POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

C^*-ALGEBRA OF A DIFFERENTIAL GROUPOID

PIOTR STACHURA
Department of Mathematical Methods in Physics
Faculty of Physics, University of Warsaw
Hoża 74, 00-682 Warszawa, Poland
E-mail: stachura@fuw.edu.pl

1. Introduction. The results contained in this paper are the basis of the author's thesis. The essential ideas are due to S. Zakrzewski. We present a construction of a covariant functor from the category of differential groupoids to the category of C^* algebras in the sense of [10]. However our definition of morphism of differential groupoids is different from the standard one, i.e. a mapping which satisfies the obvious compatibility condition with respect to the groupoid structure. Let us argue that there is no hope to construct such a functor with the standard notion of morphism. The main problem can be shown in the discrete case so let us assume that all sets are with discrete topology.

Let (Γ, m, E, s) be a groupoid (see the next section for the notation) and let $\mathcal{A}(\Gamma)$ denote a linear space of complex functions with compact support (i.e. for $f \in \mathcal{A}(\Gamma)$ we have $f(x) \neq 0$ for a finite number of x). There are natural notions of convolution and star operation in $\mathcal{A}(\Gamma)$ which make it a *-algebra. Namely $(f_1 f_2)(x) := \sum_{yz=x} f_1(y)f_2(z) = \sum_{y\in F_l(x)} f_1(y)f_2(s(y)x) = \sum_{z\in F_r(x)} f_1(xs(z))f_2(z)$ ($F_l(x), F_r(x)$ denote the left and right fiber containing x) and $f^*(x) := \overline{f(s(x))}$. We expect that $C^*(\Gamma)$ will be the completion of $\mathcal{A}(\Gamma)$ with respect to some C^*-norm.

The "extremal" examples of groupoids are sets and groups. For sets the above multiplication is equal to pointwise multiplication and for groups it is usual convolution. The standard definition of morphism of groupoids reduces to a mapping if groupoids are sets and to a group homomorphism if they are groups. If $h : \Gamma \longrightarrow \Gamma'$ is a group homomorphism, we can *push forward* the convolution algebra by the formula $(\hat{h}f)(x') := \sum_{x\in h^{-1}(x')} f(x)$ so $\hat{h} : \mathcal{A}(\Gamma) \longrightarrow \mathcal{A}(\Gamma')$. But if $h : \Gamma \longrightarrow \Gamma'$ is a mapping of sets, functions with pointwise multiplication can be *pulled back* by $(\hat{h}f')(x) := f'(h(x))$. In fact $\hat{h}f'$ can have

2000 *Mathematics Subject Classification*: 22A22, 46L87, 58B32.
The paper is in final form and no version of it will be published elsewhere.

noncompact support, but this is not a problem, since we know that it should belong to (some kind of) multiplier algebra of $\mathcal{A}(\Gamma)$ so $(\hat{h}f')f$ should be in $\mathcal{A}(\Gamma)$ for any $f \in \mathcal{A}(\Gamma)$ and certainly this is true. Disregarding the subtlety in this case we have $\hat{h} : \mathcal{A}(\Gamma') \longrightarrow \mathcal{A}(\Gamma)$. We expect that $C^*(h)$ will be some extension of $\hat{h}$. And here we are in trouble, since our "C^*-functor" is covariant in the first case and contravariant in the second. So to achieve our goal we *need a definition of morphism between groupoids which reduces to a group homomorphism if groupoids are groups and to a mapping in the reverse direction if groupoids are sets.* In particular this suggests that morphisms should be *relations* rather than *mappings.* Such a definition was given in [11] and extended to a differential setting in [12]. Let us briefly explain the main idea of the construction, still in the discrete setting. We suggest looking at the next section before reading the following.

Let $(\Gamma, m, E, s), (\Gamma', m', E', s')$ be groupoids. A morphism from Γ to Γ' is a relation which satisfies some obvious compatibility conditions. In particular it turns out that it defines a mapping $f_h : E' \longrightarrow E$ and for each $b \in E'$ a mapping $h^L_b : F_l(f_h(b)) \longrightarrow F_l(b)$. For a morphism $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma'$ and $f \in \mathcal{A}(\Gamma)$ we define $\hat{h}f$, a linear mapping on $\mathcal{A}(\Gamma')$, by the formula:

$$((\hat{h}f)f')(x') := \sum_{x \in F_l(b)} f(x) f'(s'(h^L_a(x))x'),$$

where $a := e'_L(z), b := f_h(a)$. By the same formula we define $\pi_h(f)f'$ where we view f' as an element of $L^2(\Gamma')$, the Hilbert space of square summable functions on Γ'. Let us also define norms on $\mathcal{A}(\Gamma)$: $||f||_l := \sup_{a\in E} \sum_{x\in F_l(a)} |f(x)|, ||f||_r := \sup_{a\in E} \sum_{x\in F_r(a)} |f(x)|$, and $||f|| := max\{||f||_l, ||f||_r\}$. It is not difficult to prove the following:

PROPOSITION 1.1. *a)* $(\mathcal{A}(\Gamma), *, ||.||)$ *is a normed *-algebra.*
b) $||\pi_h(f)|| \le ||f||$ *and* π_h *is a representation of the *-algebra* $\mathcal{A}(\Gamma)$.
c) $f_3^*(\hat{h}(f_1)f_2) = (\hat{h}(f_1^*)f_3)^* f_2$ *for any* $f_1 \in \mathcal{A}(\Gamma), f_2, f_3 \in \mathcal{A}(\Gamma')$.
d) If $k : \Gamma' \longrightarrow\!\!\!\triangleright\, \Gamma''$ *is a morphism of groupoids then* $\pi_k(\hat{h}(f_1)f_2)f_3 = \pi_{kh}(f_1)\pi_k(f_2)f_3$ *for any* $f_1 \in \mathcal{A}(\Gamma), f_2 \in \mathcal{A}(\Gamma'), f_3 \in L^2(\Gamma'')$ *with compact support, and* $\hat{k}(\hat{h}(f_1)f_2)f_3 = \hat{kh}(f_1)(\hat{k}(f_2)f_3)$ *for any* $f_1 \in \mathcal{A}(\Gamma), f_2 \in \mathcal{A}(\Gamma'), f_3 \in \mathcal{A}(\Gamma'')$.

Using these facts one can define the C^* norm on $\mathcal{A}(\Gamma)$ by: $||f||_{C^*} := \sup ||\pi_h(f)||$ where the supremum is taken over all morphisms $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma'$. This is obviously a C^*-seminorm, but one can show that there exists a faithful representation of $\mathcal{A}(\Gamma)$. The completion of $\mathcal{A}(\Gamma)$ with respect to this norm is the C^*-*algebra* of Γ and one can see that $\hat{h}$ extends to a $C^*(h) \in Mor(C^*(\Gamma), C^*(\Gamma'))$. The extension of this construction to a differential setting is the main result of the paper.

Of course in the above case we can also proceed in the standard way: first one can complete $(\mathcal{A}(\Gamma), *, ||.||)$ to get a Banach *-algebra and then take its enveloping C^*-algebra. However, in such a construction the functoriality is lost and moreover it seems that there is no natural, geometric norm on $\mathcal{A}(\Gamma)$ in the differential case.

Let us now say a few words about our motivations. One is to get “geometric models” of quantum groups, especially noncompact, from double Lie groups. If $(G; A, B)$ is a double Lie group (see section 3) we can define two differential groupoid structures on G: $G_A := (G, m_A, A, s_A)$ and $G_B := (G, m_B, B, s_B)$. It turns out that m_B^T is a morphism $G_A \longrightarrow G_A \times G_A$ which is coassociative: $(m_B^T \times id)m_B^T = (id \times m_B^T)m_B^T$. Applying our C^* functor we get a coassociative morphism $\Delta \in Mor(C^*(G_A), C^*(G_A \times G_A))$. We expect that $C^*(G_A \times G_A)$ is (some sort of) $C^*(G_A) \otimes C^*(G_A)$. In this way we get one of the main ingredients of quantum group structure on $C^*(G_A)$. It seems that also other ingredients as defined in [5] have a natural geometric interpretation in the groupoid setting. For a connection with symplectic geometry and quantisation see Appendix.

