VERONESE WEBS FOR BIHAMILTONIAN STRUCTURES OF HIGHER CORANK

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> *To the memory of Stanisław Zakrzewski,*
> *with respect and gratitude*

0. Introduction. A C^∞-manifold M is endowed with a Poisson pair if two linearly independent smooth bivectors c_1, c_2 are defined on M and $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. A bihamiltonian structure $J = \{c_\lambda\}$ is the whole 2-dimensional family of bivectors. The structure J is degenerate if rank $c_\lambda < \dim M$, $\lambda \in \mathbb{R}^2$.

An intensive study of such objects was done by I. M. Gelfand and I. S. Zakharevich ([10], [11], [12]) in a particular case of bihamiltonian structures in general position on an odd-dimensional M (the corresponding Poisson pairs are necessarily degenerate: rank $c_\lambda = 2n, \lambda \in \mathbb{R}^2 \setminus \{0\}$, if $\dim M = 2n + 1$). In [11] there was introduced a notion of a Veronese web, i.e. a 1-parameter family of 1-codimensional foliations such that the corresponding family of annihilators is represented by the Veronese curve in the cotangent space at each point. It turns out that Veronese webs form a complete system of local invariants for bihamiltonian structures of general position. More precisely, it was shown in [11] that any such structure $J = \{c_\lambda\}$ in $\mathbb{R}^{2n+1}$ admits a local reduction to a Veronese web $\mathcal{W}_J$ on a $(n + 1)$-dimensional manifold and that for any Veronese web $\mathcal{W}$ one can locally construct a bihamiltonian structure $J(\mathcal{W})$ of general position in $\mathbb{R}^{2n+1}$ with the reduction equal to $\mathcal{W}$. In the real analytic case J and $J(\mathcal{W}_J)$ are isomorphic.

The aim of this paper is to introduce a wider class of degenerate bihamiltonian structures that possess many features of the general position case and to generalize the notion

of a Veronese web for this class. We call the bihamiltonian structures from this class complete since they are intimately connected with the completely integrable systems ([2]) on M. In particular, the Poisson pairs appearing in the well known method of argument translation (see [8], [9], and Example 1.12, below) generate complete bihamiltonian structures of higher (> 1) corank.

The paper is organized as follows. In Section 1 we recall some definitions and facts about bihamiltonian structures and introduce the main definition of completeness. The last is based on one result of A. Brailov (Theorem 1.8). We show that complete bihamiltonian structures generalize the case of general position. Analyzing the corresponding Poisson pair $(c_1(x), c_2(x))$ at a point $x \in M$ we deduce that it consists of finite number of the so called Kronecker blocks (Corollary 1.15); the general position is characterized by the case of the sole block. Section 2 is devoted to distinguishing the invariants for the sum of k Kronecker blocks. In the next section we define local Veronese webs for complete bihamiltonian structures under some additional assumption of simplicity. This last means that: 1) the number of Kronecker blocks does not change from point to point and the corresponding subspaces vary smoothly "sweeping" a flag of k subbundles in the tangent bundle; 2) there are no blocks of equal dimension. The second condition allows to avoid some technical complications but in principle may be skipped (see Remark 2.5). In general, the above mentioned distributions are nonintegrable (Examples 3.4, 3.5); consequently, the bihamiltonian structure does not split to direct product of the bihamiltonian structures of corank 1, i.e. of general position. We conclude the paper by calculating the Veronese web for the method of argument translation (Section 5). In the case of normal noncompact real form of complex simple Lie algebra this web is generically a product of flat Veronese webs of codimension 1.

Recent papers [13], [14] are closely related to the subject, in particular to generalized Veronese webs. In [14] the author introduces a more general notion of a Kronecker web, which is essentially equivalent to the notion of a Veronese web (see Definition 3.2) in case of simple bihamiltonian structures. Our approach emphasizes a bit more the role of Veronese curves in the theory.

The following two questions arise from the context of this paper.

1. Does the Veronese web of a complete bihamiltonian structure determine it up to an isomorphism?

2. What is a relation between the Veronese webs introduced here and d-webs of maximal rank and codimension 2 studied in paper [6] of S. S. Chern and P. A. Griffiths? (The notion of the rank of a d-web should not be confused with that of a bihamiltonian structure; the corank of a bihamiltonian structure is equal to the codimension of the web.)

Note that the d-webs of maximal rank and codimension 1 considered in paper [7] of the same authors are intimately connected with the Veronese webs of codimension 1.

1. Bihamiltonian structures and completeness. Let M be a C^∞-manifold. In the sequel, all Poisson bivectors considered will have maximal rank on an open dense subset in M. Given a Poisson bivector c, define $\operatorname{rank} c$ as $\max_{x \in M} \operatorname{rank} c(x)$.

1.1. DEFINITION. Two linearly independent Poisson bivectors c_1, c_2 on M form a *Poisson pair* if $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

1.2. PROPOSITION. *A pair of linearly independent Poisson bivectors (c_1, c_2) is Poisson if and only if $[c_1, c_2] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket.*

1.3. DEFINITION. A *bihamiltonian structure* on M is defined as a two-dimensional linear subspace $J = \{c_\lambda\}_{\lambda \in \mathcal{S}}$ of Poisson bivectors on M parametrized by a two-dimensional vector space $\mathcal{S}$ over $\mathbb{R}$. We say that J is *degenerate* if $\operatorname{rank} c_\lambda < \dim M$ for any $c_\lambda \in J$.

