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NAMBU-POISSON TENSORS ON LIE GROUPS

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Abstract. First as an application of the local structure theorem for Nambu-Poisson tensors, we characterize them in terms of differential forms. Secondly left invariant Nambu-Poisson tensors on Lie groups are considered.

1. Introduction. In 1994, L. Takhtajan [6] gave geometric formulations of *Nambu-Poisson manifolds,* and a lot of papers have followed his work. A Nambu-Poisson manifold is defined to be a pair of a C^∞-manifold and a *Nambu-Poisson tensor* defined on it. A Nambu-Poisson tensor is, by definition, a skew-symmetric contravariant tensor field on a manifold such that the induced bracket operation satisfies the *fundamental identity,* which is a generalization of the usual Jacobi identity. It is generally difficult to judge whether a given tensor field is a Nambu-Poisson tensor or not. This is because a Nambu-Poisson tensor is written in the form of a contravariant tensor.

We begin with characterizing Nambu-Poisson tensors via differential forms. The characterization will be done by using contraction of Nambu-Poisson tensors with the volume form. The local structure theorem [5] for Nambu-Poisson tensors will be very useful to obtain some results concerning this topic.

Secondly left invariant Nambu-Poisson tensors on Lie groups are considered. And we shall study when they can be projected on suitable homogeneous spaces. These problems were studied by Bon-Yao Chu [3] in the case of symplectic structures.

2. Nambu-Poisson manifolds. First we give a definition of a Nambu-Poisson tensor, which is equivalent to that of L. Takhtajan [6]. Let M be an m-dimensional C^∞-manifold, and $\mathcal{F}$ its algebra of real valued C^∞-functions. We denote by $\Gamma(\Lambda^n TM)$ the space of global cross-sections $\eta : M \to \Lambda^n TM$. Then to each $\eta \in \Gamma(\Lambda^n TM)$, there

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corresponds the bracket defined by

$$\{f_1, \ldots, f_n\} = \eta(df_1, \ldots, df_n), \quad f_1, \ldots, f_n \in \mathcal{F}.$$

Let $A = \sum f_{i_1} \wedge \cdots \wedge f_{i_{n-1}}$ be any element of $\Lambda^{n-1}\mathcal{F}$. Since the bracket operation clearly satisfies the Leibniz rule, we can define a vector field X_A corresponding to A by the following equation:

$$X_A(g) = \sum \{f_{i_1}, \ldots, f_{i_{n-1}}, g\}, \quad g \in \mathcal{F}.$$

Such a vector field is called a *Hamiltonian vector field.* The space of Hamiltonian vector fields is denoted by $\mathcal{H}$.

DEFINITION 1. $\eta \in \Gamma(\Lambda^n TM)$ is called a *Nambu-Poisson tensor* of order n if it satisfies $\mathcal{L}(X_A)\eta = 0$ for all $X_A \in \mathcal{H}$, where $\mathcal{L}$ is the Lie derivative. Then a *Nambu-Poisson manifold* is a pair (M, η).

The above definition is clearly equivalent to the following *fundamental identity:*

$$\begin{aligned}\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = &\{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} \\ &+\{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} \\ &+ \ldots + \{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}\end{aligned}$$

for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in \mathcal{F}$. If $n = 2$, this equation is nothing but the Jacobi identity, and we have usual Poisson manifolds.

Let $\eta(p) \neq 0$, $p \in M$. Then we say that η is *regular* at p. Now we can state the following local structure theorem for Nambu-Poisson tensors [4],[5].

THEOREM 1. *Let $\eta \in \Gamma(\Lambda^n TM)$, $n \geq 3$. If η is a Nambu-Poisson tensor of order n, then for any regular point p, there exists a coordinate neighborhood U with local coordinates $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ around p such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

on U, and vice versa.

In Theorem 1, the condition $n \geq 3$ is essential. If $n = 2$, as is well-known, A.Weinstein [8] proved the splitting theorem for Poisson tensors. Comparing Weinstein's splitting theorem with our theorem, we know that the local structure of Nambu-Poisson manifolds is more rigid than that of usual Poisson manifolds.

Some simple applications of Theorem 1 are the following. Informations on the *Schouten bracket* may be found in [7], for instance.