2. Groupoids—algebraic structure. *A relation r from X to Y* is a triple $r = (R; Y, X)$, where X and Y are sets and R is a subset of $Y \times X$. R is the *graph* of r and we denote it by $Gr(r)$. A relation r from X to Y will be denoted by $r : X \longrightarrow\!\!\triangleright\, Y$ (note the special type of arrow). Relations can be composed: if $s : X \longrightarrow\!\!\triangleright\, Y$ and $r : Y \longrightarrow\!\!\triangleright\, Z$, then the composition rs is a relation from X to Z defined by $Gr(rs) := \{(z, x) \in Z \times X : \exists\, y \in Y\ [(z, y) \in Gr(r) \text{ and } (y, x) \in Gr(s)]\}$. We say that the composition rs is *simple* iff for any $(z, x) \in Gr(rs)$ there exists a unique $y \in Y$ such that $(y, x) \in Gr(s)$ and $(z, y) \in Gr(r)$. For a relation $r : X \longrightarrow\!\!\triangleright\, Y$ its *transposition* is a relation $r^T : Y \longrightarrow\!\!\triangleright\, X$ defined by $Gr(r^T) := \{(x, y) \in X \times Y : (y, x) \in Gr(r)\}$. The cartesian product of relations is also naturally defined: if $r : X \longrightarrow\!\!\triangleright\, Y$ and $s : Z \longrightarrow\!\!\triangleright\, T$ then $r \times s : X \times Z \longrightarrow\!\!\triangleright\, Y \times T$ is a relation with graph $Gr(r \times s) := \{(y, t, x, z) \in Y \times T \times X \times Z\ :\ (y, x) \in Gr(r)\ \mathit{and}\ (t, z) \in Gr(s)\}$. If $r : X \longrightarrow\!\!\triangleright\, Y$ and $A \subset X$ we denote be $r(A)$ the image of A by r: $r(A) := \{y \in Y : \exists\, x \in A\ (y, x) \in Gr(r)\}$. Let $\{1\}$ denote the one point set. Now we can formulate the basic definition:

Definition 2.1 ([11]). A *groupoid* is a quadruple (Γ, m, e, s) where Γ is a set, $m : \Gamma \times \Gamma \longrightarrow\!\!\triangleright\, \Gamma$ and $e : \{1\} \longrightarrow\!\!\triangleright\, \Gamma$ are relations, $s : \Gamma \longrightarrow \Gamma$ is an involution which satisfy:

associativity: $m(m \times id) = m(id \times m)$,
identity: $m(e \times id) = m(id \times e) = id$,
inverse: $sm = m(s \times s) \sim$ where $\sim : \Gamma \times \Gamma \ni (x, y) \mapsto (y, x) \in \Gamma \times \Gamma$,
strong positivity: for any $x \in \Gamma$ $\emptyset \neq m(s(x), x) \subset e(\{1\})$.

The relation m is called *multiplication*, s the *inverse* and $E := e(\{1\})$ the set of *identities.*

Notice that the first three conditions are formally the same as in the group case but instead of mappings we use relations. The above definition is equivalent (cf. proposition below) to the ”ordinary” definition of groupoid: *A groupoid is a small category in which every morphism is an isomorphism.* But if we think of a groupoid as a category, the natural candidates for morphisms are functors—this is not our point of view, so we prefer the definition based on relations.

PROPOSITION 2.2 ([11]). *Let (Γ, m, e, s) be a groupoid. Then:*
a) If $a, b \in E$ then $m(a,b) \neq \emptyset$ iff $a = b$ and in this case $m(a,a) = a$.
b) There exist unique mappings $e_L, e_R : \Gamma \longrightarrow E$ such that $m(e_L(x), x) = x = m(x, e_R(x))$ for any $x \in \Gamma$ and $e_L(a) = e_R(a) = a$ for any $a \in E$.
c) $m(s(x), x) = e_R(x), m(x, s(x)) = e_L(x)$.
d) $m(x,y) \neq \emptyset$ iff $e_R(x) = e_L(y)$.
e) $m(x,y) \cap E \neq \emptyset$ implies $y = s(x)$.
f) $m(x,y)$ consists of at most one point.

Now we explain our notation. The set of composable pairs will be denoted by $\Gamma^{(2)} := m^T(\Gamma) = \{(x,y) \in \Gamma \times \Gamma : e_R(x) = e_L(y)\}$. From the statements c) and f) of the above proposition it follows that m restricted to $\Gamma^{(2)}$ is a surjective mapping to Γ. The set of identities E will also be denoted by Γ^0. If it doesn't lead to any confusion we write $x = x_1 x_2$ instead of $(x; x_1, x_2) \in Gr(m)$. For $A, B \subset \Gamma$ let $AB := \{ab : a \in A, b \in B\}$. For $x \in \Gamma$, by $F_l(x)$ and $F_r(x)$ we denote the left and right fibers containing x, i.e. $F_l(x) := e_L^{-1}(e_L(x))$ and $F_r(x) := e_R^{-1}(e_R(x))$. If $a \in E$ we also write ${}_a\Gamma := F_l(a)$ and $\Gamma_a := F_r(a)$. Clearly ${}_a\Gamma \cap \Gamma_a$ is a group.

EXAMPLES 2.3. *a) Sets.* If X is a set then (X, d^T, X, id), where $d : X \longrightarrow X \times X$ is a diagonal mapping, is a groupoid. Conversely, any groupoid such that m^T is a mapping is of this type.

b) Groups. If G is a group then $(G, m_G, \{e\}, s)$, where m_G, s are group multiplication and group inverse, is a groupoid and any groupoid for which m is a mapping is a group.

c) Pair groupoids. Examples a) and b) are “extremal” examples of groupoids. The “middle” and the simplest are pair groupoids. Let X be a set. We put: $\Gamma := X \times X$, $Gr(m) := \{((x,y);(x,z),(z,y)) : x,y,z \in X\}$, $s(x,y) := (y,x)$ and $\Gamma^0 := \{(x,x), x \in X\}$. Then (Γ, m, Γ^0, s) is a groupoid.

d) Equivalence relations. If $R \subset X \times X$ is an equivalence relation on X, then (R, m, Γ^0, s) where m, Γ^0, s are as above is a groupoid.

e) Transformation groupoids. Let a group G act on a set X. We denote the action by $G \times X \ni (g,x) \mapsto gx \in X$. Define $\Gamma := G \times X$, $s(g,x) := (g^{-1}, gx)$, $E := \{e\} \times X$ and m by $Gr(m) := \{((g_1 g_2, x); (g_1, g_2 x), (g_2, x)) : g_1, g_2 \in G, x \in X\} \subset \Gamma \times \Gamma \times \Gamma$. Then (Γ, m, E, s) is a groupoid.

f) Double groups ([11]). Let $(G; A, B)$ be double group i.e. $A, B \subset G$ are subgroups, $A \cap B = \{e\}$ and $G = AB$. In this situation each element of G can be written uniquely as: $g = a_L(g) b_R(g) = b_L(g) a_R(g)$. This decomposition defines four mappings: $a_R, a_L : G \longrightarrow A$ and $b_R, b_L : G \longrightarrow B$. Let $m_A : G \times G \longrightarrow\!\!\triangleright\, G$ be the relation defined by

$$m_A(g_1, g_2) := \begin{cases} g_1 b_R(g_2) = b_L(g_1) g_2 & \text{if } a_R(g_1) = a_L(g_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Let $s_A : G \ni g \mapsto (b_L(g))^{-1} a_L(g) \in G$. Then $G_A := (G, m_A, A, s_A)$ is a groupoid.

g) And many, many more. See e.g. [4]

Morphisms of groupoids

DEFINITION 2.4 ([11]). Let (Γ, m, e, s) and (Γ', m', e', s') be groupoids. *A morphism from Γ to Γ'* is a relation $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ such that:
1. $hm = m'(h \times h)$ 2. $hs = s'h$ 3. $he = e'$.

PROPOSITION 2.5 ([11]). *Let $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ be a morphism of groupoids. Then:*
a) The compositions in the definition above are simple.
b) Let the relation $h_0 : E \longrightarrow\!\!\triangleright E'$ be defined by $Gr(h_0) := Gr(h) \cap (E' \times E)$.
Then $(e'_R \times e_R)Gr(h) = (e'_L \times e_L)Gr(h) = Gr(h_0)$ and $f_h := h_0^T$ is a mapping.
c) Let $b \in E'$ and $a := f_h(b)$. Let us define two relations $h_b^R : F_r(a) \longrightarrow\!\!\triangleright F_r(b)$ and $h_b^L : F_l(a) \longrightarrow\!\!\triangleright F_l(b)$ by $Gr(h_b^R) := Gr(h) \cap (F_r(b) \times F_r(a))$ and $Gr(h_b^L) := Gr(h) \cap (F_l(b) \times F_l(a))$. Then h_b^R, h_b^L are mappings.

Morphisms can also be characterised in terms of mappings.

PROPOSITION 2.6. *Any morphism $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ determines and is uniquely determined by mappings $f : E' \longrightarrow E$ and $g : \Gamma \times_f E' \longrightarrow \Gamma'$, where $\Gamma \times_f E' := \{(x, e') \in \Gamma \times E' : e_R(x) = f(e')\}$ which satisfy:*
a) $e_L e_R^{-1}(f(E')) = f(E')$ (then also $e_R e_L^{-1}(f(E')) = f(E')$)
b) $e'_R g(x, e') = e'$
c) $s' g(x, e') = g(s(x), e'_L g(x, e'))$
d) $\forall\, (x_1, x) \in \Gamma^{(2)}\ (x, e') \in \Gamma \times_f E' \Rightarrow\ g(x_1 x, e') = g(x_1, e'_L g(x, e'))\, g(x, e')$

Let $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ be a morphism and f, g be as above. Denote $\tilde{\Gamma} := \Gamma \times_f E'$. Define $\tilde{s} : \tilde{\Gamma} \ni (x, b) \mapsto (s(x), e'_L g(x, b)) \in \tilde{\Gamma}$, $\tilde{E} := E \times_f E'$ and a relation $\tilde{m} : \tilde{\Gamma} \times \tilde{\Gamma} \longrightarrow\!\!\triangleright \tilde{\Gamma}$ by: $Gr(\tilde{m}) := \{(x_1 x_2, b_2; x_1, e'_L g(x_2, b_2), x_2, b_2) : e_R(x_1) = e_L(x_2), (x_2, b_2) \in \tilde{\Gamma}\}$. Then $(\tilde{\Gamma}, \tilde{m}, \tilde{E}, \tilde{s})$ is a groupoid. Consider relations $h_1 : \Gamma \longrightarrow\!\!\triangleright \tilde{\Gamma}$ and $h_2 : \tilde{\Gamma} \longrightarrow\!\!\triangleright \Gamma'$ defined by: $Gr(h_1) := \{(x, b; x) : (x, b) \in \tilde{\Gamma}\}, Gr(h_2) := \{(g(x, b); x, b) : (x, b) \in \tilde{\Gamma}\}$. Clearly we have $h = h_2 h_1$, moreover h_1 is a morphism from Γ to $\tilde{\Gamma}$ and h_2 is a morphism from $\tilde{\Gamma}$ to Γ'. For h_1 the mappings between fibers are bijective, and f_{h_2} is a bijective mapping. In this way we have the following:

PROPOSITION 2.7. *If $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ is a morphism of groupoids, then there exists a groupoid $\tilde{\Gamma}$, morphisms $k : \Gamma \longrightarrow\!\!\triangleright \tilde{\Gamma}$ and $l : \tilde{\Gamma} \longrightarrow\!\!\triangleright \Gamma'$ such that $h = lk$ and:*
a) For each $a \in \tilde{E}$ the mappings k_a^R and k_a^L are bijections.
b) l is a mapping from $\tilde{\Gamma} \longrightarrow \Gamma'$ which is bijective when restricted to $\tilde{E}$.