It is clear that every Poisson pair generates a bihamiltonian structure and the transition from the latter to a Poisson pair corresponds to a choice of basis in $\mathcal{S}$. We shall write (J, c_1, c_2) for a bihamiltonian structure J with a chosen Poisson pair (c_1, c_2) generating J.

1.4. DEFINITION. Let J be a bihamiltonian structure. Introduce a *subfamily* $J_0 \subset J$ of Poisson bivectors of maximal rank R_0 (the set $J \setminus J_0$ is at most a finite sum of 1-dimensional subspaces), and a *set of functions* $\mathcal{F}_0 = \operatorname{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_c(M))$, where $Z_c(M)$ stands for the space of the Casimir functions of c on M. We take Span in order to obtain a vector space: a sum of two Casimir functions for different $c_1, c_2 \in J_0$ need not be a Casimir function.

The following proposition shows how the degenerate bihamiltonian structures can be applied for constructing the completely integrable systems.

1.5. PROPOSITION. *Let J be a degenerate bihamiltonian structure on M. A family $\mathcal{F}_0$ is involutive with respect to any $c_\lambda \in J$.*

PROOF. Let $c_1, c_2 \in J_0$ be linearly independent, $f_i \in Z_{c_i}, i = 1, 2$. Then

$$\{f_1, f_2\}_{c_\lambda} = (\lambda_1 c_1(f_1) + \lambda_2 c_2(f_1))f_2 = -\lambda_2 c_2(f_2)f_1 = 0. \tag{1.5.1}$$

Now it remains to prove that for any $c \in J_0, f_i \in Z_c, i = 1, 2$, one has $\{f_1, f_2\}_{c_\lambda} = 0$. For that purpose we first rewrite (1.5.1) as

$$c_\lambda(x)(\phi_1, \phi_2) = 0, \tag{1.5.2}$$

where $\phi_i \in \ker c_i(x)$, $i = 1, 2$, $x \in M$, and the lefthandside denotes the contraction of the bivector with two covectors. Second, we fix x such that $\operatorname{rank} c(x) = R_0$ and approximate $df_2|_x$ by a sequence of elements $\{\phi^i\}_{i=1}^\infty$, $\phi^i \in \ker c^i(x)$, where $c^i \in J_0$, $i = 1, 2, \ldots$, is linearly independent with c. Finally, by (1.5.2) we get $c_\lambda(x)(df_1|_x, \phi^i) = 0$ and by the continuity $\{f_1, f_2\}_{c_\lambda}(x) = 0$. Since the set of such points x is dense in M, the proof is finished. ∎

In fact this proposition is true for the local Casimir functions (for the germs of Casimir functions). The corresponding family of functions (germs) $\operatorname{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_c(U))$ ($\operatorname{Span}_{\mathbb{R}}(\bigcup_{c \in J_0} Z_{c,x})$ is denoted by $\mathcal{F}_0(U)$ ($\mathcal{F}_{0,x}$).

In order to obtain a completely integrable system from Casimir functions one should require additional assumptions on the bihamiltonian structure J. Of course, the condition

of completeness given below concerns the local Casimir functions (in fact their germs) and may be insufficient for obtaining the completely integrable system. However, it is of use if the local Casimir functions are restrictions of global ones (see Example 1.12, below).

Given a Poisson bivector $c_\lambda \in J$, let $S_\lambda(x)$ denote the symplectic leaf of c_λ through a point $x \in M$.

1.6. DEFINITION ([3]). Let J be a bihamiltonian structure; fix some $c_\lambda \in J$. J is called *complete at a point $x \in M$ with respect to c_λ* if the linear subspace of $T_x^* M$ generated by the differentials of the germs $f \in \mathcal{F}_{0,x}$ restricted to $S_\lambda(x)$ has dimension $\frac{1}{2} \dim S_\lambda(x)$.

1.7. PROPOSITION. *A bihamiltonian structure J is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $S_\lambda(x)$ is of maximal dimension if and only if* $\dim(\bigcap_{c_\lambda \in J_0} T_x S_\lambda(x))$ $= \frac{1}{2} \dim S_\lambda(x).$

The following theorem is due to A. Brailov (see [3], Theorem 1.1 and Remark after it).

1.8. THEOREM. *A bihamiltonian structure (J, c_1, c_2) is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $S_\lambda(x)$ is of maximal dimension if and only if the following condition holds*

$(*)$ $\operatorname{rank}(\lambda_1 c_1 + \lambda_2 c_2)(x) = R_0$ *for any* $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{0\}$.

Here the bivector $c_\lambda = (\lambda_1 c_1 + \lambda_2 c_2)(x)$ is regarded as an element of $\bigwedge^2 T_x^{\mathbb{C}} M$, where $T^{\mathbb{C}} M$ is the complexified tangent bundle, and its rank is defined as that of the associated sharp map $c_\lambda^\sharp(x) : (T_x^{\mathbb{C}} M)^* \to T_x^{\mathbb{C}} M$.

The theorem shows that J is complete with respect to a fixed $c_\lambda \in J_0$ at a point x such that the dimension $S_\lambda(x)$ is maximal if and only if $J = J_0 \bigcup \{0\}$ and J is complete at x with respect to any nontrivial $c_\lambda \in J$. This motivates the next definition.

