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# ASPECTS OF GEOMETRIC QUANTIZATION THEORY IN POISSON GEOMETRY

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**Abstract.** This is a survey exposition of the results of [14] on the relationship between the geometric quantization of a Poisson manifold, of its symplectic leaves and its symplectic realizations, and of the results of [13] on a certain kind of super-geometric quantization. A general formulation of the geometric quantization problem is given at the beginning.

**1. The general setting.** This is a report on results which I obtained in the subject during the last years [13], [14]. But, I will start with a general formulation of the geometric quantization problem. A general algebraic formulation was developed by Huebschmann [4]. In this exposition the manifolds and bundles will be finite dimensional, and everything will be of class $C^\infty$.

Let $M$ be a manifold, and $A \to M$ a Lie algebroid with the anchor $a : A \to TM$ [12]. A *generalized, A-valued, Hamiltonian structure* on $M$ is a Lie algebra structure $\{\ ,\ \}$ on $C^\infty(M)$, together with a Lie algebra morphism

$$\sharp : \mathcal{L}(M) := (C^\infty(M), \{\ ,\ \}) \longrightarrow (\Gamma A, [\ ,\ ]_A)$$

($\Gamma$ denotes the set of global cross sections). The name comes from the fact that $\forall f \in C^\infty(M)$ there is an associated *Hamiltonian vector field* $X_f := a(\sharp f)$. (By $:=$ we denote a definition.)

The classical example is the usual Poisson bracket on $C^\infty(M)$. Then $A = T^*M$ with the bracket of 1-forms (see e.g. [12]), with $\sharp f := df$, and the usual Hamiltonian vector field $X_f$. A more general example is a Jacobi bracket i.e., a bracket of the local type on $C^\infty(M)$ (see e.g. [2], [5]). Then, $A$ is the jet bundle $J^1(M, \mathbf{R}) = \mathbf{R} \oplus T^*M$ with the bracket of [3], [8], $\sharp f := j^1 f$, and $X_f$ is the Hamiltonian vector field defined in [2].

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In principle, we may expect other examples if we use either different Lie algebroids or brackets of nonlocal type.

For any generalized Hamiltonian structure the action of $X_f$ on $C^\infty(M)$ defines a representation of $\mathcal{L}(M)$ and yields cohomology spaces which will be denoted by $H^k_{ham}(M)$ (*Hamiltonian cohomology spaces*). If we also look at the de Rham cohomology spaces $H^k_{deR}(M)$, and the Lie algebroid cohomology spaces $H^k_A(M)$, we see morphisms

$$\iota : H^k_{deR}(M) \to H^k_A(M), \quad \iota_A : H^k_A(M) \to H^k_{ham}(M), \tag{1.1}$$
$$\iota_h = \iota_A \circ \iota : H^k_{deR}(M) \to H^k_{ham}(M),$$

which are defined at the cochain level by composing the arguments with $a$, $\sharp$ and $a \circ \sharp$. For instance, if $\omega$ is a closed $k$-form on $M$, $\iota_h[\omega]_{deR}$ is represented by the cocycle

$$(\iota_h\omega)(f_1, \ldots, f_k) = \omega(X_{f_1}, \ldots X_{f_k}), \tag{1.2}$$

etc.

Now, let $(M, A, \{\ ,\ \}, \sharp)$ be a *generalized Hamiltonian manifold*, and $(K, h)$ be a complex Hermitian line bundle over $M$. Then any mapping $C^\infty(M) \to End_{\mathbf{R}}(\Gamma K)$ which sends $f \in C^\infty(M)$ to

$$\hat{f}s = \nabla^A_{\sharp f}s + 2\pi\sqrt{-1}fs, \tag{1.3}$$

where $s \in \Gamma K$, and $\nabla^A$ is an $A$-*connection* on $K$, will be called a *Kostant-Souriau mapping* [6], [9].