Groupoids together with morphisms just defined form a category.

PROPOSITION 2.8 ([11]). *Let $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ and $k : \Gamma' \longrightarrow\!\!\triangleright \Gamma''$ be morphisms of groupoids. Then h and k have simple composition and kh is a morphism from Γ to Γ''.*

EXAMPLES 2.9. a) If X is a set and (Γ, m, e, s) is a groupoid then any morphism $h : X \longrightarrow\!\!\triangleright \Gamma$ is equal to f^T for some mapping $f : E \longrightarrow X$. In particular if X is a set and G is a group then morphisms $X \longrightarrow\!\!\triangleright G$ are just points of X.

b) If G, H are groups then morphisms from G to H are just group homomorphisms.

c) If $(G; A, B)$ is a double group then $m_B^T : G_A \longrightarrow\!\triangleright G_A \times G_A$ and $m_A^T : G_B \longrightarrow\!\triangleright G_B \times G_B$ are morphisms of groupoids [11].

d) For any groupoid Γ the mapping $\Gamma \ni x \mapsto (e_L(x), e_R(x)) \in E \times E$ is a morphism from Γ to the pair groupoid $E \times E$. We denote this relation by $\tilde{e}$.

e) The relation $l : \Gamma \longrightarrow\!\triangleright \Gamma \times \Gamma$ given by: $(x, y; z) \in Gr(l) \Leftrightarrow (x; z, y) \in Gr(m)$ is a morphism from Γ to the pair groupoid $\Gamma \times \Gamma$. It is called *left regular representation* [11].

The above defined morphisms differ from the standard one, but later on we will see that this definition is proper for defining the algebra of a groupoid and the functorial properties of the construction. Also we want to point out that our definition *is not a generalisation* of the usual definition. Below we show that our morphisms are related to actions of groupoids on sets.

Definition 2.10 ([7]). Let (Γ, m, e, s) be a groupoid, Y be a set and $\mu : Y \longrightarrow \Gamma^0$ be a mapping. Denote $\Gamma \times_\mu Y := \{(x, y) \in \Gamma \times Y \,:\, e_R(x) = \mu(y)\}$. *The (left) action of* Γ *on* Y is a mapping $\phi : \Gamma \times_\mu Y \ni (x, y) \mapsto \phi(x, y) \in Y$ which satisfies:

$$\mu\phi(x, y) = e_L(x), \qquad \phi(x_1 x_2, y) = \phi(x_1, \phi(x_2, y)), \qquad \phi(\mu(y), y) = y.$$

Now let Γ act on Y. Put $f := \mu$ and $g : \Gamma \times_f Y \ni (x, y) \mapsto (\phi(x, y), y) \in Y \times Y$. Then it is easy to see that these mappings satisfy the conditions given in Prop. 2.6, so they determine a morphism from Γ to the pair groupoid $Y \times Y$. Conversely, if $h : \Gamma \longrightarrow\!\triangleright Y \times Y$ is a morphism then putting: $\mu := f_h$ and $\phi(x, y) := e'_L h_y^R(x)$ we get an action of Γ on Y. Also for any morphism $h : \Gamma \longrightarrow\!\triangleright \Gamma'$ the mappings $\mu := f_h e'_L : \Gamma' \longrightarrow E$ and $\phi(x, x') := h_{a'}^R(x)x'$ where $a' := e'_L(x')$ define an action of Γ on Γ'.

Bissections. A *bissection* B is a subset of Γ such that $e_L|_B : B \longrightarrow \Gamma^0$ and $e_R|_B : B \longrightarrow \Gamma^0$ are bijections. The set of bissections of Γ will be denoted by $\mathcal{B}(\Gamma)$. Bissections form a group, which acts on Γ according to the formula: $Bx := B\{x\}$ (left action) and $xB := \{x\}B$ (right action). Bissections can also be characterized as subsets of Γ with the property that $Bs(B) = s(B)B = \Gamma^0$.

Examples 2.11. a) For any groupoid the set of identities is a bissection.

b) If Γ is a group then bissections are just group elements.

c) If $\Gamma := X \times X$ is a pair groupoid then any bissection is of the form $B := \{(f(x), x) \,:\, x \in X\}$ for some bijection $f : X \longrightarrow X$.

Morphisms of groupoids act on bissections: if $h : \Gamma \longrightarrow\!\triangleright \Gamma'$ is a morphism and B a bissection of Γ then the set $h(B)$ is a bissection of Γ'.

3. Differential groupoids. From now on, when we use the word *manifold* without any comments, we mean a Hausdorff, finite dimensional, smooth manifold with a countable basis of neighbourhoods. A *submanifold* is a nonempty, embedded submanifold (with the relative topology). *A differentiable relation* $r : X \longrightarrow\!\triangleright Y$ is a triple $r = (R; Y, X)$ such that X, Y are manifolds and R is a submanifold in $Y \times X$. The *tangent lift* of a differentiable relation $r : X \longrightarrow\!\!\triangleright Y$ is a relation $Tr : TX \longrightarrow\!\!\triangleright TY$ with a graph $Gr(Tr) := TGr(r)$. A *phase lift* of r is a relation $Pr : T^*X \longrightarrow\!\!\triangleright T^*Y$ such that:

$$(\alpha, \beta) \in Gr(Pr) \iff <\alpha, u>=<\beta, v> \text{ for any } (u, v) \in T_{(y,x)}Gr(r),$$

where $y := \pi_Y(\alpha), x := \pi_X(\beta)$ and π_X, π_Y are the canonical projections of the cotangent bundles. We say that relations $r : X \longrightarrow\!\!\triangleright Y$ and $s : Y \longrightarrow\!\!\triangleright Z$ are *transverse* (and write $r \pitchfork s$) iff Tr, Ts and Pr, Ps have simple composition, and sr is a differentiable relation. Let us also recall that *a differentiable reduction* is a differentiable relation $r : X \longrightarrow\!\!\triangleright Y$ of the form $r = fi^T$, where $i : C \longrightarrow X$ is the inclusion map of the submanifold $C \subset X$ and $f : C \longrightarrow Y$ is a surjective submersion.

Definition 3.1 ([12]). *A differential groupoid* (Γ, m, e, s) is a groupoid such that Γ is a manifold, m is a differentiable reduction, e is a differentiable relation, s is a diffeomorphism and the following transversality relations hold: $m \pitchfork (m \times id), m \pitchfork (id \times m), m \pitchfork (e \times id), m \pitchfork (id \times e)$.

It can be shown [12] that in this situation e_L, e_R are submersions.

Examples 3.2. a) Examples 2.3 a)-e) with obvious smoothness conditions are differential groupoids.

b) *Double Lie groups* ([3]). We say that a double group $(G; A, B)$ is a double Lie group iff G is a Lie group and A, B are closed subgroups of G. Then G_A, G_B are differential groupoids.

c) *Tangent and cotangent bundles.* If X is a manifold then $(TX, +, X, -)$ and $(T^*X, +, X, -)$ are differential groupoids. More generally, if (P, X) is a vector bundle then it is a groupoid in a natural way $(P, +, X, -)$.

d) *Tangent and phase lifts of differential groupoids* ([12]). If (Γ, m, e, s) is a differential groupoid then $(T\Gamma, Tm, Te, Ts)$ and $(T^*\Gamma, Pm, Pe, -Ps)$ are differential groupoids. If $\Gamma := (X, d^T, X, id)$ is a manifold groupoid then its tangent lift $T\Gamma = (TX, d^T_{TX}, TX, id)$ is again a manifold groupoid but its cotangent lift $P\Gamma = (T^*X, +, X, -)$ is a cotangent bundle with the usual groupoid structure.

e) If $\Gamma = (G, m, e, s)$ is a Lie group, then its tangent lift is a Lie group TG. But the phase lift is T^*G as a transformation groupoid: $T^*G = G \times \mathfrak{g}^*$ with coadjoint action.

Morphisms of differential groupoids

Definition 3.3 ([12]). *Let* Γ, Γ' *be differential groupoids and* $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ *a differentiable relation which is an (algebraic) morphism of groupoids. Then* h *is a morphism of differential groupoids iff* $m' \pitchfork (h \times h)$ *and* $h \pitchfork e$.