COROLLARY 1. *(1) Let η be a Nambu-Poisson tensor of order $n \geq 3$. If f is a smooth function, then $f\eta$ is again a Nambu-Poisson tensor.*

(2) If $m = n \geq 3$, then every $\eta \in \Gamma(\Lambda^n TM)$ is a Nambu-Poisson tensor.

(3) For every Nambu-Poisson tensor η, its Schouten bracket satisfies $[\eta, \eta] = 0$. (Of course the converse is not true.)

3. Characterizations of Nambu-Poisson tensors. Throughout this section, we assume that (M, η) is a Nambu-Poisson manifold with volume form Ω, and $m \geq n \geq 3$.

Put $\omega = i(\eta)\Omega$, where the right hand side is the interior product of η and Ω. Hence ω is an $(m-n)$-form.

A differential form α of degree k (locally defined) around $p \in M$ is called *decomposable* at p if there exist 1-forms $\alpha_1, \dots, \alpha_k$ (which are locally defined around p) such that $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$. We also define the decomposability of (locally defined) contravariant tensor fields in the same manner. In the following theorem, we will characterize a Nambu-Poisson tensor by using the notion of decomposability.

THEOREM 2. *Let $\eta \in \Gamma(\Lambda^n TM)$. Then η is a Nambu-Poisson tensor if and only if η satisfies the following two conditions around each regular point:*

1) ω is decomposable, and

2) there exists a locally defined 1-form θ such that $d\omega = \theta \wedge \omega$.

PROOF. If η is a Nambu-Poisson tensor and p is its regular point, then by Theorem 1, there exist local coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ around p such that

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$

Suppose that $\Omega = f dx_1 \wedge \cdots \wedge dx_m$ with respect to these coordinates. Then we have

$$\omega = f dx_{n+1} \wedge \cdots \wedge dx_m,$$

and

$$d\omega = df \wedge dx_{n+1} \wedge \cdots \wedge dx_m = \frac{df}{f} \wedge \omega.$$

Conversely assume that η satisfies above two conditions. Then there exist 1-forms $\omega_{n+1}, \dots, \omega_m$ such that $\omega = \omega_{n+1} \wedge \cdots \wedge \omega_m$. Note that $\omega_{n+1}, \dots, \omega_m$ are linearly independent around p since $\omega(p) \neq 0$. Adding n 1-forms $\omega_1, \dots, \omega_n$ to them, we can construct a basis of 1-forms around p. For $n+1 \leq i \leq m$, put

$$d\omega_i = \sum_{1 \leq j < k \leq m} g^i_{jk} \omega_j \wedge \omega_k.$$

Since

$$\begin{aligned} d\omega &= \theta \wedge \omega = \theta \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m \\ &= \sum_{l=n+1}^{m} (-1)^{l-n-1} \omega_{n+1} \wedge \cdots \wedge d\omega_l \wedge \cdots \wedge \omega_m, \end{aligned}$$

we have

$$\begin{aligned} 0 = \omega_i \wedge d\omega &= (-1)^{i-n-1} \omega_i \wedge \omega_{n+1} \wedge \cdots \wedge d\omega_i \wedge \cdots \wedge \omega_m \\ &= d\omega_i \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m. \end{aligned}$$

This means that $g^i_{jk} = 0$ for $1 \leq j < k \leq n$, and we have the following expression:

$$d\omega_i = \sum_{j=n+1}^{m} \theta_{ij} \wedge \omega_j,$$

where θ_{ij} are 1-forms. By the Frobenius theorem, there exist local functions f_{ij} and g_j

such that

$$\omega_i = \sum_{j=n+1}^{m} f_{ij} dg_j, \quad G = det(f_{ij}) \neq 0.$$

Adding n-local functions $x_1, \ldots, x_n$, we can adopt $(x_1, \ldots, x_n, g_{n+1}, \ldots, g_m)$ as local coordinates. With respect to these local coordinates, the volume form Ω can be written as

$$\Omega = F dx_1 \wedge \cdots \wedge dx_n \wedge dg_{n+1} \wedge \cdots \wedge dg_m.$$