An $A$-connection on a vector bundle is defined like a usual covariant derivative, except for the fact that one puts $u(f) := a(u)(f)$, $u \in \Gamma A$, $f \in C^\infty(M)$ (see e.g. [16]). Such a connection has a curvature $R_{\nabla^A}(u_1, u_2) \in End_{\mathbf{R}}\Gamma K$ $(u_1, u_2 \in \Gamma A)$, defined as the usual commutant of covariant derivatives. The preservation of the Hermitian metric of $K$ is defined by

$$(au)h(s_1, s_2) = h(\nabla^A_u s_1, s_2) + h(u_1, \nabla^A_u s_2), \tag{1.4}$$

and it implies that the curvature of $\nabla^A$ is purely imaginary and the operators $\hat{f}$ are skew-Hermitian up to derivatives, just as in the case of the usual Hermitian connection [6].

A Kostant-Souriau mapping is called a *geometric prequantization* if $\forall f, g \in C^\infty(M)$ one has

$$\widehat{\{f, g\}} = [\hat{f}, \hat{g}] := \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f}. \tag{1.5}$$

The prequantization is *trivial* if $K$ is trivial and $\hat{f} = X_f$, $\forall f \in C^\infty(M)$. If (1.5) holds, $(K, h, \nabla^A)$ is called a *quantization triple*.

The existence of a quantization triple is a basic question of the theory. By a simple calculation, it follows that (1.5) is equivalent to

$$(d_{ham}I)(f, g) = -\frac{1}{2\pi\sqrt{-1}} R_{\nabla^A}(\sharp f, \sharp g) \quad (f, g \in C^\infty(M)), \tag{1.6}$$

where $I$ is the tautological 1-cochain $I(f) = f$, and $d_{ham}$ is the coboundary of the Hamiltonian cohomology.

Since $\nabla^A_u s := \nabla_{au}s$ $(u \in \Gamma A,\ s \in \Gamma K)$, where $\nabla$ is a usual Hermitian connection on $K$, is a Hermitian $A$-connection, it is easy to understand that $-(1/2\pi\sqrt{-1})R_{\nabla^A}$, which is $d_A$-closed, defines a 2-dimensional cohomology class $c^A_1(K) \in H^2_A(M)$, which is the image of the usual, de Rham, Chern class of $K$ by $\iota$ of (1.1). We say that $c_1^A(K)$ is the *A-Chern class* of $K$.

Now, we can state the basic existence theorem which extends the known classical result:

1.1. THEOREM. *The generalized Hamiltonian manifold $M$ has a quantization triple iff $d_{ham}I$ reduces to a 2 $A$-cocycle $\Pi$, and the cohomology class $[\Pi]_A \in H_A^2(M)$ is the $\iota$-image of an integral de Rham cohomology class.*

PROOF. If (1.6) holds, the conditions of the theorem hold since the usual Chern class is integral. Conversely, if $\Pi$ exists, and $[\Pi]_A = \iota[\omega]_{deR}$ where $[\omega]_{deR}$ is an integral class, the Weil-Kobayashi theorem (see e.g. [6]) yields Hermitian line bundles $(K, h)$ with the usual Hermitian connection $\nabla$ such that $[\omega]_{deR}$ is the Chern class of $K$, and $\omega$ is its curvature representative. Then, if $\Pi - \iota(\omega) = d_A\lambda$ ($\lambda \in \Gamma A^*$, and $\iota(\omega)$ is computed similarly to (1.2)), $\nabla^A = \nabla - (2\pi\sqrt{-1})\lambda$ is a Hermitian $A$-connection which satisfies (1.6). ■

For a usual Poisson manifold we have

$$(d_{ham}I)(f,g) = \{f,g\} = P(df,dg)$$

where $P$ is the Poisson bivector. The $A$-cohomology is the Poisson-Lichnerowicz cohomology of $(M,P)$, and the quantization condition of Theorem 1.1 is that $[P]$ is the image of an integral de Rham class. This means that there exist a closed integral 2-form $\lambda$ and a vector field $A$ such that

$$P + L_A P = \sharp_P \lambda \tag{1.7}$$

($L$ is the Lie derivative) [11]. (See Kotov [7] for interesting examples.) In the particular case of a symplectic manifold the quantization condition reduces to the integrality of the symplectic form [6], [9].

