NAMBU-POISSON TENSORS ON LIE GROUPS

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Abstract. First as an application of the local structure theorem for Nambu-Poisson tensors, we characterize them in terms of differential forms. Secondly left invariant Nambu-Poisson tensors on Lie groups are considered.

1. Introduction. In 1994, L. Takhtajan [6] gave geometric formulations of Nambu-Poisson manifolds, and a lot of papers have followed his work. A Nambu-Poisson manifold is defined to be a pair of a $C^\infty$-manifold and a Nambu-Poisson tensor defined on it. A Nambu-Poisson tensor is, by definition, a skew-symmetric contravariant tensor field on a manifold such that the induced bracket operation satisfies the fundamental identity, which is a generalization of the usual Jacobi identity. It is generally difficult to judge whether a given tensor field is a Nambu-Poisson tensor or not. This is because a Nambu-Poisson tensor is written in the form of a contravariant tensor.

We begin with characterizing Nambu-Poisson tensors via differential forms. The characterization will be done by using contraction of Nambu-Poisson tensors with the volume form. The local structure theorem [5] for Nambu-Poisson tensors will be very useful to obtain some results concerning this topic.

Secondly left invariant Nambu-Poisson tensors on Lie groups are considered. And we shall study when they can be projected on suitable homogeneous spaces. These problems were studied by Bon-Yao Chu [3] in the case of symplectic structures.

2. Nambu-Poisson manifolds. First we give a definition of a Nambu-Poisson tensor, which is equivalent to that of L. Takhtajan [6]. Let $M$ be an $m$-dimensional $C^\infty$-manifold, and $\mathcal{F}$ its algebra of real valued $C^\infty$-functions. We denote by $\Gamma(\Lambda^n TM)$ the space of global cross-sections $\eta : M \to \Lambda^n TM$. Then to each $\eta \in \Gamma(\Lambda^n TM)$, there
corresponds the bracket defined by
\[ \{ f_1, \ldots, f_n \} = \eta(df_1, \ldots, df_n), \quad f_1, \ldots, f_n \in \mathcal{F}. \]
Let \( A = \sum f_1 \wedge \cdots \wedge f_{n-1} \) be any element of \( \Lambda^{n-1} \mathcal{F} \). Since the bracket operation clearly satisfies the Leibniz rule, we can define a vector field \( X_A \) corresponding to \( A \) by the following equation:
\[ X_A(g) = \sum \{ f_1, \ldots, f_{n-1}, g \}, \quad g \in \mathcal{F}. \]
Such a vector field is called a Hamiltonian vector field. The space of Hamiltonian vector fields is denoted by \( \mathcal{H} \).

**Definition 1.** \( \eta \in \Gamma(\Lambda^n TM) \) is called a Nambu-Poisson tensor of order \( n \) if it satisfies \( L(X_A)\eta = 0 \) for all \( X_A \in \mathcal{H} \), where \( L \) is the Lie derivative. Then a Nambu-Poisson manifold is a pair \( (M, \eta) \).

The above definition is clearly equivalent to the following fundamental identity:
\[ \{ f_1, \ldots, f_{n-1}, \{ g_1, \ldots, g_n \} \} = \{ \{ f_1, \ldots, f_{n-1}, g_1 \}, g_2, \ldots, g_n \} + \cdots + \{ g_1, \ldots, g_{n-1}, \{ f_1, \ldots, f_{n-1}, g_n \} \} \]
for all \( f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in \mathcal{F} \). If \( n = 2 \), this equation is nothing but the Jacobi identity, and we have usual Poisson manifolds.

Let \( \eta(p) \neq 0, \ p \in M \). Then we say that \( \eta \) is regular at \( p \). Now we can state the following local structure theorem for Nambu-Poisson tensors [4],[5].

**Theorem 1.** Let \( \eta \in \Gamma(\Lambda^n TM) \), \( n \geq 3 \). If \( \eta \) is a Nambu-Poisson tensor of order \( n \), then for any regular point \( p \), there exists a coordinate neighborhood \( U \) with local coordinates \( (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \) around \( p \) such that
\[ \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \]
on \( U \), and vice versa.

In Theorem 1, the condition \( n \geq 3 \) is essential. If \( n = 2 \), as is well-known, A.Weinstein [8] proved the splitting theorem for Poisson tensors. Comparing Weinstein’s splitting theorem with our theorem, we know that the local structure of Nambu-Poisson manifolds is more rigid than that of usual Poisson manifolds.

Some simple applications of Theorem 1 are the following. Informations on the Schouten bracket may be found in [7], for instance.