1.9. DEFINITION. Let (J, c_1, c_2) be a bihamiltonian structure. The structure J (the pair (c_1, c_2)) is *complete at a point $x \in M$* if condition $(*)$ of Theorem 1.8 holds at x. J $((c_1, c_2))$ is called *complete* if it is so at any point from some open and dense subset in M.

1.10. PROPOSITION. *Let J be complete on M and let $x \in M$ be a point of completeness. Then there exists a neighbourhood $U \ni x$ such that the foliation $\mathcal{L}$ defined on U by $\mathcal{F}_0(U)$ is lagrangian in any $S_\lambda(y), \lambda \neq 0, y \in U$ (by Proposition 1.7 this foliation can be defined as the intersection of the foliations of symplectic leaves for $c_\lambda \in J_0$).*

1.11. DEFINITION. Call $\mathcal{L}$ a *bilagrangian foliation* of J.

1.12. EXAMPLE (Method of argument translation, see [8], [3]). Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual space. Fix a basis $\{e_1, \ldots, e_n\}$ in $\mathfrak{g}$ with the structure constants $\{c_{ij}^k\}$; write $\{e^1, \ldots, e^n\}$ for the dual basis in $\mathfrak{g}^*$. The standard linear Poisson bivector on $\mathfrak{g}^*$ is defined as

$$c_1(x) = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\{x_k\}$ are linear coordinates in $\mathfrak{g}^*$ corresponding to $\{e^1, \ldots, e^n\}$. In more invariant terms c_1 is described as an operator dual to the Lie-multiplication map $[\,,\,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$. It is well-known that the symplectic leaves of c_1 are the coadjoint orbits in $\mathfrak{g}^*$. Now define c_2 as a bivector with constant coefficients $c_2 = c(a)$, where a is a fixed point on any leaf of maximal dimension. It turns out that c_1, c_2 form a Poisson pair and it is easy to describe the set I of points x for which condition $(*)$ fails. Consider the complexification $(\mathfrak{g}^*)^{\mathbb{C}} \cong (\mathfrak{g}^{\mathbb{C}})^*$ and the sum $\operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^*$ of symplectic leaves of nonmaximal dimension for the complex linear bivector $c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$, are the corresponding complex coordinates in $(\mathfrak{g}^*)^{\mathbb{C}}$. Then I is equal to the intersection of the sets $\mathfrak{g}^* \subset (\mathfrak{g}^*)^{\mathbb{C}}$ and $\overline{a, \operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^*}$, where $\overline{a, \operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^*}$ denotes the cone of complex 2-dimensional subspaces passing through a and $\operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^*$.

In particular, (c_1, c_2) is complete for a semisimple $\mathfrak{g}$ ($\operatorname{codim} \operatorname{Sing}(\mathfrak{g}^{\mathbb{C}})^* \geq 3$, see [1], Corollary 4.42, and the codimension of I in $\mathfrak{g}^*$ is not less than 2). Note that this gives rise to completely integrable systems since the local Casimir functions on $\mathfrak{g}^*$ are restrictions of global ones, i.e. invariants of the coadjoint action.

1.13. EXAMPLE (Bihamiltonian structure of general position on an odd-dimensional manifold, see [11]). Consider a pair of bivectors (a_1, a_2), $a_i \in \bigwedge^2 V, i = 1, 2$, where V is a $(2m + 1)$-dimensional vector space; (a_1, a_2) is in general position if and only if is represented by the Kronecker block of dimension $2m + 1$, i.e.

$$\begin{aligned} a_1 &= p_1 \wedge q_1 + p_2 \wedge q_2 + \cdots + p_m \wedge q_m \\ a_2 &= p_1 \wedge q_2 + p_2 \wedge q_3 + \cdots + p_m \wedge q_{m+1} \end{aligned} \tag{1.13.0}$$

in an appropriate basis $p_1, \ldots p_m, q_1, \ldots q_{m+1}$ of V. A bihamiltonian structure J on a $(2m + 1)$-dimensional M is in general position if and only if the pair $(c_1(x), c_2(x))$ is so for any $x \in M$. Such a J is complete. In general, a complete Poisson pair at a point is a direct sum of the Kronecker blocks as the corollary of the next theorem shows. This theorem is a reformulation of the classification result for pairs of 2-forms in a vector space ([10], [12]).

1.14. THEOREM. *Given a finite-dimensional vector space V over $\mathbb{C}$ and a pair of bivectros (c_1, c_2), $c_i \in \bigwedge^2 V$, there exists a direct decomposition $V = \oplus V_j$, $c_i = \sum c_i^{(j)}$, $c_i^{(j)} \in \bigwedge^2 V_j$, $i = 1, 2$, such that each triple $(V_j, c_1^{(j)}, c_2^{(j)})$ is from the following list:*

(a) *the Jordan block: $\dim V_j = 2n_j$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to*

$$\begin{pmatrix} 0 & A_i \\ -A_i^T & 0 \end{pmatrix}, \quad i = 1, 2,$$

where $A_1 = I_{n_j}$ (the unit $n_j \times n_j$-matrix) and $A_2 = J_{n_j}^{\lambda}$ (the Jordan block with eigenvalue λ);

(b) *the Kronecker block: $\dim V_j = 2n_j + 1$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to*

$$\begin{pmatrix} 0 & B_i \\ -B_i^T & 0 \end{pmatrix}, \quad i = 1, 2,$$

$$\text{where } B_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

$((n_j + 1) \times n_j$-matrices; in case $n_j = 0$ put $c_1 = c_2 = 0$).