For a Jacobi manifold $M$ with the bracket

$$\{f,g\} = \Lambda(df,dg) + f(Eg) - g(Ef)$$

where $\Lambda$ is a bivector field and $E$ is a vector field on $M$ (see e.g. see [2]), $H_A^*$ is the Jacobi-Lichnerowicz cohomology of [8]. One has $X_f = \sharp_\Lambda df + fE$ and $(d_{ham}I)(f,g) = \Lambda(df,dg)$, which is a $J^1(M,\mathbf{R})$-cocycle. In fact, $\Lambda = d_{J^1(M,\mathbf{R})}(1,0)$ [8]. Hence, $[\Lambda] = 0$, and the trivial line bundle $M \times \mathbf{C}$, with the connection

$$\nabla^{J^1(M,\mathbf{R})}_{(f,u)}(1) := -2\pi\sqrt{-1}f \qquad (f \in C^\infty(M), u \in T^*M)$$

and the metric $h(1,1) = 1$, form a quantization triple with the trivial prequantization. Thus, in this case, we still have to look for existence conditions of non trivial prequantizations. Such conditions are given in [8].

**2. Comparison theorems in the Poisson case.** In the remaining part of this exposition we consider only the case of a Poisson manifold $(M,P)$. Theorem 1.1 gives us the quantization condition for $\mathcal{L}(M) := (C^\infty(M), \{\ ,\ \}_P)$. But there are a lot of other interesting questions to be studied. We will be interested in the study of relationships between the quantization of $M$ and the quantization of the symplectic leaves of $P$, on one hand, and that of the symplectic realizations of $(M,P)$ on the other hand. The results below appeared in [14].

It is natural to look for quantization conditions of a symplectic leaf $S$ of $P$ in the geometry of the immersion $S \to P$. This problem is open. A hint is given by

2.1. PROPOSITION. *Assume that $S$ has a normal bundle $NS$ which is of the form $\mathcal{C}/_S$, where $\mathcal{C}$ is a distribution on an open neighborhood $U$ of $S$ in $M$, such that the annihilator $Ann(\mathcal{C}) \subseteq T^*M$ is involutive with respect to the $P$-bracket of 1-forms. Then, if $(M,P)$ has a quantization triple $(K,h,\nabla^{T^*M})$, this triple induces a quantization triple of $S$.*

PROOF. By "normal distribution" we mean that $TM = TS \oplus NS$. I also recall that a distribution $\mathcal{C}$ as in the proposition is called a cofoliation [12]. The Hermitian bundle $(K,h)$ has the natural restriction to $S$. Furthermore, $\forall X \in TS$ there is a unique *$\mathcal{C}$-based dual 1-form* $\alpha \in Ann\,(NS)$ such that $X = \sharp_P\alpha$, and the formula

$$D_X s = \nabla^{T^*M}_\alpha \tilde{s} \qquad (\tilde{s} \in \Gamma(K/_U),\ s = \tilde{s}/_S) \tag{2.1}$$

defines a Hermitian connection on $K/_S$. If $\alpha,\beta$ are the $\mathcal{C}$-based dual forms of $X,Y \in TS$, and $\tilde{\alpha},\tilde{\beta} \in Ann(\mathcal{C})$ are extensions to $U$, we get

$$[X,Y] = [\sharp_P\alpha, \sharp_P\beta] = \sharp_P\{\tilde{\alpha},\tilde{\beta}\}. \tag{2.2}$$

Thus $\{\tilde{\alpha},\tilde{\beta}\}/_S$ is the $\mathcal{C}$-based dual form of $[X,Y]$, and we deduce that $D$ and $\nabla$ have the same curvature operators when computed on $\mathcal{C}$-based dual elements. ∎