Proposition 3.4 ([12]). *If* $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ *is a morphism of differential groupoids then:*
a) $f_h : E' \longrightarrow E$ *is a smooth mapping.*
b) $\Gamma *_h E' := \{(x, b) \in \Gamma \times E' : e_L(x) = f_h(b)\}$ *is a submanifold of* $\Gamma \times E'$ *(and of* $\Gamma \times \Gamma'$*).*
c) The mapping: $Gr(h) \ni (y, x) \mapsto (x, e'_L(y)) \in \Gamma *_h E'$ *is a diffeomorphism.*

Using this proposition one shows that the sets: $\Gamma *_h \Gamma' := \{(x,y) \in \Gamma \times \Gamma' \,:\, e_L(x) = f_h(e'_L(y))\}$ and $\Gamma \times_h \Gamma' := \{(x,y) \in \Gamma \times \Gamma' \,:\, e_R(x) = f_h(e'_L(b))\}$ are submanifolds of $\Gamma \times \Gamma'$. Let us define a mapping $m_h : \Gamma \times_h \Gamma' \ni (x,y) \mapsto m'(h^R_b(x), y) \in \Gamma'$, where $b := e'_L(y)$. It turns out that m_h is a surjective submersion. It is illustrated in figure 1. Also we define the diffeomorphism $\mathfrak{t}_h : \Gamma \times_h \Gamma' \ni (x,y) \mapsto (x, m_h(x,y)) \in \Gamma *_h \Gamma'$.

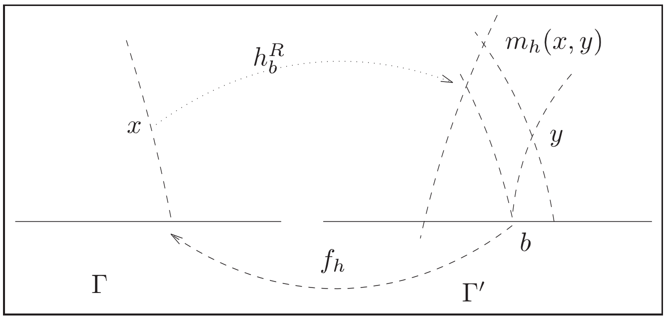

Fig. 1

Examples 3.5. a) If $f : X \longrightarrow Y$ is a smooth mapping then T^*f considered as a relation: $T^*Y \longrightarrow\!\!\triangleright T^*X$ is a morphism of differential groupoids. The same is true for $(Tf)^T : TY \longrightarrow\!\!\triangleright TX$. Note that here TX, TY are considered as manifold groupoids, not vector bundle groupoids. Unless f is a local diffeomorphism, $(Tf)^T$ is not a morphism of $(TY, +, Y, -)$ and $(TX, +, X, -)$.

b) Let a Lie group G act on a manifold X. We form the transformation groupoid $\Gamma := G \times X$. Let Y be a manifold and $h : \Gamma \longrightarrow\!\!\triangleright Y \times Y$ be a morphism to the pair groupoid of Y. Consider the smooth mapping $\Phi : G \times Y \ni (g,y) \mapsto e'_L h^R(g, f_h(y); y) \in Y$. Then Φ defines an action of G on Y. Moreover f_h is equivariant, i.e. $g f_h(y) = f_h \Phi(g,y)$. Conversely, if G acts on X and Y with equivariant mapping $f : Y \longrightarrow X$ then h defined by: $Gr(h) := \{(gy, y; g, f(y))) \,:\, y \in Y, g \in G\}$ is a morphism from Γ to $Y \times Y$.

c) Let X, Y be manifolds and $\Gamma := X \times X$, $\Gamma' := Y \times Y$ be the corresponding pair groupoids. Then using Props. 2.6 and 3.4 one can see that any morphism $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ is determined by a smooth surjection $f : Y \longrightarrow X$ and a smooth mapping $g : X \times Y \longrightarrow Y$ which satisfy for any $x, x_1 \in X, y \in Y$: a) $fg(x,y) = x$, b) $g(f(y), g(x,y)) = y$, c) $g(x,y) = g(x, g(x_1, y))$. Then $Gr(h) := \{(g(x,y), y; x, f(y)) \,:\, x \in X, y \in Y\}$. One can show that f is a submersion and for $x_0 \in X$, $Z := f^{-1}(x_0)$ the mapping: $\phi : X \times Z \ni (x,y) \mapsto g(x,y) \in Y$ is a diffeomorphism. In this way for any morphism $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ there exists a diffeomorphism $\phi : Y \longrightarrow X \times Z$ and $Gr((\phi \times \phi)h) = \{(x, z, x_1, z; x, x_1) \,:\, x, x_1 \in X, z \in Z\}$.

The next proposition shows that differential groupoids with the morphisms defined above form a category.

PROPOSITION 3.6 ([12]). *Let $\Gamma, \Gamma', \Gamma''$ be differential groupoids and let $h : \Gamma \longrightarrow\!\triangleright \Gamma'$ and $k : \Gamma' \longrightarrow\!\triangleright \Gamma''$ be morphisms. Then $h \pitchfork k$ and $kh : \Gamma \longrightarrow\!\triangleright \Gamma''$ is a morphism.*

A submanifold $B \subset \Gamma$ is a *bissection* iff $e_L|_B$ and $e_R|_B$ are diffeomorphisms. If $h : \Gamma \longrightarrow\!\triangleright \Gamma'$ is a morphism of differential groupoids and B is a bissection then $h(B)$ is a bissection.

4. Construction of the $*$-algebra of a differential groupoid. In this section we introduce the $*$-algebra of a differential groupoid. The way we do it is rather nonstandard and at first sight may be regarded as too complicated, nevertheless it is very convenient for the further development.

Let Γ be a differential groupoid and let $\Omega^{1/2}(e_L)$ ($\Omega^{1/2}(e_R)$) be the smooth bundle of complex half densities along the left (right) fibers of Γ. Following A. Connes [2] our basic object is the linear space of compactly supported smooth sections of the bundle $\Omega^{1/2}(e_L) \otimes \Omega^{1/2}(e_R)$. We denote this space by $\mathcal{A}(\Gamma)$. Its elements will be called *bidensities* and usually denoted by ω. So $\omega(x) = \lambda(x) \otimes \rho(x) \in \Omega^{1/2} T^l_x\Gamma \otimes \Omega^{1/2} T^r_x\Gamma$, where we used notation: $T^l_x\Gamma := T_x(F_l(x)), T^r_x\Gamma := T_x(F_r(x))$. In the following we also write $\Omega^{1/2}_L(x) := \Omega^{1/2} T^l_x\Gamma$ and $\Omega^{1/2}_R(x) := \Omega^{1/2} T^r_x\Gamma$.

We also use the following notation: if M, N are manifolds, $F : M \longrightarrow N$ and Ψ is some geometric object on M which can be pushed-forward by F, then we denote the push-forward of ψ simply by $F\psi$. What it really means will be clear from the context.

The groupoid inverse induces the star operation on $\mathcal{A}$ as follows

$$\omega^*(x)(v \otimes w) := \overline{\omega(s(x))(s(w) \otimes s(v))}, v \in \Lambda^{max} T^l_x\Gamma, w \in \Lambda^{max} T^r_x\Gamma$$

(for any vector space V by $\Lambda^{max}V$ we denote the maximal exterior power of V). This is a well defined antilinear involution (since s is an involutive diffeomorphism which interchanges left and right fibers).

We are going to show that with any morphism $h : \Gamma \longrightarrow\!\triangleright \Gamma'$ is associated a mapping $\hat{h} : \mathcal{A}(\Gamma) \longrightarrow L\mathcal{A}(\Gamma')$ ($L\mathcal{A}(\Gamma')$ denotes linear endomorphisms of $\mathcal{A}(\Gamma')$), which "well behaves" with respect to composition of morphisms and the $*$-operation. Then putting $h = id$ we get an algebra structure on $\mathcal{A}(\Gamma)$. Before this we define some special sections of $\Omega^{1/2}(e_L) \otimes \Omega^{1/2}(e_R)$ which are very convenient for computations.

$*$-invariant bidensities. Since left (right) translations are diffeomorphisms of left (right) fibers, we can define left (right) invariant sections of $\Omega^{1/2}(e_L)(\Omega^{1/2}(e_R))$, namely a section λ is left invariant iff for any $(x, y) \in \Gamma^{(2)}$, $\lambda(xy)(xv) = \lambda(y)(v)$, $v \in \Lambda^{max} T^l_y\Gamma$. In the same way right invariant half densities are defined. Any left invariant half density is determined by its value on Γ^0 and conversely any section of $\Omega^{1/2}(e_L)|_{\Gamma^0}$ can be uniquely extended to a left invariant half density on Γ.

So let $\tilde{\lambda}$ be a nonvanishing, real, half density on Γ^0 along the left fibers (one constructs such a density by covering Γ^0 with maps adapted to the submersion e_L and using an appropriate partition of unity to glue them together). We define:

$$\lambda_0(x)(v) := \tilde{\lambda}(e_R(x))(s(x)v), v \in \Lambda^{max} T^l_x \Gamma,$$

then λ_0 is a left invariant, nonvanishing section of $\Omega^{1/2}(e_L)$. Now $\tilde{\rho} := \tilde{\lambda} s$ is a nonvanishing, real, half density on Γ^0 along the right fibers, and ρ_0 defined by: $\rho_0(x)(v) := \tilde{\rho}(e_L(x)(vs(x)), v \in \Lambda^{max} T^r_x \Gamma$ is a right invariant, nonvanishing, real half density along the right fibers. Let $\omega_0 := \lambda_0 \otimes \rho_0$; this is a real, nonvanishing bidensity. *From now on the symbol ω_0 will always mean the bidensity constructed in this way.* When ω_0 is chosen any element $\omega \in \mathcal{A}(\Gamma)$ can be written uniquely as $\omega = f\omega_0$ for some smooth, complex function f with compact support. Note the following:

LEMMA 4.1. *If $\omega = f\omega_0$ then $\omega^* = f^*\omega_0$ where $f^*(x) := \overline{f(s(x))}$.*

Choosing λ_0 in fact we choose some left Haar system in the sense of [6] on our groupoid. But all our constructions and in particular our C^* algebra are independent of this choice.