1.15. COROLLARY. *Let (J, c_1, c_2) be a bihamiltonian structure. It is complete at a point $x \in M$ if and only if the pair $(c_1(x), c_2(x))$, $c_i(x) \in \bigwedge^2(T_x^{\mathbb{C}} M), i = 1, 2$, does not contain the Jordan blocks in its decomposition.*

PROOF. The statement follows from the definition of completeness. ∎

1.16. REMARK. In the absence of Jordan blocks Theorem 1.14 is valid also over the reals.

2. Complete bihamiltonian structure at a point. Now, we shall examine a linear bihamiltonian structure $(J, c_1, c_2), c_i \in \bigwedge^2 V$, where V is a vector space over $\mathbb{R}$, such that the decomposition $V = \oplus_{j=1}^k V_j$, $c_i = \sum_{j=1}^k c_i^{(j)}$ (see Theorem 1.14 and Remark 1.16) consists of k Kronecker blocks $V_1, \ldots, V_k, \dim V_j = 2n_j + 1, n_1 < \ldots < n_k$.

The aim is to isolate the invariants and to introduce the infinitesimal approximation to Veronese webs (these last will be defined in the next section).

It turns out that the decomposition to Kronecker blocks is noninvariant. To illustrate this let us consider $V = \text{Span}\{e, p, q_1, q_2\}, c_1 = p \wedge q_1, c_2 = p \wedge q_2$. Here $V = V_1 \oplus V_2$, where $V_1 = \text{Span}\{e\}, V_2 = \text{Span}\{p, q_1, q_2\}$, but instead V_1 one can choose any direct complement to V_2. However, there is a canonically defined filtration associated to J.

Let $P_{c_\lambda} \subset V$ ($P_{c_\lambda^{(j)}} \subset V_j$) be the characteristic subspace, i.e. the symplectic leaf through 0, of $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ ($c_\lambda^{(j)} = \lambda_1 c_1^{(j)} + \lambda_2 c_2^{(j)}$), $(\lambda_1, \lambda_2) \in \mathbb{R}^2$; let $L = \cap_{\lambda \neq 0} P_{c_\lambda}$ ($L_j = \cap_{\lambda \neq 0} P_{c_\lambda^{(j)}}$) be the bilagrangian subspace, i.e. the leaf through 0 of the bilagrangian foliation, corresponding to the bihamiltonian structure J ($\{c_\lambda^{(j)}\}_{\lambda \in \mathbb{R}^2}$), see Definition 1.11.

Put

$$\Phi_i = \sum_{\text{distinct } \lambda_1, \ldots, \lambda_i \in \mathbb{P}(\mathbb{R}^2)} P_{c_{\lambda_1}} \cap \ldots \cap P_{c_{\lambda_i}}, \quad i = 1, 2, \ldots, \Phi_0 = V.$$

2.1. THEOREM. *The following relations hold:*

$$\Phi_0 = \Phi_1 = \cdots = \Phi_{n_1} \supset \Phi_{n_1+1} = \Phi_{n_1+2} = \cdots = \Phi_{n_2} \supset \cdots \supset \Phi_{n_{k-1}+1} = \cdots = \Phi_{n_k} \supset$$

$$\supset \Phi_{n_k+1} = \Phi_{n_k+2} = \ldots =: \Phi_{n_{k+1}},$$

where

$$\Phi_{n_j} = \sum_{l<j} L_l \oplus \sum_{l \geq j} V_l, \quad j = 1, \ldots, k+1$$

(we put $V_l = 0, l > k$). In particular, the filtration is stabilized from $i > n_k$, $\Phi_{n_k+1} = \oplus L_j = L$, and the numbers $n_1, \ldots, n_k$ are invariants of J.

2.2. REMARK. The filtration $F_0 = \Phi_{n_1}^\perp \subset \cdots \subset F_{k-1} = \Phi_{n_k}^\perp \subset F_k = L^\perp = (V/L)^*$ ($\perp$ stands for the annihilator sign) appears in [14] and is called there isotypic. We shall refer to this notion below.

Before we begin to prove the theorem we recall the following definition.

2.3. DEFINITION. ([11]) Let $\mathcal{S}, V$ be vector spaces of dimensions 2 and $n+1, n \geq 0$, respectively. A *Veronese inclusion* of $\mathbb{P}(\mathcal{S})$ in $\mathbb{P}(V)$ is a map $i : \mathbb{P}(\mathcal{S}) \to \mathbb{P}(V)$ such that there exists a linear isomorphism $\phi : \mathbb{P}(V) \to \mathbb{P}(S^n \mathcal{S})$ making the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{S}) & \xrightarrow{i} & \mathbb{P}(V) \\
\| & & \uparrow \phi \\
\mathbb{P}(\mathcal{S}) & \xrightarrow{\mathbb{P}(S^n(\cdot))} & \mathbb{P}(S^n \mathcal{S}).
\end{array}
$$

The image $i(\mathbb{P}(\mathcal{S}))$ is called a *Veronese curve*.