We can say more on the relationship between quantization and symplectic realizations. We need the slightly more general notion of a *presymplectic realization* $(V,\sigma)$ of a Poisson manifold $(M,P)$, which we define to be a presymplectic manifold $(V,\sigma)$, with a surjective submersion $r : V \to M$ such that $\mathcal{F} := ker\,\sigma \subseteq ker(dr)$, and $\forall\varphi,\psi \in C^\infty(M)$ one has

$$\{\varphi,\psi\}_P \circ r = \{\varphi\circ r, \psi\circ r\}_\sigma$$

(the last bracket exists since $\varphi\circ r, \psi\circ r$ are $\mathcal{F}$-projectable functions). We also need to extend the notion of a quantization triple from symplectic to presymplectic manifolds. Namely, it will be a triple $(K,h,\nabla)$ on $V$ where the usual connection $\nabla$ on $K$ satisfies the condition $R_\nabla = -2\pi\sqrt{-1}\sigma$.

2.2. THEOREM. *Let $r : (V,\sigma) \to (M,P)$ be a presymplectic realization with connected fibers. Assume that $(V,\sigma)$ has a quantization triple $(K,h,\nabla)$ such that the holonomy of $\nabla$ along paths in the fibers of $r$ is zero. Then the fibers of $r$ are $\sigma$-isotropic, $P$ is regular, and $(M,P)$ has a quantization triple $(K',h',\nabla')$ where $K = r^{-1}(K')$, $h = r^*h'$, and $\nabla'$ is a partial connection along the symplectic leaves of $P$ which is well defined by $\nabla$.*

PROOF. The zero holonomy hypothesis implies that the curvature $R_\nabla(X,Y) = 0$, $\forall X,Y \in ker(dr)$, and the quantization condition $R_\nabla = -2\pi\sqrt{-1}\sigma$ shows that the fibers of $r$, which define a foliation $\mathcal{R}$ of $V$, are isotropic submanifolds of $(V,\sigma)$. That is $r$ is an *isotropic realization*, and so are the local symplectic realizations of $(M,P)$ given by the local transversal submanifolds of $\mathcal{R}$ in $(V,\sigma)$. By a result of Dazord [1], [12], $P$ has a constant rank.

The zero holonomy hypothesis for $\nabla$ also implies that $K$ has an $\mathcal{R}$-foliated bundle structure with respect to which $\nabla$ is a *Bott connection*. This means that there exists a choice of leafwise $\nabla$-parallel local bases with $\mathcal{R}$-projectable transition functions. Namely, these bases may be taken arbitrarily along $\mathcal{R}$-transversal neighborhoods, then moved by the path independent $\nabla$-parallelism along the fibers of $\mathcal{R}$. These bases, and their transition functions project to $M$, where they produce the Hermitian line bundle $(K', h')$ required.

For $f \in C^\infty(M)$, the Hamiltonian vector fields are related by $dr(X^\sigma_{f\circ r}) = X^P_f$ hence, $X^\sigma_{f\circ r}$ is an $r$-projectable vector field. Then, if $s$ is a projectable cross section of $K$, the Bott property of $\nabla$ and the quantization condition yield

$$\nabla_Y \nabla_{X^\sigma_{f\circ r}} s = \nabla_{X^\sigma_{f\circ r}} \nabla_Y s - \nabla_{[X^\sigma_{f\circ r}, Y]} s - R_\nabla(X^\sigma_{f\circ r}, Y) = 0,$$

$\forall Y \in T\mathcal{R}$. This shows that $\nabla_{X^\sigma_{f\circ r}} s$ is also $r$-projectable, and ensures the existence of the *partial connection*

$$\nabla'_{X^P_f} s' := pr(\nabla_{X^\sigma_{f\circ r}} s) \qquad (s' = pr(s) \in \Gamma K'), \tag{2.3}$$

where $pr$ is the projection induced by $r$. Such a partial connection is equivalent to a $T^*M$-connection.

The quantization condition for the triple $(K', h', \nabla')$ follows from the fact that $r$ is a Poisson mapping. ∎

Applications of Theorem 2.2 can be obtained for realizations $r$ which have simply connected, isotropic fibers. In this case, the quantization condition for a triple implies $R_\nabla(X, Y) = 0$ for $X, Y \in T\mathcal{R}$ hence, zero holonomy.