**Corollary 1.**
1. Let \( \eta \) be a Nambu-Poisson tensor of order \( n \geq 3 \). If \( f \) is a smooth function, then \( f\eta \) is again a Nambu-Poisson tensor.
2. If \( m = n \geq 3 \), then every \( \eta \in \Gamma(\Lambda^n TM) \) is a Nambu-Poisson tensor.
3. For every Nambu-Poisson tensor \( \eta \), its Schouten bracket satisfies \( [\eta, \eta] = 0 \). (Of course the converse is not true.)

3. **Characterizations of Nambu-Poisson tensors.** Throughout this section, we assume that \( (M, \eta) \) is a Nambu-Poisson manifold with volume form \( \Omega \), and \( m \geq n \geq 3 \).
Put \( \omega = i(\eta)\Omega \), where the right hand side is the interior product of \( \eta \) and \( \Omega \). Hence \( \omega \) is an \((m-n)\)-form.

A differential form \( \alpha \) of degree \( k \) (locally defined) around \( p \in M \) is called decomposable at \( p \) if there exist 1-forms \( \alpha_1, \ldots, \alpha_k \) (which are locally defined around \( p \)) such that \( \alpha = \alpha_1 \wedge \cdots \wedge \alpha_k \). We also define the decomposability of (locally defined) contravariant tensor fields in the same manner. In the following theorem, we will characterize a Nambu-Poisson tensor by using the notion of decomposability.

**Theorem 2.** Let \( \eta \in \Gamma(A^nTM) \). Then \( \eta \) is a Nambu-Poisson tensor if and only if \( \eta \) satisfies the following two conditions around each regular point:

1) \( \omega \) is decomposable, and

2) there exists a locally defined 1-form \( \theta \) such that \( d\omega = \theta \wedge \omega \).

**Proof.** If \( \eta \) is a Nambu-Poisson tensor and \( p \) is its regular point, then by Theorem 1, there exist local coordinates \((x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)\) around \( p \) such that

\[
\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.
\]

Suppose that \( \Omega = fdx_1 \wedge \cdots \wedge dx_m \) with respect to these coordinates. Then we have

\[
\omega = fdx_{n+1} \wedge \cdots \wedge dx_m,
\]

and

\[
d\omega = df \wedge dx_{n+1} \wedge \cdots \wedge dx_m = \frac{df}{f} \wedge \omega.
\]

Conversely assume that \( \eta \) satisfies above two conditions. Then there exist 1-forms \( \omega_{n+1}, \ldots, \omega_m \) such that \( \omega = \omega_{n+1} \wedge \cdots \wedge \omega_m \). Note that \( \omega_{n+1}, \ldots, \omega_m \) are linearly independent around \( p \) since \( \omega(p) \neq 0 \). Adding \( n \) 1-forms \( \omega_1, \ldots, \omega_n \) to them, we can construct a basis of 1-forms around \( p \). For \( n+1 \leq i \leq m \), put

\[
d\omega_i = \sum_{1 \leq j < k \leq m} g_{jk}^i \omega_j \wedge \omega_k.
\]

Since

\[
d\omega = \theta \wedge \omega = \theta \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m
\]

\[
= \sum_{i=n+1}^m (-1)^{i-n-1} \omega_{n+1} \wedge \cdots \wedge d\omega_i \wedge \cdots \wedge \omega_m,
\]

we have

\[
0 = \omega_i \wedge d\omega = (-1)^{i-n-1} \omega_i \wedge \omega_{n+1} \wedge \cdots \wedge d\omega_i \wedge \cdots \wedge \omega_m
\]

\[
= d\omega_i \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m.
\]

This means that \( g_{jk}^i = 0 \) for \( 1 \leq j < k \leq n \), and we have the following expression:

\[
d\omega_i = \sum_{j=n+1}^m \theta_{ij} \wedge \omega_j,
\]

where \( \theta_{ij} \) are 1-forms. By the Frobenius theorem, there exist local functions \( f_{ij} \) and \( g_j \).
such that
\[ \omega_i = \sum_{j=n+1}^{m} f_{ij} dg_j, \quad G = \det(f_{ij}) \neq 0. \]

Adding \( n \)-local functions \( x_1, \ldots, x_n \), we can adopt \( (x_1, \ldots, x_n, g_{n+1}, \ldots, g_m) \) as local coordinates. With respect to these local coordinates, the volume form \( \Omega \) can be written as
\[ \Omega = F dx_1 \wedge \cdots \wedge dx_n \wedge dg_{n+1} \wedge \cdots \wedge dg_m. \]