Here S^n denotes the n-th symmetric power; the standard model of the mapping $S^n(\cdot)$ is described as follows. Let $\mathcal{S}$ be a space of linear functions f in two variables t_1, t_2. Then $S^n \mathcal{S}$ is a space of homogeneous polynomials in t_1, t_2 and $S^n(f) = f^n$. For $n = 0$ the map i is not an inclusion, but we shall use the defined term in this situation as well.

PROOF. We now prove Theorem 2.1. It is sufficient to show the following equalities

$$
\Phi_i V_j := \sum_{\text{distinct } \lambda_1, \ldots, \lambda_i} P_{c_{\lambda_1}^{(j)}} \cap \ldots \cap P_{c_{\lambda_i}^{(j)}} = \begin{cases} L_j & i \geq n_j + 1 \\ V_j & i < n_j + 1. \end{cases} \tag{2.3.1}
$$

One has

$$
(\Phi_i V_j)^\perp = \bigcap_{\text{distinct } \lambda_1, \ldots, \lambda_i} (P_{c_{\lambda_1}^{(j)}}^\perp + \cdots + P_{c_{\lambda_i}^{(j)}}^\perp).
$$

The 1-dimentional annihilator $P_{c_\lambda^{(j)}}^\perp \in \mathbb{P}((V_j/L_j)^*)$ sweeps an appropriate Veronese curve (see [11]). On the other hand, images of distinct points under a Veronese inclusion $\mathbb{P}(\mathcal{S}) \to \mathbb{P}(V)$ are linearly independent provided their number does not exceed $\dim V$ (cf. [7], I.A). So now, the first of equalities 2.3.1 follows from the fact that $P_{c_{\lambda_1}^{(j)}}^\perp + \cdots + P_{c_{\lambda_i}^{(j)}}^\perp = (V_j/L_j)^*$ for any set of distinct $\lambda_1, \ldots, \lambda_i, i \geq n_j + 1$.

For the second one, we notice that for any set of points $\lambda_1, \ldots, \lambda_{n_j+1}$

$$
\bigcap_{s=1}^{n_j+1} (P_1 + \cdots + \hat{P}_s + \cdots + P_{n_j+1}) = \{0\},
$$

where we put $P_s = P_{c_{\lambda_s}}^\perp$ and ^ means omitting the corresponding term. Thus we proved it for $i = n_j$; this implies 2.3.1 also for $i < n_j$. ∎

2.4. COROLLARY. *Let* $0 = F_1 \subset \cdots \subset F_{k-1} \subset F_k = (V/L)^*$ *be the isotypic filtration (see 2.2). Put* $F_j P_{c_\lambda}^\perp = F_j \cap P_{c_\lambda}^\perp, j = 1, \ldots, k$. *Then: 1)* $A_j^\lambda := F_j P_{c_\lambda}^\perp / F_{j-1} P_{c_\lambda}^\perp$ *is a one-dimensional subspace in* $A_j := F_j/F_{j-1}$; *2) the mapping* $\mathbb{P}(\mathbb{R}^2) \ni \lambda \mapsto A_j^\lambda \in \mathbb{P}(A_j)$ *is a Veronese inclusion for any* $j = 1, \ldots, k$.

PROOF. We first notice that $F_j = \oplus_{i=1}^{j}(V_i/L_i)^*, j = 1, \ldots, k$, as Theorem 2.1 implies. Under the identification $(V/L)^* = \oplus_{i=1}^{k}(V_i/L_i)^*$

$$F_j P_{c_\lambda}^\perp = \oplus_{i=1}^{j} P_{c_\lambda^{(i)}}^{\perp_i},$$

where $\perp_i$ stands for the annihilator of a subspace in V_i/L_i. Thus there are linear isomorphisms $A_j \cong (V_j/L_j)^*$ and $A_j^\lambda \cong P_{c_\lambda^{(j)}}^{\perp_j}, j = 1, \ldots, k$, and 2) follows from the analogous fact for a sole Kronecker block ([11]). ∎

2.5. REMARK. We note that analogues of Theorem 2.1 and Corollary 2.4 can be proved also without the restriction that $n_1, \ldots, n_k$ are distinct. In order to do that one should use a "multiple" version of a Veronese inclusion, i.e. a map $\phi : \mathbb{P}(\mathcal{S}) \to G(k, V^{(l+1)k})$ ($G(k, V)$ denotes the Grassmannian of k-planes in a $(l+1)k$-dimensional vector space V) such that there exist a decomposition $V = \oplus_{j=1}^{k} V_j, \dim V_j = l + 1$, and Veronese inclusions $i_j : \mathbb{P}(\mathcal{S}) \to \mathbb{P}(V_j), j = 1, \ldots, k$, with the property $\phi(v) = \mathrm{Span}_{\mathbb{R}}\{i_1(v), \ldots, i_k(v)\}$, where $i_j(v)$ is considered as a 1-dimensional subspace in V. This definition can be also used to adapt the notion of a Veronese web (see Definition 3.2, below) to a more general situation.