If the fibers of $r$ are isotropic but not simply connected, $K$ is foliated but $K'$ may not exist, since $\nabla$ may still have holonomy. In this case we may try to quantize $f \in C^\infty(M)$ by an operator on $\Gamma_{fol}K :=$ the space of projectable cross sections of $K$ or on some cohomology space with values in the sheaf of germs of the projectable cross sections of $K$. This might yield a generalized geometric quantization of $(M, P)$. For instance, if $G$ is a Lie group, any coadjoint orbit $\mathcal{O}$ with a connected isotropy subgroup $G_0$ is isotropically realized by an exact presymplectic structure of $G$ (see e.g. [15]). Hence, $\mathcal{O}$ might have a generalized geometric prequantization but, it may not have a usual geometric quantization.

In order to get an application of Theorem 2.2, let us look at a symplectic realization $r : (V, \sigma) \to (M, P)$ which also realizes a dual Poisson manifold by $r' : (V, \sigma) \to (M', P')$ (see e.g. [12]). Assume that the fibers of $r$ and $r'$ are connected and have connected intersections. Then every symplectic leaf $S$ of $P$ is of the form $S = r(r'^{-1}(y))$ for some $y \in M'$, and $\sigma$ induces a presymplectic structure on $r'^{-1}(y)$ with the kernel foliation $r'^{-1}(y) \cap r^{-1}(x)$ $(x \in S)$ (see e.g. [12]). Hence, $(r'^{-1}(y), \sigma)$ is an isotropic presymplectic realization of $S$, and we have

2.3. Corollary. *If $(r'^{-1}(y), \sigma)$ has a quantization triple $(K, h, \nabla)$ and if $r'^{-1}(y) \cap r^{-1}(x)$ are simply connected $\forall x \in S$, $S$ has an induced quantization triple $(K', h', \nabla')$.*

This Corollary may be used in the case of a symplectic groupoid by taking $(r, r')$ to be the source and target projection of the groupoid (see e.g. [12]).

It is well known that geometric quantization of symplectic manifolds involves one more essential ingredient, *polarizations*. In the Poisson case, we define a polarization to be a subsheaf $\mathcal{P}$ of the sheaf $\mathcal{S}$ of the Poisson algebras of germs of the complex valued functions on $(M, P)$ such that the stalks of $\mathcal{P}$ are abelian subalgebras. The usual *Lagrangian* character is lost but, we still may construct a vector space

$$\Gamma_0 K := \{s \in \Gamma K \ / \ \nabla_{X_\varphi} s = 0, \ \forall \varphi \in \mathcal{P}\}, \tag{2.4}$$

and quantize the functions $f \in C^\infty(M)$ such that $\{\varphi, f\}$ is in $\mathcal{P}$ whenever $\varphi \in \mathcal{P}$ by operators on $\Gamma_0 K$ in a "physics consistent" way [11], [14].

Furthermore, if the germs in $\mathcal{P}$ are of the form $\varphi(y^1, \ldots, y^h, z^1, \ldots, z^{k-h})$ where $y^a$ are real valued, $z^\alpha$ are complex valued, and $(y^a, z^\alpha)$ are functionally independent, the polarization will be *regular of rank* $(k, h)$. By a theorem of Nirenberg, this happens iff the equations $d\varphi = 0$, $\forall \varphi \in \mathcal{P}$, define a *Nirenberg integrable subbundle* $\Delta \subseteq TM \otimes \mathbf{C}$. In more detail, this means that $\Delta$ is involutive, and $\Delta + \bar{\Delta}$ is of constant dimension and involutive (the bar denotes complex conjugation). In this case, $\{\varphi, \psi\} = 0 \ \forall \varphi, \psi \in \mathcal{P}$ is equivalent to the fact that $P(\alpha, \beta) = 0$, $\forall \alpha, \beta \in Ann\, \Delta$ i.e., $\Delta$ is a *coisotropic distribution* (see e.g. [12]). Therefore, a regular polarization may be identified with a coisotropic, Nirenberg integrable, complex distribution on $(M, P)$.