Action of groupoid morphisms on bidensities. Now, for a morphism $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$ we construct the mapping $\hat{h}$. It can be shown that:

1. The set $\Gamma \times_h \Gamma'_a := \{(x, y) \in \Gamma \times \Gamma' : e_R(x) = f_h(e'_L(y)), e'_R(y) = a\}$ is a submanifold of $\Gamma \times_h \Gamma'$.

2. The mapping: $\pi_2 : \Gamma \times_h \Gamma'_a \ni (x, y) \mapsto y \in \Gamma'_a$ is a surjective submersion and $\pi_2^{-1}(y)$ is diffeomorphic to $F_r(f_h(e'_L(y)))$.

3. The mapping $\mathfrak{t}_h : \Gamma \times_h \Gamma'_a \ni (x, y) \mapsto (x, m_h(x, y)) \in \Gamma *_h \Gamma'_a := \{(x, y) \in \Gamma \times \Gamma'_a : e_L(x) = f_h(e'_L(y))\}$ is a diffeomorphism.

4. $\tilde{\pi}_2 : \Gamma *_h \Gamma'_a \ni (x, y) \mapsto y \in \Gamma'_a$ is a surjective submersion and $\tilde{\pi}_2^{-1}(y)$ is diffeomorphic to $F_l(f_h(e'_L(y)))$.

Before we go further, let us recall some facts about densities. Let V be a finite dimensional vector space. For $p \geq 0$ we denote the linear space of complex p-densities on V by $\Omega^p(V)$. If $V = V_1 \oplus V_2$ and ν_1, ν_2 are p-densities on V_1, V_2 then the formula $(\nu_1 \otimes \nu_2)(v_1 \wedge v_2) := \nu_1(v_1)\nu_2(v_2)$ for $v_1 \in \Lambda^{max} V_1, v_2 \in \Lambda^{max} V_2$ defines an isomorphism $\Omega^p(V) = \Omega^p(V_1) \otimes \Omega^p(V_2)$. Also we have $\Omega^p(V) = \Omega^p(V_1) \otimes \Omega^p(V/V_1)$ defined by choosing some $V_2 \subset V$ complementary to V_1. The isomorphism does not depend on the choice made. In this way if $F : V \longrightarrow W$ is a linear surjection, we have a canonical isomorphism $\Omega^p(V) = \Omega^p(ker\, F) \otimes \Omega^p(W)$. This fact is constantly used in the following. Now we go back to groupoid morphisms.

Let $(x, y) \in \Gamma \times_h \Gamma'_a$ and $\mathfrak{t}_h(x, y) =: (x, z), b := e'_L(z)$. Due to point 2, we have an isomorphism: $i_1 : \Omega^{1/2}_R(x) \otimes \Omega^{1/2}_R(y) \longrightarrow \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_a)$. From point 3, $\mathfrak{t}_h : \Omega^{1/2} T_{(x,y)}(\Gamma \times_h \Gamma'_a) \longrightarrow \Omega^{1/2} T_{(x,z)}(\Gamma *_h \Gamma'_a)$ is an isomorphism and from 4, $i_2 : \Omega^{1/2}_L(x) \otimes \Omega^{1/2}_R(z) \longrightarrow \Omega^{1/2} T_{(x,z)}(\Gamma *_h \Gamma'_a)$ is an isomorphism. So $(i_2)^{-1} \mathfrak{t}_h i_1(\rho_x \otimes \rho_y) =: \lambda_x \otimes \rho_z$ for some $\lambda_x \otimes \rho_z \in \Omega^{1/2}_L(x) \otimes \Omega^{1/2}_R(z)$. Moreover the mapping $F_l(y) \ni u \mapsto h^R_{e'_L(y)}(x)u \in F_l(z)$ is a diffeomorphism, so for $\lambda_y \in \Omega^{1/2}_L(y)$, $h^R_{e'_L(y)}(x)\lambda_y \in \Omega^{1/2}_L(z)$.

Now let $\omega = \lambda \otimes \rho \in \mathcal{A}(\Gamma)$ $\omega' = \lambda' \otimes \rho' \in \mathcal{A}(\Gamma')$. Then $(i_2)^{-1} \mathfrak{t}_h i_1(\rho(x) \otimes \rho'(y)) =:$

$\tilde{\lambda_x} \otimes \tilde{\rho_z}$ and $h^R_{e'_L(y)}(x)\lambda'(y) =: \tilde{\lambda'_z}$. So the expression: $[\lambda(x)\tilde{\lambda_x}] \otimes \tilde{\lambda'_z} \otimes \tilde{\rho_z}$ defines a one-density on $F_l(f_h(b))$ with values in the one dimensional vector space $\Omega_L^{1/2}(z) \otimes \Omega_R^{1/2}(z)$.

Let us define

$$(\hat{h}(\omega)\omega')(z) := \int_{F_l(f_h(b))} [\lambda\tilde{\lambda}] \otimes \tilde{\lambda'_z} \otimes \tilde{\rho_z}.$$

Choose $\omega_0 = \lambda_0 \otimes \rho_0$, $\omega'_0 = \lambda'_0 \otimes \rho'_0$. Then $\omega = f_1\,\omega_0, \omega' = f_2\,\omega'_0$ and $(i_2)^{-1}\mathfrak{t}_h i_1(\rho_0(x) \otimes \rho'_0(y)) =: t_h(x,y)\lambda_0(x) \otimes \rho'_0(z)$ for some smooth, nonvanishing function $t_h : \Gamma \times_h \Gamma' \longrightarrow R$ and $h^R_{e'_L(y)}(x)\lambda'_0(y) = \lambda'_0(z)$.

We get the explicit expression:

$$(\hat{h}(\omega)\omega')(z) := \left[\int_{F_l(f_h(b))} \lambda_0^2(x) f_1(x) t_h(x,y) f_2(y)\right] \omega'_0(z) =: (f_1 *_h f_2)(z)\,\omega'_0(z),$$

where y is defined by $\mathfrak{t}_h(x,y) = (x,z)$, i.e. $y = s'(h^L_b(x))z$.

The next proposition is crucial for the construction, it describes how the mapping $\hat{h}$ behaves with respect to composition of morphisms. We cannot simply write: $\hat{k}(\hat{h}\omega) = \hat{kh}\omega$ since the left hand side is not defined. Instead of this equality we prove another one, which is formally the same as for morphisms of C^*-algebras.

PROPOSITION 4.2. *Let $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma', k : \Gamma' \longrightarrow\!\!\!\triangleright\, \Gamma''$ be morphisms of differential groupoids. Then*

$$\hat{k}(\hat{h}(\omega_1)\omega_2)\omega_3 = \hat{kh}(\omega_1)(\hat{k}(\omega_2)\omega_3) \quad \textit{for any } \omega_1 \in \mathcal{A}(\Gamma), \omega_2 \in \mathcal{A}(\Gamma'), \omega_3 \in \mathcal{A}(\Gamma'').$$

PROOF. The proof of this proposition is based on the following lemma which describes how the functions t_h, t_k and t_{kh} are related. The lemma is proven by investigation of various submanifolds defined by the morphisms h, k, kh and various natural identifications among (half)densities at corresponding points.

LEMMA 4.3. *Let $(x,y,z) \in \Gamma \times \Gamma' \times \Gamma''$ satisfy: $e_R(x) = f_h(e'_L(y)), e'_R(y) = f_k(e''_L(z))$ and let y', z' be defined by: $\mathfrak{t}_h(x,y) = (x,y')$ and $\mathfrak{t}_k(y,z) = (y,z')$. Then $t_h(x,y)t_k(y',z) = t_{kh}(x,z')t_k(y,z)$.*

REMARK 4.4 (the algebra structure on $\mathcal{A}(\Gamma)$). Take $h = id : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma$, then $\Gamma \times_h \Gamma = \Gamma^{(2)}$ and put: $\omega_1\omega_2 := \hat{id}(\omega_1)\omega_2$. Due to the above proposition, this product is associative. Chosen ω_0 we can write $\omega_1 = f_1\omega_0, \omega_2 = f_2\omega_0$. In this situation $t_{id} \equiv 1$ and the explicit formula for the product is: $\omega_1\omega_2 =: (f_1 * f_2)\omega_0$ and

$$(f_1 * f_2)(x) := \int_{F_l(x)} \lambda_0^2(y) f_1(y) f_2(s(y)x) = \int_{F_r(x)} \rho_0^2(y) f_1(xs(y)) f_2(y).$$

The equality is implied by the fact that right and left fibers are diffeomorphic by s.

That the multiplication is compatible with the star operation (i.e. $\mathcal{A}$ is in fact a $*$-algebra) can be shown directly, but it will follow from more general facts. The multiplication is nondegenerate, i.e. if $\omega'\omega = 0$ for $\omega \in \mathcal{A}(\Gamma)$ then $\omega' = 0$.