If we put $y_1 = \int \frac{F}{G} dx_1,\ y_2 = x_2, \ldots, y_n = x_n, y_{n+1} = g_{n+1}, \ldots, y_m = g_m$, then Ω is rewritten as

$$\Omega = G dy_1 \wedge \cdots \wedge dy_m.$$

With respect to these new local coordinates $(y_1, \ldots, y_m)$, ω has the following expression:

$$\begin{aligned} \omega &= \omega_{n+1} \wedge \cdots \wedge \omega_m = G dy_{n+1} \wedge \cdots \wedge dy_m \\ &= i(\eta)(G dy_1 \wedge \cdots \wedge dy_m). \end{aligned}$$

Thus we obtain that $\eta = \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}$. Using Theorem 1 once again, we know that η is a Nambu-Poisson tensor. ∎

REMARK 1. The above criterion for Nambu-Poisson tensors does not depend on the choice of volume form.

Suppose that $m = n + 1$. Since every 1-form is clearly decomposable, we have

COROLLARY 2. *If $m = n + 1$, then η is a Nambu-Poisson tensor if and only if $\omega \wedge d\omega = 0$.*

4. Nambu-Poisson tensors on Lie groups. Let G be an m-dimensional connected Lie group, $m \geq 3$. First we shall determine the form of left invariant Nambu-Poisson tensors on G. Denote by $\mathfrak{g}$ the Lie algebra of left invariant vector fields on G.

PROPOSITION 3. *Let η be a (non-zero) left invariant Nambu-Poisson tensor of order $n \geq 3$ on a Lie group G. Then η is globally decomposable.*

PROOF. Let e be the unit element of G. By Theorem 1, η has the following expression around $e \in G : \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$, where $(x_1, \ldots, x_m)$ is some coordinate neighborhood around e. Then there exist uniquely n elements $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that $(X_i)_e = (\frac{\partial}{\partial x_i})_e,\ 1 \leq i \leq n$. Since η is left invariant, we immediately have $\eta = X_1 \wedge \cdots \wedge X_n$ on G. ∎

By the above proposition, any left invariant Nambu-Poisson tensor η of order n can be written as a decomposable element of $\Lambda^n \mathfrak{g}$.

PROPOSITION 4. *Let G be an m-dimensional Lie group.*

(i) Let $\mathfrak{h}$ be an n-dimensional Lie subalgebra of $\mathfrak{g}$, $n \geq 3$. For a basis $< X_1, \ldots, X_n >$ of $\mathfrak{h}$, put $\eta = X_1 \wedge \cdots \wedge X_n$. Then η is a left invariant Nambu-Poisson tensor of order n on G.

(ii) Conversely given a left invariant Nambu-Poisson tensor $\eta = X_1 \wedge \cdots \wedge X_n \in \Lambda^n \mathfrak{g}$ on G, then $\mathfrak{h} = < X_1, \ldots, X_n >$ is a Lie subalgebra of $\mathfrak{g}$.

PROOF. *(i)* Let $< X_1, \ldots, X_n, X_{n+1}, \ldots X_m >$ be a basis of $\mathfrak{g}$ obtained by extending a basis of $\mathfrak{h}$, and let $< \omega_1, \ldots, \omega_m >$ be the dual basis of $< X_1, \ldots, X_m >$. Put $\Omega = \omega_1 \wedge \cdots \wedge \omega_m$. Then Ω is a left invariant volume form on G. Define a left invariant $(m-n)$-form ω by $\omega = i(\eta)\Omega = \omega_{n+1} \wedge \cdots \wedge \omega_m$. Let $\{C^i_{pq}\}$ be the structure constants of $\mathfrak{g}$. Since $\mathfrak{h}$ is a Lie subalgebra, $C^i_{pq} = 0$ $(n+1 \leq i \leq m,\ 1 \leq p, q \leq n)$. Then

$$d\omega_i = -\sum_{n+1 \leq p < q \leq m} C^i_{pq}\omega_p \wedge \omega_q - \sum_{1 \leq r \leq n} \sum_{n+1 \leq s \leq m} C^i_{rs}\omega_r \wedge \omega_s.$$

