3.6).

In the one-dimensional case, the Diff(S^1)-modules of differential operators and the related higher order analogues of the Schwarzian derivative was studied in [3].

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References

[1] L. V. Ahlfors, *Cross-ratios and Schwarzian derivatives in* $\mathbb{R}^n$, in: Complex Analysis, Birkhäuser, Boston, 1989.

[2] R. Bott, *On the characteristic classes of groups of diffeomorphisms*, Enseign. Math. 23 (1977), 209–220.

[3] S. Bouarroudj and V. Ovsienko, *Three cocycles on* $\mathrm{Diff}(S^1)$ *generalizing the Schwarzian derivative*, Internat. Math. Res. Notices 1998, No.1, 25–39.

[4] C. Duval and V. Ovsienko, *Space of second order linear differential operators as a module over the Lie algebra of vector fields*, Adv. in Math. 132 (1997), 316–333.

[5] C. Duval and V. Ovsienko, *Conformally equivariant quantization*, Preprint CPT, 1998.

[6] A. A. Kirillov, *Infinite dimensional Lie groups: their orbits, invariants and representations. The geometry of moments*, Lect. Notes in Math., 970 Springer-Verlag 1982, 101–123.

[7] S. Kobayashi and C. Horst, *Topics in complex differential geometry*, in: Complex Differential Geometry, Birkhäuser Verlag, 1983, 4–66.

[8] S. Kobayashi and T. Nagano, *On projective connections*, J. of Math. and Mech. 13 (1964), 215–235.

[9] P. B. A. Lecomte, P. Mathonet and E. Tousset, *Comparison of some modules of the Lie algebra of vector fields*, Indag. Math., N.S., 7 (1996), 461–471.

[10] P. B. A. Lecomte and V. Ovsienko, *Projectively invariant symbol calculus*, Lett. Math. Phys., to appear.

[11] R. Molzon and K. P. Mortensen, *The Schwarzian derivative for maps between manifolds with complex projective connections*, Trans. of the AMS 348 (1996), 3015–3036.

[12] B. Osgood and D. Stowe, *The Schwarzian derivative and conformal mapping of Riemannian manifolds*, Duke Math. J. 67 (1992), 57–99.

[13] V. Ovsienko, *Lagrange Schwarzian derivative and symplectic Sturm theory.* Ann. Fac. Sci. Toulouse Math. 6 (1993), no. 1, 73–96.

[14] V. Retakh and V. Shander, *The Schwarz derivative for noncommutative differential algebras. Unconventional Lie algebras*, Adv. Soviet Math. 17 (1993), 139–154.

[15] S. Tabachnikov, *Projective connections, group Vey cocycle, and deformation quantization.* Internat. Math. Res. Notices 1996, No. 14, 705–722.

[16] E. J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig - Teubner - 1906.