2.6. DEFINITION. An *infinitesimal Veronese web* of type $(n_1, \ldots, n_k), n_1 < \ldots < n_k$, on a vector space $W, \dim W = n_1 + \cdots + n_k + k$, is a 1-parameter family $\{\mathcal{W}_\lambda\}_{\lambda \in \mathbb{P}(\mathcal{S})}$ of linear subspaces $\mathcal{W}_\lambda \subset W, \mathrm{codim}\, \mathcal{W}_\lambda = k$, satisfying the following conditions:

(i) there is a filtration $0 = F_0 \subset \ldots \subset F_{k-1} \subset F_k = W^*$ of the dual space with $\dim F_j/F_{j-1} = n_j + 1, j = 1, \ldots, k$;

(ii) it induces the filtration $0 = F_0\mathcal{W}_\lambda^\perp \subset \ldots \subset F_{k-1}\mathcal{W}_\lambda^\perp \subset F_k\mathcal{W}_\lambda^\perp = \mathcal{W}_\lambda^\perp, F_j\mathcal{W}_\lambda^\perp = F_j \cap \mathcal{W}_\lambda^\perp, j = 1, \ldots, k$, of the annihilator $\mathcal{W}_\lambda^\perp \subset W^*$ so that $\dim F_j\mathcal{W}_\lambda^\perp = j$; in particular $A_j^\lambda := F_j\mathcal{W}_\lambda^\perp/F_{j-1}\mathcal{W}_\lambda^\perp$ can be considered as a 1-dimensional subspace in $A_j := F_j/F_{j-1}$;

(iii) the map $\mathbb{P}(\mathcal{S}) \ni \lambda \mapsto A_j^\lambda \in \mathbb{P}(A_j)$ is a Veronese inclusion, $j = 1, \ldots, k$.

2.7. PROPOSITION. *Let (J, c_1, c_2) be as above. Then the vector space $W = V/L$ has a structure of an infinitesimal Veronese web of type $(n_1, \ldots, n_k)$.*

PROOF. The proof follows from Corollary 2.4. ∎

3. Simple bihamiltonian structures and their Veronese webs. In this section we shall define objects that generalize the Veronese webs introduced in [11] for the bihamiltonian structures of general position. We shall show that any complete bihamiltonian structure from the class defined below admits the local reduction to such an object.

3.1. DEFINITION. Let J be a complete bihamiltonian structure on M. A *type* of J at $x \in M$ is the vector $(n_1, \ldots, n_k)(x)$, where $2n_1(x) + 1, \ldots, 2n_k(x) + 1$ are dimensions of the Kronecker blocks in the decomposition of $(c_1(x), c_2(x))$ for some Poisson pair (c_1, c_2) generating J (these dimensions do not depend on this pair, see Theorem 2.1). If this vector is independent of x we call it a type of J and say that J is *regular* (cf. Example 3.8, below). If, moreover, all $n_j, j = 1, \ldots, k$, are different we call J *simple*.

3.2. DEFINITION. Consider a manifold U diffeomorphic to an open set in $\mathbb{R}^N$, where $N = (n_1 + 1) + \cdots + (n_k + 1), n_1 < \ldots < n_k$, and a family $\mathcal{W} = \{\mathcal{W}_\lambda\}_{\lambda \in \mathbb{P}(\mathcal{S})}$ of k-codimensional foliations on U parametrized by the projectivizaton of a two-dimensional vector space $\mathcal{S}$. We call $\mathcal{W}$ a *Veronese web* of type $(n_1, \ldots, n_k)$ if the following conditions are satisfied:

(i) there is a bundle filtration $0 = F_0 \subset \ldots \subset F_{k-1} \subset F_k = T^*U$ such that rank $F_j/F_{j-1} = n_j + 1, j = 1, \ldots, k$;

(ii) it induces the bundle filtration $0 = F_0 \mathcal{W}_\lambda^\perp \subset \ldots \subset F_{k-1} \mathcal{W}_\lambda^\perp \subset F_k \mathcal{W}_\lambda^\perp = \mathcal{W}_\lambda^\perp$, $F_j \mathcal{W}_\lambda^\perp := F_j \cap \mathcal{W}_\lambda^\perp, j = 1, \ldots, k$, of the annihilating bundle $\mathcal{W}_\lambda^\perp := (T\mathcal{W}_\lambda)^\perp \subset T^*U$ so that rank $F_j \mathcal{W}_\lambda^\perp = j$; in particular $A_j^\lambda(x) := F_j \mathcal{W}_{\lambda,x}^\perp/F_{j-1}\mathcal{W}_{\lambda,x}^\perp$ can be considered as a 1-dimensional subspace in $A_j(x) := F_{j,x}/F_{j-1,x}$ for any $x \in U$;

(iii) the map $\mathbb{P}(\mathcal{S}) \ni \lambda \mapsto A_j^\lambda(x) \in \mathbb{P}(A_j(x))$ is a Veronese inclusion for any $x \in U, j = 1, \ldots, k$.