Thus we have

$$\begin{aligned}
d\omega &= \sum_{t=1}^{m-n} (-1)^{t-1}\omega_{n+1} \wedge \cdots \wedge d\omega_{n+t} \wedge \cdots \wedge \omega_m \\
&= \sum_{t=1}^{m-n} (-1)^{t-1}\omega_{n+1} \wedge \cdots \wedge \Bigg(- \sum_{n+1 \leq p < q \leq m} C^{n+t}_{pq}\omega_p \wedge \omega_q \\
&\quad - \sum_{1 \leq r \leq n} \sum_{n+1 \leq s \leq m} C^{n+t}_{rs}\omega_r \wedge \omega_s \Bigg) \wedge \cdots \wedge \omega_m \\
&= \sum_{t=1}^{m-n} \sum_{1 \leq r \leq n} (-1)^t C^{n+t}_{r,n+t}\omega_{n+1} \wedge \cdots \wedge (\omega_r \wedge \omega_{n+t}) \wedge \cdots \wedge \omega_m \\
&= \Bigg(- \sum_{t=1}^{m-n} \sum_{1 \leq r \leq n} C^{n+t}_{r,n+t}\omega_r \Bigg) \wedge \omega_{n+1} \wedge \cdots \wedge \omega_m,
\end{aligned}$$

and we can write $d\omega$ as $d\omega = \theta \wedge \omega$. Hence by Theorem 2, we know that η is a Nambu-Poisson tensor.

(ii) We use the same notations as *(i)*. Then by writing down the condition $d\omega = \theta \wedge \omega$, we can get easily that $C^i_{pq} = 0$ $(n+1 \leq i \leq m,\ 1 \leq p, q \leq n)$. This means that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. ■

By Proposition 4, to each Lie subalgebra of $\mathfrak{g}$, there corresponds a left invariant Nambu-Poisson tensor of order n up to constant multiple. Conversely if a left invariant Nambu-Poisson tensor η has two expressions: $\eta = X_1 \wedge \cdots \wedge X_n = Y_1 \wedge \cdots \wedge Y_n$, then by E.Cartan's lemma, we know that $< X_1, \ldots, X_n > = < Y_1, \ldots, Y_n >$. Thus we have

COROLLARY 3. *There is a one to one correspondence up to constant multiple between the set of left invariant Nambu-Poisson tensors of order n on G and the set of n-dimensional Lie subalgebras of $\mathfrak{g}$.*

Let G be an m-dimensional connected Lie group and H an n- dimensional closed subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Let $\pi : G \to G/H$ be the natural projection. The mapping $\bar{\omega} \to \pi^*\bar{\omega}$ establishes a 1-1 correspondence between the G-invariant p-forms on G/H and the left invariant p-forms ω on G which satisfy

(a) $i(X)\omega = 0$ for all $X \in \mathfrak{h}$,

(b)$\mathcal{L}(X)\omega = 0$ for all $X \in \mathfrak{h}$ [2].

If $\bar{\omega}$ is a G-invariant $(m-n)$-form (i.e. G-invariant volume form) on G/H. Then $\omega = \pi^*\bar{\omega}$ is a left invariant $(m-n)$-form on G. Since ω is closed and decomposable, ω induces a left invariant Nambu-Poisson tensor η of order n on G by the equation $i(\eta)\Omega = \omega$. It is clear that η is equal to the left invariant Nambu-Poisson tensor corresponding to the Lie algebra $\mathfrak{h}$ of H up to constant multiple. Define $\mathfrak{h}_\omega = \{X \in \mathfrak{g} \mid i(X)\omega = 0\}$. Then $\mathfrak{h}_\omega$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_\omega = \mathfrak{h}$. The maximal integral submanifold H_ω through e is the identity component of H. Since H is closed, H_ω is also a closed subgroup of G.

Conversely let us give a left invariant Nambu-Poisson tensor η of order $n \geq 3$. Then as we have seen in Proposition 4, η determines an n-dimensional Lie subalgebra $\mathfrak{h}$, and η also induces the left invariant $(m-n)$-form ω on G by $i(\eta)\Omega = \omega$. In the following theorem, we give a sufficient condition for ω to be projected down to the G-invariant volume form of G/H. This is essentially due to S.S.Chern [1].