If we put \( y_1 = \int \frac{F}{G} dx_1, \ y_2 = x_2, \ldots, y_n = x_n, y_{n+1} = g_{n+1}, \ldots, y_m = g_m, \) then \( \Omega \) is rewritten as
\[ \Omega = G dy_1 \wedge \cdots \wedge dy_m. \]

With respect to these new local coordinates \( (y_1, \ldots, y_m) \), \( \omega \) has the following expression:
\[ \omega = \omega_{n+1} \wedge \cdots \wedge \omega_m = G dy_{n+1} \wedge \cdots \wedge dy_m \]
\[ = i(\eta)(G dy_1 \wedge \cdots \wedge dy_m). \]
Thus we obtain that \( \eta = \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}. \) Using Theorem 1 once again, we know that \( \eta \) is a Nambu-Poisson tensor.

**Remark 1.** The above criterion for Nambu-Poisson tensors does not depend on the choice of volume form.

Suppose that \( m = n + 1 \). Since every 1-form is clearly decomposable, we have

**Corollary 2.** If \( m = n + 1 \), then \( \eta \) is a Nambu-Poisson tensor if and only if \( \omega \wedge d\omega = 0 \).

### 4. Nambu-Poisson tensors on Lie groups

Let \( G \) be an \( m \)-dimensional connected Lie group, \( m \geq 3 \). First we shall determine the form of left invariant Nambu-Poisson tensors on \( G \). Denote by \( \mathfrak{g} \) the Lie algebra of left invariant vector fields on \( G \).

**Proposition 3.** Let \( \eta \) be a (non-zero) left invariant Nambu-Poisson tensor of order \( n \geq 3 \) on a Lie group \( G \). Then \( \eta \) is globally decomposable.

**Proof.** Let \( e \) be the unit element of \( G \). By Theorem 1, \( \eta \) has the following expression around \( e \in G : \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \), where \( (x_1, \ldots, x_m) \) is some coordinate neighborhood around \( e \). Then there exist uniquely \( n \) elements \( X_1, \ldots, X_n \) of \( \mathfrak{g} \) such that \( (X_i)_e = (\frac{\partial}{\partial x_i})_e \), \( 1 \leq i \leq n \). Since \( \eta \) is left invariant, we immediately have \( \eta = X_1 \wedge \cdots \wedge X_n \) on \( G \).

By the above proposition, any left invariant Nambu-Poisson tensor \( \eta \) of order \( n \) can be written as a decomposable element of \( \Lambda^n \mathfrak{g} \).

**Proposition 4.** Let \( G \) be an \( m \)-dimensional Lie group.

(i) Let \( \mathfrak{h} \) be an \( n \)-dimensional Lie subalgebra of \( \mathfrak{g} \), \( n \geq 3 \). For a basis \( \langle X_1, \ldots, X_n \rangle \) of \( \mathfrak{h} \), put \( \eta = X_1 \wedge \cdots \wedge X_n \). Then \( \eta \) is a left invariant Nambu-Poisson tensor of order \( n \) on \( G \).

(ii) Conversely given a left invariant Nambu-Poisson tensor \( \eta = X_1 \wedge \cdots \wedge X_n \in \Lambda^n \mathfrak{g} \) on \( G \), then \( \mathfrak{h} = \langle X_1, \ldots, X_n \rangle \) is a Lie subalgebra of \( \mathfrak{g} \).
PROOF. (i) Let \(< X_1, \ldots, X_n, X_{n+1}, \ldots, X_m >\) be a basis of \(g\) obtained by extending a basis of \(\mathfrak{h}\), and let \(< \omega_1, \ldots, \omega_m >\) be the dual basis of \(< X_1, \ldots, X_m >\). Put \(\Omega = \omega_1 \wedge \cdots \wedge \omega_m\). Then \(\Omega\) is a left invariant volume form on \(G\). Define a left invariant \((m - n)\)-form \(\omega\) by \(\omega = i(\eta)\Omega = \omega_{n+1} \wedge \cdots \wedge \omega_m\). Let \(\{C_{pq}^r\}\) be the structure constants of \(g\). Since \(\mathfrak{h}\) is a Lie subalgebra, \(C_{pq}^r = 0\) \((n + 1 \leq i \leq m, 1 \leq p, q \leq n)\). Then

\[
d\omega_i = - \sum_{n+1 \leq p < q \leq m} C_{pq}^r \omega_p \wedge \omega_q - \sum_{1 \leq r \leq n} \sum_{n+1 \leq s \leq m} C_{s,r}^i \omega_r \wedge \omega_s,
\]