3.3. THEOREM. *Let J be a simple bihamiltonian structure of type $\mathbf{n} = (n_1, \ldots, n_k)$, $n_1 < \cdots < n_k$, and let $x \in M$ be a point of completeness for J. Write $\mathcal{V}_\lambda$ for the foliation of symplectic leaves of $c_\lambda \in J$. Then there exists a neighbourhood $\tilde{U} \ni x$ such that $U = \tilde{U}/\mathcal{L}$ (see 1.10) is diffeomorphic to an open set in $\mathbb{R}^N$ and $\{\mathcal{V}_\lambda|_{\tilde{U}}/\mathcal{L}\}_{\lambda \in \mathbb{P}(\mathcal{S})}$ is a Veronese web of type $\mathbf{n}$ on U.*

PROOF. The theorem follows from Proposition 2.7. ∎

3.4. EXAMPLE. Let $U = \mathbb{R}^3(x, y, z)$, $\alpha_1 = xdy - dz, \alpha_2^\lambda = \lambda_1 dx + \lambda_2 dy, k = 2, n_1 = 0, n_2 = 1$. Put $\Gamma(F_1) = \mathrm{Span}\{\alpha_1\}, \Gamma((T\mathcal{W}_\lambda)^\perp) = \mathrm{Span}\{\alpha_1, \alpha_2^\lambda\}$, where Γ stands for the space of sections and Span is taken over the ring of functions. Then $\Gamma(F_1(T\mathcal{W}_\lambda)^\perp) = \Gamma(F_1)$. Since $T\mathcal{W}_\lambda \subset TU$ is a subbundle of rank 1, it is indeed tangent to the 1-dimensional foliation $\mathcal{W}_\lambda$. Explicitely, $\Gamma(T\mathcal{W}_\lambda) = \mathrm{Span}\{\lambda_2 v_1 - \lambda_1 v_2\}$, where $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. On $\tilde{U} = \mathbb{R}(p) \times U$ one defines the corresponding bihamiltonian structure as $\{\frac{\partial}{\partial p} \wedge (\lambda_1 v_1 + \lambda_2 v_2)\}_{(\lambda_1, \lambda_2) \in \mathbb{R}^2}$.

3.5. EXAMPLE. Let $U = \mathbb{R}^5(x, y, z, s, t)$, $\alpha_1^\lambda = \lambda_1(xdy - dz) + \lambda_2(sdy - dt), \alpha_2^\lambda = \lambda_1^2 dx + \lambda_1 \lambda_2 ds + \lambda_2^2 dy, \Gamma(F_1) = \mathrm{Span}\{xdy - ds, sdy - dt\}, \Gamma((T\mathcal{W}_\lambda)^\perp)) = \mathrm{Span}\{\alpha_1^\lambda, \alpha_2^\lambda\}, k = 2, n_1 = 1, n_2 = 2$. Then $\Gamma(F_1(T\mathcal{W}_\lambda)^\perp)) = \mathrm{Span}\{\alpha_1^\lambda\}$. The 3-distribution in TU annihilated by the 1-forms $\alpha_1^\lambda, \alpha_2^\lambda$ is integrable since

$$d\alpha_1^\lambda = \begin{cases} \alpha_2^\lambda \wedge \frac{1}{\lambda_1}dy & \text{if } \lambda_1 \neq 0 \\ -\alpha_2^\lambda \wedge \frac{1}{\lambda_2}ds & \text{if } \lambda_1 = 0. \end{cases}$$

Explicitly, $\Gamma(T\mathcal{W}_\lambda) = \mathrm{Span}\{\lambda_2 \frac{\partial}{\partial x} - \lambda_1 \frac{\partial}{\partial s}, \lambda_2 \frac{\partial}{\partial s} - \lambda_1 v, \lambda_2 \frac{\partial}{\partial z} - \lambda_1 \frac{\partial}{\partial t}\}$, where $v = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + s\frac{\partial}{\partial t}$, and on $\tilde{U} = \mathbb{R}^3(p_1, p_2, p_3) \times U$ the corresponding bihamiltonian structure is $\{\frac{\partial}{\partial p_1} \wedge (\lambda_2 \frac{\partial}{\partial x} - \lambda_1 \frac{\partial}{\partial s}) + \frac{\partial}{\partial p_2} \wedge (\lambda_2 \frac{\partial}{\partial s} - \lambda_1 v) + \frac{\partial}{\partial p_3} \wedge (\lambda_2 \frac{\partial}{\partial z} - \lambda_1 \frac{\partial}{\partial t})\}_{(\lambda_1, \lambda_2) \in \mathbb{R}^2}$.

3.6. REMARK. Of course, it is more convenient to describe Veronese webs in terms of a bundle direct decomposition $B_1 \oplus \ldots \oplus B_k = T^*U$ such that $F_j = \oplus_{i=1}^j B_i$ rather than in terms of the isotypic filtration itself. In the above examples we used implicitly such a decomposition. However, one should remember that it is not unique. For instance,

in Example 3.5 one has $B_1 = F_1, \Gamma(B_2) = \text{Span}\{dx, ds, dy\}$. But one could take $\tilde{\alpha}_2^\lambda = \lambda_1^2(dx + xdy - dz) + \lambda_1\lambda_2(ds + sdy - dt) + \lambda_2^2 dy = \alpha_2^\lambda + \lambda_1\alpha_1^\lambda$ instead of α_2^λ and $\Gamma(\tilde{B}_2) = \text{Span}\{dx + xdy - dz, ds + sdy - dt, dy\}$. Although this does not change the web, the corresponding decomposition is changed. In [14] the author gives an involved analysis of this nonuniqueness.