It seems that there is no natural, geometric, norm on $\mathcal{A}(\Gamma)$, but one can introduce the family of useful norms "indexed" by ω_0 [6]. So choose λ_0 and write $\omega = f\omega_0$. We define

$$||\omega||_l := \sup_{a\in\Gamma^0}\int_{F_l(a)}\lambda_0^2\,|f|,\;\; ||\omega||_r := \sup_{a\in\Gamma^0}\int_{F_r(a)}\rho_0^2\,|f|,\;\; ||\omega|| := max\{||\omega||_l, ||\omega||_r\}.$$

(We do not explicitly write the dependence on λ_0 to make notation simpler.) The next lemma shows that the definitions are correct.

LEMMA 4.5. *The functions $||.||_l, ||.||_r, ||.||$ are norms and give $\mathcal{A}$ the structure of a normed algebra. Moreover $||\omega^*||_l = ||\omega||_r$ so $||\omega^*|| = ||\omega||$ (in fact $(\mathcal{A}, *, ||,||)$ is a normed *-algebra as we will see later on).*

REMARK 4.6. We can try to define a "geometric" norm on $\mathcal{A}(\Gamma)$ as follows. Recall that the orbit of a point $a \in E$ is the set $e_L(\Gamma_a) = e_R({}_a\Gamma)$. It is known [4] that for each $a \in \Gamma^0$ the set $\Gamma_a \cap {}_a\Gamma$ is a submanifold in Γ and a Lie group. Since right and left translations are diffeomorphisms of the fibers it is clear that all sets ${}_a\Gamma \cap \Gamma_b$ for a, b in the same orbit are diffeomorphic submanifolds. Also we have that $e_L|_{\Gamma_a} : \Gamma_a \longrightarrow \Gamma^0$ and $e_R|_{{}_a\Gamma} : {}_a\Gamma \longrightarrow \Gamma^0$ are of constant rank so orbits are immersed submanifolds. Now suppose that each orbit in Γ^0 is a submanifold; let us denote the orbit through a by O_a. In this case $(\Gamma_a, O_a, e_L|_{\Gamma_a})$ and $({}_a\Gamma, O_a, e_R|_{{}_a\Gamma})$ are locally trivial differential fibrations, with the fibers diffeomorphic to the Lie group ${}_a\Gamma \cap \Gamma_a$. If λ is a half density on Γ along the left fibers then its restriction to ${}_a\Gamma$ can be written as $\lambda(x) = \mu(e_R(x)) \otimes \nu(x)$ for μ a half density on O_a and ν a half density on ${}_a\Gamma$ along the fibers of $e_R|_{{}_a\Gamma}$. In the same way if ρ is a half density along the right fibers then its restriction to Γ_a can be written as $\rho(x) = \mu_1(e_L(x)) \otimes \nu_1(x)$ for μ_1 a half density on the orbit and ν_1 a half density along the fibers of $e_L|_{\Gamma_a}$. So for $\omega = \lambda \otimes \rho \in \mathcal{A}(\Gamma)$ and $x \in \Gamma$ with $e_L(x) = a, e_R(x) = b$ we have $\omega(x) = \mu(e_R(x)) \otimes \mu_1(e_L(x)) \otimes \nu(x) \otimes \nu_1(x)$ but since fibers of $e_R|_{\Gamma_a}$ and $e_L|_{{}_b\Gamma}$ are the same $\nu(x) \otimes \nu_1(x)$ is a density on $T_x(\Gamma_a \cap {}_b\Gamma)$. Let S denote the set of orbits and define the following quantity:

$$||\omega||_{geom} := \sup_{s\in S}\sqrt{\int_{s\times s}|\mu_1|^2(a)\otimes|\mu|^2(b)\left(\int_{{}_a\Gamma\cap\Gamma_b}|\nu\nu_1|\right)^2}.$$

This quantity is finite and for $\omega \in \mathcal{A}(\Gamma)$ we have $||\omega||_{geom} \le ||\omega||$ where the norm on the right side is introduced above.

Now we show that $\mathcal{A}$ is a normed $*$-algebra and morphisms define $*$-homomorphisms. As in Prop. 4.2 the equality we prove is taken from the definition of conjugation of a linear mapping on a C^*-algebra [8].

PROPOSITION 4.7. *Let $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$ be a morphism of differential groupoids. Then*

$$\omega_3^*(\hat{h}(\omega_1)(\omega_2)) = (\hat{h}(\omega_1^*)\omega_3)^*\omega_2 \quad \textit{for any } \omega_1 \in \mathcal{A}(\Gamma), \omega_2, \omega_3 \in \mathcal{A}(\Gamma').$$

REMARK 4.8 ($*$-algebra structure on $\mathcal{A}$). Take $h = id$. Then we have $\omega_3^*(\omega_1\omega_2) = (\omega_3^*\omega_1)\omega_2 = (\omega_1^*\omega_3)^*\omega_2$ for any bidensities $\omega_1, \omega_2, \omega_3$. So $(\omega_3^*\omega_1) = (\omega_1^*\omega_3)^*$ (multiplication is nondegenerate) and since $*$ is an involution we have that $(\omega_1\omega_2)^* = \omega_2^*\omega_1^*$.

Representation of the $*$-algebra of a groupoid associated with a morphism. Let $h : \Gamma \longrightarrow\!\!\triangleright\, \Gamma'$ be a morphism of differential groupoids. Then it defines a representation of the $*$-algebra $\mathcal{A}(\Gamma)$ in the Hilbert space $L^2(\Gamma')$ of square integrable half densities on Γ' as follows.

Let Ψ be a smooth half density on Γ' with compact support and $\omega \in \mathcal{A}(\Gamma)$, $\omega = \lambda \otimes \rho$. Let $(x, y) \in \Gamma \times_h \Gamma'$ and $\mathfrak{t}_h(x, y) =: (x, z)$. As in the definition of $\hat{h}$, $\rho(x) \otimes \Psi(y)$ can be viewed as a half density on $T_{(x,y)}(\Gamma \times_h \Gamma')$ and $\mathfrak{t}_h(\rho(x) \otimes \Psi(y))$ is a half density on $T_{(x,z)}(\Gamma *_h \Gamma')$. Since $\Omega_L^{1/2}(x) \otimes \Omega^{1/2} T_z\Gamma' \simeq \Omega^{1/2} T_{(x,z)}(\Gamma *_h \Gamma')$ this half density can be written as $\tilde{\lambda_x} \otimes \Psi_x(z)$ for $\tilde{\lambda_x}$ a half density on $T_x(F_l(x))$ and $\Psi_x(z)$ a half density on $T_z(\Gamma')$. Then $\lambda(x)\tilde{\lambda_x} \otimes \Psi_x(z)$ is a 1-density on $T_x^r\Gamma$ with values in half densities on $T_z(\Gamma')$. Integrating it we get a half density on $T_z(\Gamma')$. Let us define:

$$(\pi_h(\omega)\,\Psi)(z) := \int_{F_l(f_h(a))} [\lambda(x)\tilde{\lambda}_x] \otimes \Psi_x(z).$$

Choose ω_0 and write $\omega = f\,\omega_0$. Since e'_R is a surjective submersion we have: $\Omega^{1/2} T_w\Gamma' \simeq \Omega_R^{1/2}(w) \otimes \Omega^{1/2} T_{e'_R(w)} E'$ for any $w \in \Gamma'$. In this way, if we choose ρ'_0 and ν_0 a nonvanishing, real half density on E' then $\rho'_0 \otimes \nu_0$ defines a nonvanishing, real, half density on Γ'. So any other smooth half density with compact support Ψ can be written as $\Psi = \psi\,\rho'_0 \otimes \nu_0 =: \psi\,\Psi_0$ for some smooth, complex function ψ with compact support. It is easy to see that: $\mathfrak{t}_h(\rho_0(x) \otimes \rho'_0(y) \otimes \nu_0(a)) = t_h(x, y)\lambda_0(x) \otimes \rho'_0(z) \otimes \nu_0(a)$ where t_h is as in the definition of $\hat{h}$. So the explicit formula is:

$$(\pi_h(\omega)\Psi)(z) = \left[\int_{{}^b\Gamma} \lambda_0^2(x)\, f(x)\, t_h(x, y)\, \psi(y)\right] \Psi_0(z),$$

where $b := f_h(e'_L(z))$, $\mathfrak{t}_h(x, y) = (x, z)$. Note that formally the expression is the same as in Prop. 4.2.

PROPOSITION 4.9. *a) Let $h : G \longrightarrow\!\!\triangleright\, \Gamma'$ be a morphism of differential groupoids. Let $||.||$ be a norm on $\mathcal{A}(\Gamma)$ associated with the chosen ω_0. The correspondence: $\mathcal{A}(\Gamma) \ni \omega \mapsto \pi_h(\omega)$ is a representation of the normed $*$-algebra $\mathcal{A}(\Gamma)$ in $L^2(\Gamma')$.*
b) If $k : \Gamma' \longrightarrow\!\!\triangleright\, \Gamma''$ is a morphism then: $\pi_k(\hat{h}(\omega_1)\omega_2)\Psi = \pi_{kh}(\omega_1)\pi_k(\omega_2)\Psi$ for any $\omega_1 \in \mathcal{A}(\Gamma), \omega_2 \in \mathcal{A}(\Gamma'), \Psi$ a smooth half density on Γ'' with compact support.