Thus we have

\[
d\omega = \sum_{1 \leq t \leq n} (-1)^{t-1} \omega_{n+1} \wedge \cdots \wedge d\omega_{n+t} \wedge \cdots \wedge \omega_m
\]

\[
= \sum_{1 \leq t \leq n} (-1)^{t-1} \omega_{n+1} \wedge \cdots \wedge \left( - \sum_{n+1 \leq p < q \leq m} C_{pq}^{n+t} \omega_p \wedge \omega_q \right) \wedge \cdots \wedge \omega_m
\]

\[
= \sum_{1 \leq t \leq n} \sum_{1 \leq r \leq n} (-1)^{t} C_{r,n+t}^{n+1} \omega_{n+1} \wedge \cdots \wedge (\omega_r \wedge \omega_{n+t}) \wedge \cdots \wedge \omega_m
\]

\[
= \left( - \sum_{1 \leq t \leq n} \sum_{1 \leq r \leq n} C_{r,n+t}^{n+1} \omega_r \right) \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m,
\]

and we can write \(d\omega = d\omega = \theta \wedge \omega\). Hence by Theorem 2, we know that \(\eta\) is a Nambu-Poisson tensor.

(ii) We use the same notations as (i). Then by writing down the condition \(d\omega = \theta \wedge \omega\), we can get easily that \(C_{pq}^r = 0\) \((n + 1 \leq i \leq m, 1 \leq p, q \leq n)\). This means that \(\mathfrak{h}\) is a Lie subalgebra of \(g\). 

By Proposition 4, to each Lie subalgebra of \(g\), there corresponds a left invariant Nambu-Poisson tensor of order \(n\) up to constant multiple. Conversely if a left invariant Nambu-Poisson tensor \(\eta\) has two expressions: \(\eta = X_1 \wedge \cdots \wedge X_n = Y_1 \wedge \cdots \wedge Y_n\), then by E.Cartan’s lemma, we know that \(< X_1, \ldots, X_n > = < Y_1, \ldots, Y_n >\). Thus we have

COROLLARY 3. There is a one to one correspondence up to constant multiple between the set of left invariant Nambu-Poisson tensors of order \(n\) on \(G\) and the set of \(n\)-dimensional Lie subalgebras of \(g\).

Let \(G\) be an \(m\)-dimensional connected Lie group and \(H\) an \(n\)-dimensional closed subgroup of \(G\). Denote by \(g\) and \(h\) the Lie algebras of \(G\) and \(H\) respectively. Let \(\pi : G \to G/H\) be the natural projection. The mapping \(\omega \to \pi^* \omega\) establishes a 1-1 correspondence between the \(G\)-invariant \(p\)-forms on \(G/H\) and the left invariant \(p\)-forms \(\omega\) on \(G\) which satisfy

(a) \(i(X)\omega = 0\) for all \(X \in \mathfrak{h}\),

(b) \(\mathcal{L}(X)\omega = 0\) for all \(X \in \mathfrak{h}\) [2].
If $\omega$ is a $G$-invariant $(m - n)$-form (i.e., $G$-invariant volume form) on $G/H$. Then $\omega = \pi^*\omega$ is a left invariant $(m - n)$-form on $G$. Since $\omega$ is closed and decomposable, $\omega$ induces a left invariant Nambu-Poisson tensor $\eta$ of order $n$ on $G$ by the equation $i(\eta)\Omega = \omega$. It is clear that $\eta$ is equal to the left invariant Nambu-Poisson tensor corresponding to the Lie algebra $h$ up to constant multiple. Define $h_\omega = \{X \in g \mid i(X)\omega = 0\}$. Then $h_\omega$ is a Lie subalgebra of $g$ and $h_\omega = h$. The maximal integral submanifold $H_\omega$ through $e$ is the identity component of $H$. Since $H$ is closed, $H_\omega$ is also a closed subgroup of $G$.

Conversely let us give a left invariant Nambu-Poisson tensor $\eta$ of order $n \geq 3$. Then as we have seen in Proposition 4, $\eta$ determines an $n$-dimensional Lie subalgebra $\mathfrak{h}$, and $\eta$ also induces the left invariant $(m - n)$-form $\omega$ on $G$ by $i(\eta)\Omega = \omega$. In the following theorem, we give a sufficient condition for $\omega$ to be projected down to the $G$-invariant volume form of $G/H$. This is essentially due to S.S.Chern [1].