The notion of a polarization can be adapted to presymplectic manifolds $(V, \sigma)$. Namely, in this case, we will take $\mathcal{P}$ to be a subsheaf with abelian stalks of the sheaf of Poisson algebras of the germs of complex valued, $ker\,\sigma$-projectable functions on $V$. The regular case, and its identification with coisotropic, Nirenberg integrable distributions are obtained in the same way as above. (In the presymplectic case a coisotropic distribution is one which contains its $\sigma$-orthogonal distribution hence, it must contain $T(ker\,\sigma)$.)

With these definitions in place, we get

2.4. THEOREM. *Let $\pi : (M_1, P_1) \to (M_2, P_2)$ be a Poisson mapping. Then:*
*1) Any polarization $\mathcal{P}_2$ of $M_2$ pulls back to a polarization $\mathcal{P}_1$ of $M_1$.*
*2) If $\pi$ is surjective, any $\mathcal{P}_1$ of $M_1$ can be pushed forward to some (possibly zero) $\mathcal{P}_2$ of $M_2$.*
*3) If $\pi$ is a submersion and $\mathcal{P}_2$ is regular, its pullback $\mathcal{P}_1$ is regular.*
*4) If $\pi$ is a submersion, $\mathcal{P}_1$ is regular, and the germs of $\mathcal{P}_1$ are constant along the fibers of $\pi$, then the pushed forward polarization $\mathcal{P}_2$ is also regular.*

PROOF. 1) The pullback of $\mathcal{P}_2$ is defined by

$$\mathcal{P}_1 = \{germs\, \psi \circ \pi \ / \ germ\, \psi \in \mathcal{P}_2\}. \tag{2.5}$$

2) The push forward of $\mathcal{P}_1$ is

$$\mathcal{P}_2 = \{germs\, \psi \in C^\infty(M_2) \ / \ germ\, \psi \circ \pi \in \mathcal{P}_1\}. \tag{2.6}$$

3) Identify $\mathcal{P}_2$ with a coisotropic, Nirenberg integrable distribution $\Delta_2 \subseteq TM_2 \otimes \mathbf{C}$, and check that, then, $\mathcal{P}_1$ identifies with $\Delta_1 := (d\pi)^{-1}(\Delta_2)$. $\Delta_1$ exists because $\pi$ is a submersion, and $\Delta_1 \supseteq T(\text{fibers of } \pi)$.

4) Identify $\mathcal{P}_1$ with a $\Delta_1$, and use local coordinates $(y^a, z^\alpha, \bar{z}^\alpha, y^u)$ on $M_1$ such that $Ann\, \Delta_1 = span\{dy^a, dz^\alpha\}$. It turns out that $\Delta_1$ is $\pi$-projectable and $d\pi(\Delta_1) = \Delta_2$ defines the pushed forward polarization $\mathcal{P}_2$. ∎

The same proof as in the last part of Theorem 2.4 shows that a (regular) polarization of a presymplectic manifold $(V, \sigma)$ can be pushed forward to a (regular) polarization of the symplectic reduction $V/(ker\, \sigma)$. Accordingly, if $(M, P)$ has a dual Poisson manifold $(M', P')$, with the symplectic realizations

$$r : (V, \sigma) \to (M, P), \quad r' : (V, \sigma) \to (M', P'),$$

and if the symplectic leaf $S$ of $(M, P)$ is $S = r(r'^{-1}(y))$ $(y \in M')$, any (regular) polarization of $r'^{-1}(y)$ (conceivably induced by a polarization of $(V, \sigma)$) projects to a (regular) polarization of $S$.

A more interesting consequence of Theorem 2.4 is

2.5. COROLLARY. *Any polarization $\mathcal{P}$ of the Poisson manifold $(M, P)$ pulls back to polarizations $\mathcal{P}_S$ of the symplectic leaves $S$ of $P$. If $\mathcal{P}$ is regular and given by the coisotropic, Nirenberg integrable distribution $\Delta$, and if*

$$codim_S(TS \cap \Delta) = codim_M \Delta, \tag{2.7}$$

*then $\mathcal{P}_S$ is regular and given by the distribution $\Delta' = TS \cap \Delta$.*

This follows by using the immersion of $S$ in $P$ as the Poisson mapping of Theorem 2.4. The condition on codimensions is needed to ensure that the functions $(y^a, z^\alpha)/_S$, where $dy^a = 0$, $dz^\alpha = 0$ are the local equations of $\Delta$, are functionally independent. Then, $dy^a/_S = 0, dz^\alpha/_S = 0$ define the distribution $\Delta'$.