EXAMPLES 4.10. *a) Reduced C^*-algebra of a differential groupoid.* Let l be the morphism from Γ to the pair groupoid $\Gamma \times \Gamma$ defined in Example 2.9 f), i.e. $(x, y; z) \in Gr(l) \iff (x; z, y) \in Gr(m)$. Let π_l be a representation of $\mathcal{A}(\Gamma)$ in $L^2(\Gamma \times \Gamma) = L^2(\Gamma) \otimes L^2(\Gamma)$ associated with the morphism l. Short computation shows that $\pi_l = \pi_{id} \otimes I$. So $||\pi_l(\omega)|| = ||\pi_{id}(\omega)||$ and it can be shown that the function $\mathcal{A}(\Gamma) \ni \omega \mapsto ||\pi_{id}(\omega)||$ is a C^*-norm on $\mathcal{A}(\Gamma)$. The completion of $\mathcal{A}(\Gamma)$ in this norm will be called *the reduced C^*-algebra of* Γ and denoted by $C^*_{red}(\Gamma)$.

b) Modular function. Let $\tilde{e}$ be a morphism from Γ to the pair groupoid $E \times E$ defined in Example 2.9 e), i.e. $Gr(\tilde{e}) = \{(e_L(x), e_R(x); x) : x \in \Gamma\}$. It is easy to see that $\mathfrak{t}_{\tilde{e}}(x; e_R(x), e) = (x; e_L(x), e)$. Choose some $\omega_0 = \lambda_0 \otimes \rho_0$ and some real, nonvanishing half density ν_0 on E. Such a choice defines a function $t_{\tilde{e}}(x; e_R(x), e)$. From Lemma 4.3, this function does not depend on e and if we define $\Delta(x) := t_{\tilde{e}}(x, e_R(x))$ then $\Delta(xy) = \Delta(x)\Delta(y)$ for any composable $x, y \in \Gamma$. Δ is called *a modular function of* Γ (it depends on the chosen λ_0, ν_0). Δ can also be described in the following way. When ω_0, ν_0 are chosen, the expressions: $\psi_r(x) := \rho_0(x) \otimes \nu_0(e_R(x))$ and $\psi_l(x) := \lambda_0(x) \otimes \nu_0(e_L(x))$ define smooth, nonvanishing, real half densities on Γ. Then Δ is defined by: $\psi_l =: \Delta\psi_r$.

Nondegeneracy of morphisms. In the context of morphisms of C^*-algebras an important role is played by the nondegeneracy condition. The following proposition is used to prove that the action of morphisms on bidensities is nondegenerate (i.e. if $\hat{h}(\omega)\omega' = 0$ for any ω then $\omega' = 0$).

PROPOSITION 4.11. *Let $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$ be a morphism of differential groupoids. Then for any $\omega' \in \mathcal{A}(\Gamma')$ there exists a sequence $\omega_n \in \mathcal{A}(\Gamma)$ such that:* $\lim_{n\to\infty} \hat{h}(\omega_n)\omega' = \omega'$. *(The limit is in the norm given by some ω'_0.)*

REMARK 4.12. a) The action of morphisms on bidensities is nondegenerate. Indeed, if $\hat{h}(\omega)\omega' = 0$ for any ω then taking ω_n as above we have: $0 = \lim_{n\to\infty} \hat{h}(\omega_n)\omega' = \omega'$.
b) The representation π_h associated with the morphism h is nondegenerate.

Note that we don't claim that $\hat{h}\omega_n$ is an approximate identity for $\mathcal{A}(\Gamma')$, since ω_n depend on the chosen ω'. Also $\pi_h(\omega_n)$ does not converge strongly or weakly to the identity on $L^2(\Gamma')$.

5. C^*-algebra of a differential groupoid. Let Γ be a differential groupoid. The results from the previous sections show that the following definition is meaningful.

DEFINITION 5.1. *The C^*-algebra of a differential groupoid* Γ is the completion of $\mathcal{A}(\Gamma)$ with respect to the norm: $||\omega|| := \sup_h ||\pi_h(\omega)||$, where the supremum is taken over all morphisms $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$.

Now we can state the main result, i.e. functoriality of our construction.

PROPOSITION 5.2. *a) For any morphism $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$, π_h extends to a nondegenerate representation of $C^*(\Gamma)$ and $\hat{h}$ extends to $C^*(h) \in Mor(C^*(\Gamma), C^*(\Gamma'))$.*
b) $C^*(kh) = C^*(k)C^*(h)$.

The definition of the C^* norm given above is rather abstract and the C^* algebra obtained seems untreatable. However, we can restrict ourselves to a smaller class of morphisms—namely to the morphisms into pair groupoids. Let $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma'$ be a morphism of differential groupoids and let $l' : \Gamma' \longrightarrow\!\!\!\!\triangleright\, \Gamma' \times \Gamma'$ be a left regular representation as defined in 2.9 f) ($\Gamma' \times \Gamma'$ is a pair groupoid, not the product of groupoids). Then $\tilde{h} := l'h$ is a morphism from Γ to the pair groupoid $\Gamma' \times \Gamma'$. Then one can show that the (semi)norms coming from π_h and $\pi_{\tilde{h}}$ are equal.

EXAMPLES 5.3. *a) Pair groupoids.* Let $\Gamma := X \times X, \Gamma' := Y \times Y$ be pair groupoids. Due to the structure of morphisms in this case as explained in Example 3.5b) we can assume that $Y = X \times Z$ for some manifold Z and $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma'$ is given by $Gr(h) := \{(x, z, x', z; x, x') : x, x' \in X, z \in Z\}$. Using this fact one can show that for pair groupoids the C^* norm on $\mathcal{A}(\Gamma)$ is equal to the norm coming from the left regular representation. The completion of $\mathcal{A}(\Gamma)$ in this norm is the algebra of compact operators.

Groups. Let $\Gamma = G$ be a Lie group. $\mathcal{A}(G)$ is by definition the *-algebra of compactly supported, smooth, complex densities on G. For any U-strongly continuous, unitary representation of G on the Hilbert space H the formula

$$(x, \pi_U(\nu)x) := \int_G \nu(g)(x, U(g)x), \nu \in \mathcal{A}(G), x \in H$$

defines a nondegenerate *-representation of $\mathcal{A}(G)$. $C^*(G)$ is the completion of $\mathcal{A}(G)$ with respect to the norm $||\nu|| := \sup_U ||\pi_U(\nu)||$.

Since for a group, left and right fibers are equal it is clear that any $\omega \in \mathcal{A}(\Gamma)$ is a density on G by the assignment: $\mathcal{A}(\Gamma) \ni \lambda \otimes \rho \mapsto \lambda\rho \in \mathcal{A}(G)$. Let X be a manifold and $\Gamma' := X \times X$ the corresponding pair groupoid. By a slight modification of arguments used in Example 3.5 a) morphisms $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma'$ are in one to one correspondence with smooth actions of G on X: $Gr(h) := \{(gx, x; g) : x \in X, G \in G\}$. Choose λ_0 a real, smooth, nonvanishing left invariant half density on G and ψ_0 a smooth, real, nonvanishing half density on X. Then short computations show that for $\omega = f\lambda_0 \otimes \rho_0, \Psi = \psi\psi_0 \otimes \psi_0$ we have:

$$(\pi_h(\omega)\Psi)(x_1, x_2) = \left[\int_G \lambda_0^2(g) f(g) t_h(g; g^{-1}x_1, x_2)\psi(g^{-1}x_1, x_2)\right] \psi_0(x_1) \otimes \psi_0(x_2).$$

Since $L^2(\Gamma') = L^2(X) \otimes L^2(X)$ this representation is of the form $\pi_h(\omega) = \tilde{\pi}_h(\omega) \otimes I$ for

$$(\tilde{\pi}_h(\omega)\Psi)(x) := \left[\int_G \lambda_0^2(g) f(g) \frac{\rho_0}{\lambda_0}(g) \frac{g\psi_0}{\psi_0}(x)\psi(g^{-1}x)\right] \psi_0(x),$$

where $\Psi = \psi\psi_0$ is a smooth half density on X with compact support.

On the other hand, the action of G on X defines a strongly continuous unitary representation of G on $L^2(X)$ by the formula: $U_g\Psi := g\Psi$ for Ψ a smooth, compactly supported half density on X. If $\Psi = \psi\psi_0$ then $(U_g\Psi)(x) = \psi(g^{-1}x)\frac{g\psi_0}{\psi_0}(x)\psi_0(x)$. If $\omega = f\lambda_0 \otimes \rho_0$ then $\nu := f\lambda_0\rho_0 = f\frac{\rho_0}{\lambda_0}\lambda_0^2$ and $\tilde{\pi}_h(\omega)\Psi = \pi_U(\nu)\Psi$. In this way $||\omega||_{C^*(\Gamma)} \leq ||\nu||_{C^*(G)}$ and the C^*-algebra of a Lie group G viewed as a differential groupoid is something "between" the reduced C^*-algebra of G and the algebra $C^*(G)$ where G is treated as a locally compact topological group.

Transformation groupoids. Let $\Gamma := G \times X$ be a transformation groupoid. By $C_0(X)$ we denote the C^*-algebra of complex, continuous functions on X vanishing at infinity. The action of G on X induces a strongly continuous homomorphism $\alpha : G \ni g \mapsto \alpha_g \in Aut(C_0(X))$, where $Aut(C_0(X))$ is the group of *-isomorphisms of $C_0(X)$), namely $(\alpha_g f)(x) := f(g^{-1}x)$. So $(G, C_0(X), \alpha)$ is a C^* dynamical system [1]. Let Y be a manifold and $\Gamma' := Y \times Y$ the corresponding pair groupoid. By Example 3.5 a) morphisms $h : \Gamma \longrightarrow\!\!\!\triangleright\, \Gamma'$ are in one to one correspondence with smooth actions $G \times Y \ni (g, y) \mapsto gy \in Y$ together with smooth equivariant mapping $F : Y \longrightarrow X$. The graph of h is then

$Gr(h) = \{(gy, y; g, F(y)) : y \in Y, g \in G\}$. The action of G on Y induces a strongly continuous, unitary representation $G \ni g \mapsto U_g \in B(L^2(Y))$. The mapping F defines a nondegenerate representation π of $C_0(X)$ on $L^2(Y)$ by the formula: $(\pi(f)\psi)(y) := f(F(y))\psi(y)$, $f \in C_0(X)$ and ψ a smooth, compactly supported half density on Y. The pair (π, U) is a covariant representation of $(G, C_0(X), \alpha)$.