3.7. DEFINITION. A Veronese web admits the following local description. One can choose linear coordinates (λ_1, λ_2) on $\mathcal{S}$ and a local coframe $\alpha_1^1, \ldots, \alpha_{n_1+1}^1, \ldots, \alpha_1^k, \ldots, \alpha_{n_k+1}^k$ such that $\alpha_1^j, \ldots, \alpha_{n_j+1}^j \in \Gamma(F_j), j = 1, \ldots, k$, and the annihilator $(T\mathcal{W}_\lambda)^\perp \subset T^*U$ is generated by $\alpha_\lambda^1, \ldots, \alpha_\lambda^k$, where $\alpha_\lambda^j = \lambda_1^{n_j}\alpha_1^j + \lambda_1^{n_j-1}\lambda_2\alpha_2^j + \cdots + \lambda_2^{n_j}\alpha_{n_k+1}^j$ (Veronese curve). If in a neighbourhood of any $x \in U$ there exists a holonomic coframe with the above properties, the Veronese web is called *flat*.

In particular, all bundles in the isotypic filtration of a flat web are completely integrable as differential systems and, moreover, such a web splits into a direct product of flat Veronese webs of codimension 1.

The webs from Examples 3.4, 3.5 are not flat, since the bundles F_1 are nonintegrable.

We conclude the section by an example of a complete bihamiltonian structure that is not regular.

3.8. EXAMPLE. Let $M = \mathbb{R}^6$ with coordinates $(p_1, p_2, q_1, \ldots, q_4), c_1 = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}, c_2 = \frac{\partial}{\partial p_1} \wedge (\frac{\partial}{\partial q_2} + q_1\frac{\partial}{\partial q_3}) + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_4}$. Here we have: two 3-dimensional Kronecker blocks on $M \setminus H, H = \{q_1 = 0\}$; the 5-dimensional Kronecker block and the 1-dimensional zero block on the hyperplane H.

4. Veronese webs for the argument translation method.

The notations from Subsection 1.12 will be used below. We consider a normal (déployable in the terminology of Bourbaki, [5], IX,3) real form $\mathfrak{g}$ of a complex simple Lie algebra. Let $m_1, \ldots, m_r, r = \text{rank}(\mathfrak{g})$ be the exponents of $\mathfrak{g}$.

4.1. THEOREM. *Let (c_1, c_2) be the Poisson pair from Example 1.12. Then the Veronese web $\{\mathcal{W}_\lambda\}_{\lambda \in \mathbb{R}^2}$ of the corresponding bihamiltonian structure J is of type $(m_1, \ldots, m_r)$ and is flat (Definition 3.7) in a neighbourhood of any point $\pi(x)$, where $x \in (\mathfrak{g}^* \setminus I)$ and π denotes the canonical projection $\pi : \mathfrak{g}^* \setminus I \to (\mathfrak{g}^* \setminus I)/\mathcal{L}$ (cf. 1.10, 3.3).*

PROOF. Let $g_1(x), \ldots, g_r(x), \deg g_j = m_j + 1$, be a set of algebraically independent global homogeneous polynomial Casimir functions for c (see [4], VIII,8). Here we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form. Note that $g_1, \ldots, g_r$ are functionally independent on $\mathfrak{g} \setminus \text{Sing } \mathfrak{g}$, where Sing $\mathfrak{g}$ is the set of adjoint orbits of nonmaximal dimension. Indeed, their restrictions to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ are algebraically independent and invariant with respect to the Weyl group W. Now, we can apply the result of R. Steinberg ([16]) to deduce the nondegeneracy for the Jacobi matrix of $g_1|_{\mathfrak{h}}, \ldots, g_r|_{\mathfrak{h}}$ at a regular point.

Consider the subspace $d\mathcal{F}_0 \subset \Gamma(T^*\mathfrak{g}^*)$ generated by the differentials of functions from the involutive set $\mathcal{F}_0$ (see 1.4) corresponding to J. It turns out that $d\mathcal{F}_0$ is generated by $\{dg_j|_{\lambda_1 x + \lambda_2 a}, (\lambda_1, \lambda_2) \in \mathbb{R}^2, j = 1, \ldots, r\}$. If $g_j^i(a, x), i = 0, \ldots, m_j + 1, j = 1, \ldots, r$, are

the coefficients of the Taylor expansions $g_j(x + \lambda a), j = 1, \ldots, r$, with respect to $\lambda \in \mathbb{R}$, then one also has

$$d\mathcal{F}_0 = \mathrm{Span}\{dg_j^i(a, x), i = 0, \ldots, m_j, j = 1, \ldots, r\}. \qquad (4.1.1)$$

Moreover, these differentials are linearly independent at any $x \in \mathfrak{g}^* \setminus I$. This follows from the fact that J is complete at $\mathfrak{g}^* \setminus I$, from (4.1.1), and from the formula $\sum_{j=1}^r m_j = \frac{1}{2}(\dim \mathfrak{g} - r)$ (cf. [15], formula (F1), p. 289).

Thus, we can regard $g_j^i(a, x), i = 0, \ldots, m_j, j = 1, \ldots, r$ as coordinates on the reduced space $(\mathfrak{g}^* \setminus I)/\mathcal{L}$. Finally, $(T\mathcal{W}_\lambda)^\perp, \lambda = (\lambda_1, \lambda_2)$, is generated by

$$\lambda_1^{m_j} dg_j^0(a, x) + \lambda_1^{m_j - 1}\lambda_2 dg_j^1(a, x) + \cdots + \lambda_2^{m_j} dg_j^{m_j}(a, x), \quad j = 1, \ldots, r. \ \blacksquare$$

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