**3. Super-geometric quantization.** The problem of geometric quantization may be generalized as follows. Assume that $(M, P)$ is a Poisson manifold with a quantization triple $(K, h, \nabla)$. Is it possible to embed the space $\Gamma K$ in a superspace, and extend the Kostant-Souriau formula (1.3) in such a way that the quantization condition (1.5) holds if the commutator of its right hand side is replaced by the corresponding super-commutator? Some results on this problem were given in [13], and I am reporting on them briefly here.

As a superspace, we take the most natural one, namely,

$$SK = S^+K \oplus S^-K, \tag{3.1}$$

where

$$S^+K := \oplus_{i \geq 0} \wedge^{2i} (M, K), \quad S^-K := \oplus_{i \geq 0} \wedge^{2i+1} (M, K), \tag{3.2}$$

and $\wedge^*(M, K)$ denote spaces of $K$-valued differential forms.

The Kostant-Souriau formula (1.3) can be extended to $K$-valued differential forms by

$$\hat{f}A = L^{\nabla}_{X_f} A + 2\pi\sqrt{-1} fA \quad (A \in SK), \tag{3.3}$$

where $L^{\nabla}_X := D_\nabla i(X) + i(X) D_\nabla$, and $D_\nabla$ is the covariant-exterior differential [10]. The commutation condition (1.5) still holds. But, what we want now is to find an odd (i.e., grade-parity switching) endomorphism $l(f) : SK \to SK$, defined for $f \in C^\infty(M)$, such that the operators

$$\tilde{f}A := \hat{f}A + 2\pi\sqrt{-1}\, l(f)(A) \quad (A \in SK) \tag{3.4}$$

satisfy the commutation condition

$$\widetilde{\{f,g\}} = {}^s[\tilde{f},\tilde{g}] := \tilde{f}\circ\tilde{g} - (-1)^{deg\,f.deg\,g}\tilde{g}\circ\tilde{f}. \tag{3.5}$$

If this happens, $f \to \tilde{f}$ is a *super-geometric prequantization.* (But not a geometric super-prequantization, since we do not act on a supermanifold.)

The following result follows by technical computations [13].

3.1. Proposition. *The operation $*$ defined by*

$$f * \theta = [\hat{f},\theta] = \hat{f}\theta - \theta\hat{f} \qquad f \in C^\infty(M), \theta \in End(SK))$$

*is a representation of the Poisson-Lie algebra $\mathcal{L}(M)$ on $End(SK)$ which leaves the odd subspace $End_-(SK)$ invariant. Formula (3.4) is a super-geometric prequantization iff $l$ is an $End(SK)$-valued 1-cocycle with respect to the representation $*$, and $l^2(f) = 0$.*

For instance, if $c \in End_-(SK)$, and $[\hat{f},c]^2 = 0$, then $l(f) := [\hat{f},c]$ is a coboundary of the required type. In particular, if $\theta$ and $V$ are a complex valued 1-form and vector field, respectively, $c = e(\theta) + i(V)$, where $e, i$ denote exterior and interior products, provides such a coboundary.

As in the classical theory, we can extend the operators $\tilde{f}$ to a similarly defined superspace $S(K \otimes \mathcal{D})$, where $\mathcal{D}$ is the bundle whose cross sections are the half-densities (or half-forms) on $M$, by acting on $\mathcal{D}$ by Lie derivatives, and try to get a pre-Hilbert metric on $S_c(K \otimes \mathcal{D})$ ($c$ means compact supports) such that $\tilde{f}$ would be skew-Hermitian. A natural guess is

$$< \alpha_1 \otimes s_1 \otimes \rho_1, \alpha_2 \otimes s_2 \otimes \rho_2 > := \int_M g(\alpha_1,\alpha_2)h(s_1,s_2)\rho_1\bar{\rho}_2, \tag{3.6}$$

where $\alpha_a \in \wedge^k M$, $s_a \in \Gamma K$, $\rho_a \in \Gamma\mathcal{D}$ $(a = 1, 2)$, and $g$ is a Riemannian metric on $M$. Then, the operators $\tilde{f}$ will be skew-Hermitian if $f$ is such that the Hamiltonian vector field $X_f$ is a Killing vector field of $g$ and $l(f)$ is Hermitian for the metric $gh$.