Now we go back to the groupoid Γ. Choose $\mu_0 \neq 0$ a real half density on T_eG. Since $T^r_{(g,x)}\Gamma = T_gG$, it defines a right invariant, nonvanishing, half density on Γ by the formula $\rho_0(g,x)(v_g) := \mu_0(v_g g^{-1})$ where $v_g \in \Lambda^{max}T_gG = \Lambda^{max}T^r_{(g,x)}\Gamma$. The corresponding left invariant half density is given by $\lambda_0(g,x)(v) = \mu_0(g^{-1}\pi(v))$ where $v \in \Lambda^{max}T^l_{(g,x)}\Gamma$ and $\pi : \Gamma \times X \longrightarrow G$ is a projection. Let λ be the corresponding left invariant density on G, i.e. $\lambda(g) := g\mu_0^2$ and let Δ be the corresponding modular function.

Let $K(G, C_0(X))$ be the *-algebra of compactly supported, continuous functions from G to $C_0(X)$ with the usual structure [1]. We define the mapping $\mathcal{A}(\Gamma) \ni \omega \mapsto \hat{\omega} \in K(G, C_0(X))$ by $\hat{\omega}(g)(x) := \Delta(g)^{-1/2} f(g, g^{-1}x)$ for $\omega = f\omega_0$. Straightforward computations show that this is an injective *-homomorphism.

One can show that $||\pi_h(\omega)|| = ||\rho(\hat{\omega})||$ for ρ a covariant representation of $K(G, C_0(X))$ defined by the morphism h. In this way we see that $C^*(G \times X)$ is a kind of "smooth" crossed product, which can be "smaller" than the universal crossed product $C^*(G, C_0(X), \alpha)$.

Since bissections act in a natural way on $\mathcal{A}(\Gamma)$ and are "transported" by morphisms one can expect that they define unitary multipliers of $C^*(\Gamma)$. And indeed this is true. Also there is a natural action of smooth functions on Γ^0 on bidensities which can be extended to an action of continuous functions on Γ^0 on $C^*(\Gamma)$. It turns out that those functions are affiliated with $C^*(\Gamma)$ [8].

6. Appendix: Lie groupoids and C*-algebras. *These are notes left by S. Zakrzewski.*

In this work we construct a functor from the category of smooth groupoids (with suitably defined morphisms) to the category of C*-algebras (with morphisms defined as in the context of *locally compact noncommutative spaces*, cf. for instance [9]). (...)

In the rest of this introduction, let us explain the role of our construction in establishing relations between 'classical' and 'quantum' theories. Recall that symplectic manifolds correspond to (play a similar role as) Hilbert spaces (possibly projective) and symplectic diffeomorphisms correspond to unitaries.

In order to have a procedure which relates some concrete symplectic manifolds to some concrete Hilbert spaces and some concrete symplectic diffeomorphisms to unitaries one has to consider a more special situation. Suppose we are given a manifold Q (playing the role of 'configurations'). We have then immediately the corresponding phase space $S = T^*Q$ and also the Hilbert space $H = L^2(Q)$ (of square-integrable complex half-densities on Q). To any diffeomorphism ϕ of Q there corresponds a symplectomorphism $u := \phi_*$ (the push-forward of covectors) and also a unitary operator $U := \phi_*$ (the push-forward of half-densities). It is clear that in these circumstances we have a 1–1 correspondence between such u's and U's, illustrated by the diagrams in figure 2.

$$\begin{tikzcd} S = T^*Q \arrow[rr, dashed] & & H = L^2(Q) \\ & Q \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} u = \phi_* \arrow[rr, dashed] & & U = \phi_* \\ & \phi \in Diff(Q) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Fig. 2

We see that the transition from the classical level to the quantum level is possible in this case due to the common 'configuration level' as shown in figure 3.

$$\begin{tikzcd} \text{classical level} \arrow[rr, dashed] & & \text{quantum level} \\ & \text{configuration} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Fig. 3

The symplectic diffeomorphisms of T^*Q which are just the natural lifts of diffeomorphisms of Q are said to be *point transformations*. It turns out that not only these can be 'quantized'. Namely, as a second step consider *phase shifts* of T^*Q, that is, symplectic diffeomorphisms v of T^*Q of the form

$$T^*Q \ni \xi \mapsto \xi + df(\pi(\xi)) \in T^*Q,$$

where f is a smooth function on Q and $\pi\colon T^*Q \to Q$ is the cotangent bundle projection. It is natural to associate with f also the unitary operator V in $L^2(Q)$ of multiplication by e^{if}. Symbolically, we have the situation as in figure 4:

$$\begin{tikzcd} v = +df \arrow[rr, dashed] & & V = \cdot e^{if} \\ & f : Q \to R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Fig. 4

It means that we can associate (projective) unitaries with phase shifts. Moreover, symplectic diffeomorphisms of the form vu form a group, which can be then naturally mapped into the unitary opearators by the rule $vu \mapsto VU$ (it works modulo the phase factor). What we here obtain is (essentially) the quantization of symplectic diffeomorphisms which preserve the natural polarization of the cotangent bundle (i.e. map fibers onto fibers). In fact, to construct (projective) H from S, the polarization is sufficient (the change of the lagrangian section playig the role of the 'zero section' is then implemented by the corresponding unitary transformations of type V).

We may summarize the above discussion as follows. A symplectic manifold S may serve to construct a quantum-mechanical Hilbert space H if S comes from configurations, $S = T^*Q$, or if at least S is equipped with a projection on Q with lagrangian fibers (which essentially means that S is equipped with a polarization). Then a symplectic diffeomorphism of S may serve to construct a unitary operator in H if it is a point transformation, or, at least if it preserves the polarization.

Now the point is that structures important for quantum mechanics such as operator algebras (in particular, C*-algebras) have also classical counterparts, namely symplectic groupoids. In this context, the above diagrams have the form given in figure 5 whose concrete realization is given in figure 6.

$$\begin{tikzcd}
\begin{array}{c}\text{symplectic}\\\text{groupoids}\end{array} \arrow[rr, dashed] & & C^*\text{-algebra} \\
& \text{Lie groupoids} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 5

$$\begin{tikzcd}
T^*\Gamma \arrow[rr, dashed] & & C^*(\Gamma) \\
& \Gamma \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 6

Here Γ is a Lie groupoid, $T^*\Gamma$ is its cotangent (symplectic) groupoid and $C^*(\Gamma)$ is the C*-algebra of Γ. Similarly, for morphisms, we shall have (as a result of the present paper)

$$\begin{tikzcd}
Ph(h) \in Mor(T^*(\Gamma), T^*(\Gamma')) \arrow[rr, dashed] & & C^*(h) \in Mor(C^*(\Gamma), C^*(\Gamma')) \\
& h \in Mor(\Gamma, \Gamma') \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 7

This corresponds to the 'point case'. There is also the second step, admitting also 'phase shifts'. It consists in considering symplectic groupoids which are projectable on Lie groupoids (in the sense that Γ is the cotangent bundle of some manifold Q and the multiplication relation projects onto a Lie groupoid multiplication relation on Q). It turns out that the symplectic groupoid structure of Γ is the cotangent lift of the Lie groupoid structure, shifted by a '2-cocycle' and the previous diagram is generalized to

$$\begin{tikzcd}
\begin{array}{c}\text{symplectic groupoids}\\\text{projectable on Lie groupoids}\end{array} \arrow[rr, dashed] & & C^*\text{-algebras} \\
& \begin{array}{c}\text{Lie groupoids}\\\text{with 2-cocycle}\end{array} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 8

The situation with 2-cocycles will be described in another paper.

References

[1] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics. I*, Springer, Berlin, 1981.

[2] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[3] J.-H. Lu and A. Weinstein, *Poisson Lie Groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom. 31 (1990), 501–526.

[4] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, LMS Lecture Note Series 124, Cambridge Univ. Press, 1987.

[5] T. Masuda, Y. Nakagami and S. L. Woronowicz, *A C^*-algebraic framework for the quantum groups*, to appear.

[6] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math. 793 (1980).

[7] A. Weinstein, *Geometric Models for Noncommutative Algebras*, Lecture Notes in Math. 227, preliminary version.

[8] S. L. Woronowicz, *Unbounded Elements Affiliated with C^*-Algebras and Non-Compact Quantum Groups*, Comm. Math. Phys. 136 (1991), 399–432.

[9] S. L. Woronowicz, *From Multiplicative Unitaries to Quantum Groups*, Int. Journal of Math. 7, No.1 (1996), 127–149.

[10] S. L. Woronowicz, *Pseudospaces, pseudogroups and Pontriyagin duality*, Proc. of the International Conference on Math. Phys., Lausanne 1979, Lecture Notes in Math. 116.

[11] S. Zakrzewski, *Quantum and classical pseudogroups I*, Comm. Math. Phys. 134 (1990), 347–370.

[12] S. Zakrzewski, *Quantum and classical pseudogroups II*, Comm. Math. Phys. 134 (1990), 371–395.