**Theorem 5.** Let $G$ be an $n$-dimensional connected unimodular Lie group, and $\eta$ a left invariant Nambu-Poisson tensor of order $n \geq 3$ on $G$. Then there corresponds an $n$-dimensional Lie subalgebra $\mathfrak{h}$. Denote by $H$ the connected Lie subgroup corresponding to $\mathfrak{h}$. If $H$ is closed and unimodular, then $\omega$ is projected down to the $G$-invariant volume form of $G/H$.

**Proof.** It is clear that $i(X)\omega = 0$ for all $X \in \mathfrak{h}$. Since $G$ and $H$ are unimodular, it holds that $\text{Tr}d_{\mathfrak{g}}(X) = \text{Tr}d_{\mathfrak{h}}(X) = 0$ for all $X \in \mathfrak{h}$. Let $C_{pq}^r$ be the structure constants of $\mathfrak{g}$. Then this implies that $\sum_{\alpha=n+1}^m C_{\alpha\beta}^\gamma = 0, (i = 1, \ldots, n)$. In view of the proof of Proposition 4, we know that $d\omega = 0$. Hence two conditions (a) and (b) are satisfied so that $\omega$ is projectable.

Another easy sufficient condition for $\omega$ to be projectable is the following. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\text{ad}(X)$ is $\mathfrak{h}$-valued for $X \in \mathfrak{h}$, and we easily obtain that $d\omega = 0$. Thus we have

**Proposition 6.** Let $\eta$ be a left invariant Nambu-Poisson tensor of order $n$ on $G$. Suppose that $\mathfrak{h}$ induced by $\eta$ is an ideal of $\mathfrak{g}$ and the connected Lie group $H$ which corresponds to $\mathfrak{h}$ is a closed subgroup of $G$. Then $\omega$ is projected down to the $G$-invariant volume form of $G/H$.

Here let us give one example of the pair of Lie groups $(G, H)$ such that $\omega$ cannot be projected down to any $G$-invariant volume form of $G/H$. In this case, of course, $H$ is not unimodular. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ and let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n} + \mathfrak{t}$ be the usual Iwasawa decomposition. Take $\mathfrak{a} + \mathfrak{n}$ as $\mathfrak{h}$. Then $\mathfrak{h}$ is not an ideal but a Lie subalgebra of $\mathfrak{g}$. Let $A$ and $N$ be the connected Lie groups corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then $A$ and $N$ are closed Lie subgroups of $SL(3, \mathbb{R})$, and $H$ is diffeomorphic to $A \times N$. Hence $H$ is a closed subgroup of $SL(3, \mathbb{R})$. Now we can find a basis $\langle X_1, \ldots, X_8 \rangle$ of $\mathfrak{g}$ such that $\mathfrak{a} = \langle X_1, X_2 \rangle$ and $\mathfrak{n} = \langle X_3, X_4, X_5 \rangle$. Put $\eta = X_1 \wedge \cdots \wedge X_5$. Then $\omega = i(\eta)\Omega$ can be written as $\omega = \omega_6 \wedge \omega_7 \wedge \omega_8$ with respect to the dual basis $\langle \omega_1, \ldots, \omega_8 \rangle$ of $\langle X_1, \ldots, X_8 \rangle$. Then we know that $i(\mathfrak{h})d\omega \neq 0$. Hence $\omega$ cannot be projected down to any $G$-invariant form of $G/H$. 
Let $\omega$ be a left invariant closed form on a Lie group $G$. Put $h_\omega = \{ X \in \mathfrak{g} \mid i(X)\omega = 0 \}$. Then $h_\omega$ is a Lie subalgebra of $\mathfrak{g}$. Denote by $H_\omega$ the connected Lie subgroup corresponding to $h_\omega$. Bon-Yao Chu [3] proved the following:

**Proposition 7.** On a simply connected Lie group, if a left invariant 2-form $\omega$ is closed, the corresponding connected Lie subgroup $H_\omega$ is closed in $G$.

Applying the above result to our cases, we can easily obtain:

**Proposition 8.** Let $G$ be an $(n+2)$-dimensional simply connected Lie group and $\eta$ a left invariant Nambu-Poisson tensor of order $n$ on $G$. Denote by $\mathfrak{h}$ the Lie subalgebra induced by $\eta$. Put $\omega = i(\eta)\Omega$, where $\Omega$ is a left invariant volume form of $G$. If $d\omega = 0$, then the connected Lie subgroup $H$ corresponding to $\mathfrak{h}$ is closed in $G$. In particular if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $H$ is a closed normal subgroup of $G$.

**References**