Furthermore, polarizations will also have to be used here. In fact, it turns out that one is even more compelled to reduce the class of quantizable functions via polarizations than one was in the classical theory, in order to be able to define the operators $l(f)$ of the previous scheme. I could obtain well behaved operators $l(f)$ only on restricted classes of functions. After such a restiction, we will speak of *super-geometric quantization* rather than prequantization.

For an example let us look at a phase space $M = T^*N$ with the symplectic structure

$$\omega = -d\theta + p^*F,$$

where $p : M \to N$ is the natural projection, $\theta$ is the Liouville form, and $F = d\lambda$ ($\lambda \in \wedge^1 N$) is an exact electromagnetic term. We also assume that $N$ is endowed with a Riemannian metric $g$. Then, we may take the trivial bundle $K$, and the $K$-valued forms are just differential forms on $M$. Let $\nabla$ be the flat connection of $K$ which is defined by the global connection form $2\pi\sqrt{-1}(\theta - \lambda)$. Let $\mathcal{P}$ be the polarization defined on $M$ by the germs $\varphi \circ p$, $\varphi \in C^\infty(N)$. With this notation in place, one gets

3.2. Proposition. *On $M$, consider the "observables" $f = \mu(Y) + \varphi \circ p$, where $\varphi \in C^\infty(N)$, $Y$ is a Killing vector field of $g$ on $N$, and $\mu(Y)$ is the momentum of $Y$ ($\mu(Y) =$*

$\sum_i p_i Y^i$, $p_i$ = *momentum coordinates on* $M$). *Then, the formula*

$$\tilde{f}A = -L_Y A + 2\pi\sqrt{-1}(\varphi + \mu(Y))A + 2\pi\sqrt{-1}(d\varphi - i(Y)F) \wedge A + 2\pi\sqrt{-1}i(d\varphi - i(Y)F)A \qquad (A \in \wedge^* N \otimes \mathbf{C}) \tag{3.7}$$

*defines a super-geometric quantization on* $\wedge^* N \otimes \mathbf{C}$ *with the odd-even grading. This quantization is by skew-Hermitian operators with respect to the global scalar product induced by* $g$ *on the compactly supported forms on* $N$.

Proof. The technique of polarizations asks for the use of arguments in $\wedge^* N \otimes \mathbf{C}$ (see e.g. see the earlier formula (2.4)). The first two terms of (3.7) come from the extended Kostant-Souriau formula (3.3). The remaining part behaves like a correct odd extension to a super-geometric quantization on the indicated family of functions $f$. This may be checked by technical computations for the cases where $f$ is the pullback of a function on $N$, and $f$ is the momentum $\mu(Y)$ of a Killing vector field $Y$ on $(N, g)$. ■

In [13], another example is also given namely, a symplectic manifold $(M, \sigma)$ endowed with a Kähler polarization associated to a $\sigma$-compatible complex structure $J$, and with a $J$-holomorphic 1-form $\theta$. In this case, the line bundle $K$ must be holomorphic, and, if one puts

$$l(f) = e(L_{X_f}\theta) + i([X_f, \sharp_g \bar{\theta}]),$$

where $g$ is the metric defined by $(\sigma, J)$ on $M$, one gets a correct super-geometric quantization for functions $f$ which satisfy the following conditions: i) $X_f$ preserves the Kähler polarization, ii)$\sharp_g L_{X_f} \bar{\theta}$ is a $J$-holomorphic vector field.

The proof involves manipulation of known formulas of complex differential geometry [13].

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