THEOREM 5. *Let G be an m-dimensional connected unimodular Lie group, and η a left invariant Nambu-Poisson tensor of order $n \geq 3$ on G. Then there corresponds an n-dimensional Lie subalgebra $\mathfrak{h}$. Denote by H the connected Lie subgroup corresponding to $\mathfrak{h}$. If H is closed and unimodular, then ω is projected down to the G-invariant volume form of G/H.*

PROOF. It is clear that $i(X)\omega = 0$ for all $X \in \mathfrak{h}$. Since G and H are unimodular, it holds that $\mathrm{Tr}ad_{\mathfrak{g}}(X) = \mathrm{Tr}ad_{\mathfrak{h}}(X) = 0$ for all $X \in \mathfrak{h}$. Let C^r_{pq} be the structure constants of $\mathfrak{g}$. Then this implies that $\sum_{\alpha=n+1}^m C^\alpha_{i\alpha} = 0, (i = 1, \ldots, n)$. In view of the proof of Proposition 4, we know that $d\omega = 0$. Hence two conditions (a) and (b) are satisfied so that ω is projectable. ∎

Another easy sufficient condition for ω to be projectable is the following. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $ad(X)$ is $\mathfrak{h}$-valued for $X \in \mathfrak{h}$, and we easily obtain that $d\omega = 0$. Thus we have

PROPOSITION 6. *Let η be a left invariant Nambu-Poisson tensor of order n on G. Suppose that $\mathfrak{h}$ induced by η is an ideal of $\mathfrak{g}$ and the connected Lie group H which corresponds to $\mathfrak{h}$ is a closed subgroup of G. Then ω is projected down to the G-invariant volume form of G/H.*

Here let us give one example of the pair of Lie groups (G, H) such that ω cannot be projected down to any G-invariant volume form of G/H. In this case, of course, H is not unimodular. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathrm{R})$ and let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n} + \mathfrak{k}$ be the usual Iwasawa decomposition. Take $\mathfrak{a} + \mathfrak{n}$ as $\mathfrak{h}$. Then $\mathfrak{h}$ is not an ideal but a Lie subalgebra of $\mathfrak{g}$. Let A and N be the connected Lie groups corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively.Then A and N are closed Lie subgroups of $SL(3, \mathrm{R})$, and H is diffeomorphic to $A \times N$. Hence H is a closed subgroup of $SL(3, \mathrm{R})$. Now we can find a basis $< X_1, \ldots, X_8 >$ of $\mathfrak{g}$ such that $\mathfrak{a} =< X_1, X_2 >$ and $\mathfrak{n} =< X_3, X_4, X_5 >$. Put $\eta = X_1 \wedge \cdots \wedge X_5$. Then $\omega = i(\eta)\Omega$ can be written as $\omega = \omega_6 \wedge \omega_7 \wedge \omega_8$ with respect to the dual basis $< \omega_1, \ldots, \omega_8 >$ of $< X_1, \ldots, X_8 >$. Then we know that $i(\ \mathfrak{h}\ )d\omega \neq 0$. Hence ω cannot be projected down to any G-invariant form of G/H.

Let ω be a left invariant closed form on a Lie group G. Put $\mathfrak{h}_{\omega} = \{X \in \mathfrak{g} \,|\, i(X)\omega = 0\}$. Then $\mathfrak{h}_\omega$ is a Lie subalgebra of $\mathfrak{g}$. Denote by H_ω the connected Lie subgroup corresponding to $\mathfrak{h}_\omega$. Bon-Yao Chu [3] proved the following:

Proposition 7. *On a simply connected Lie group, if a left invariant 2-form ω is closed, the corresponding connected Lie subgroup H_ω is closed in G.*

Applying the above result to our cases, we can easily obtain:

Proposition 8. *Let G be an $(n+2)$-dimensional simply connected Lie group and η a left invariant Nambu-Poisson tensor of order n on G. Denote by $\mathfrak{h}$ the Lie subalgebra induced by η. Put $\omega = i(\eta)\Omega$, where Ω is a left invariant volume form of G. If $d\omega = 0$, then the connected Lie subgroup H corresponding to $\mathfrak{h}$ is closed in G. In particular if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then H is a closed normal subgroup of G.*

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