CLASSIFICATION OF ALMOST SPHERICAL PAIRS OF COMPACT SIMPLE LIE GROUPS

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Abstract. All homogeneous spaces $G/K$ ($G$ is a simple connected compact Lie group, $K$ a connected closed subgroup) are enumerated for which arbitrary Hamiltonian flows on $T^*(G/K)$ with $G$-invariant Hamiltonians are integrable in the class of Noether integrals and $G$-invariant functions.

1. Introduction. Let $G$ be a compact connected Lie group and $K$ its closed connected subgroup. Denote by $X$ a symplectic manifold on which $G$ acts in a Hamiltonian fashion. Let $P : X \to g^*$, where $g$ is the Lie algebra of $G$, be the moment mapping. The functions of type $h \circ P$, for $h : g^* \to \mathbb{R}$, are called collective. Such $h \circ P$ are integrals for any flow on $X$ with $G$-invariant Hamiltonian (Noether’s theorem). A completely integrable system consisting of (dim $X/2$ independent real-analytic commuting with respect to the Poisson bracket) functions of this type is called a collective completely integrable system [GS1]. All symmetric spaces $G/K$ admit a collective completely integrable system on the phase space $T^*(G/K)$ ( [Ti, Mi, My1, GS2] and [IW]). Moreover, the following conditions are equivalent [GS1, GS2, My2, PM]:

1) on the phase space $T^*(G/K)$ there exists a collective completely integrable system (and, consequently, every Hamiltonian system with a $G$-invariant Hamiltonian $H$ is integrable);

2) the algebra of $G$-invariant functions on $T^*(G/K)$ is commutative;

3) the subgroup $K$ of $G$ is spherical; i.e., the quasiregular representation of $G$ on

2000 Mathematics Subject Classification: Primary 22E46; Secondary 17B.
Research supported by INTAS Grant 00-418906.
The paper is in final form and no version of it will be published elsewhere.
the Hilbert space $L^2(G/K)$ (or on the space $C[G/K]$ of regular functions on the affine algebraic variety $G/K$) has simple spectrum;

4) $P$ separates (at least generically) the $G$-orbits, i.e. $P^{-1}(P(x)) \subset G \cdot x$.

The classification of spherical subgroups of simple compact (connected) Lie groups was obtained in [Kr] by M. Kramer (1979), of semisimple compact connected Lie groups in our paper [My2] (see also [PM]) and by a different method in [Br] by M. Brion. In the case of a real noncompact Lie group $G$ the generalization of these results was obtained by M. Chumak [Ch].

The spherical pairs of Lie groups were studied in many papers and have many beautiful features. But, since the Poisson structure on $X$ is nondegenerate, then every $G$-invariant Hamiltonian $H$ locally has the form $h \circ P$; i.e., for integrability we don’t use the function $H$.

Let $N_{\text{max}}$ be the maximal number of independent real-analytic commuting (with respect to the Poisson bracket) functions on $X = T^*(G/K)$ of type $h \circ P$. If $N_{\text{max}} = (\dim X/2) - 1$ we will call the corresponding system of functions an almost collective completely integrable system and the subgroup $K$ an almost spherical subgroup of $G$. In this case every Hamiltonian system with a $G$-invariant Hamiltonian $H$, in particular, the geodesic flow, is also integrable; for the integrability we can use $H$ or another $G$-invariant function.

In this paper we enumerate all such homogeneous spaces $G/K$ with a simple compact Lie group $G$. Crucial ingredients in our classification are the dimensional criterion

$$\dim K \geq \frac{1}{2}(\dim G - \text{rank } G) - 1,$$

Theorem 5, and inequality (9). Roughly speaking this criterion says that $K$ has to be a “large” subgroup of $G$, Theorem 5 allows us to find only “maximal large” almost spherical subgroups $K$ and to use the already known classification of spherical pairs of semisimple compact Lie groups. We show by inspection that simple groups $G$ do not have many maximal large subgroups and there is only one pair $(G, K)$, where the almost spherical subgroup $K$ is maximal in $G$. With the exception of this one case, an almost spherical subgroup $K$ of a simple group $G$ is not maximal. Moreover, if $K \subset S \subset G$, where a connected subgroup $S$ is a spherical maximal subgroup of $G$, then for the isotropy subgroup $S'^2$ of an arbitrary point $x_2 \in G/S$ in some neighborhood of the point $\{S\} \subset G/S$ the following inequality holds:

$$\dim(S/K) \leq \frac{1}{2}(\dim S'^2 + \text{rank } S'^2) + 1.$$

A connected closed subgroup $K$ of a compact simple group $G$ is said to be of height at most $n$ in $G$ if there exist a chain of distinct connected (closed) subgroups $K \subset S^{(n)} \subset \ldots \subset S' \subset G$ and every chain of distinct connected (closed) subgroups between $K$ and $G$ is of length at most $n$. It is proved that if $K$ is almost spherical, then $K$ is of height at most $n$ in $G$, where $n \leq 2$. There are eighteen types of such pairs $(G, K)$, one of which is of height 0, four are of height 2, while the remaining are of height 1. In the latter case $K \subset S \subset G$, where $G/S$ is a symmetric space.
2. Almost spherical pairs and the existence of invariant flows

2.1. Moment map. Let $G$ be a compact real Lie group, $K$ its closed subgroup and $M = G/K$. The natural action of $G$ on the quotient space $M$ extends to an action of $G$ on $T^*M$. This $G$-action on $T^*M$ is symplectic since it preserves the canonical 1-form $\lambda$ (the form $\omega = \omega^{pdq}$), and thus also the symplectic 2-form $d\lambda$. For each vector $\xi$ belonging to the Lie algebra $\mathfrak{g}$ of $G$ the 1-parameter subgroup $\exp t\xi$ induces the Hamiltonian vector field $\xi$ on $T^*M$ with the Hamiltonian function $f_\xi = \lambda(\xi) = df_\xi = -\hat{\xi}d\lambda$. The mapping $\xi \mapsto f_\xi$ of $\mathfrak{g}$ into the algebra $C^\infty(T^*M)$ (with the Poisson bracket) is an equivariant algebra homomorphism: $f(g^{-1}m) = f_{\Ad g(\xi)}(m)$ and hence the action of $G$ on $T^*M$ is Poisson [GS3]. This action defines the moment map $\Phi : T^*M \to \mathfrak{g}^*$ from $T^*M$ to the dual space of the Lie algebra $\mathfrak{g}$ by $P(x)(\xi) = f_\xi(x)$. For arbitrary smooth functions $h_1$ and $h_2$ on $\mathfrak{g}^*$ we have $\{h_1 \circ P, h_2 \circ P\} = \{h_1, h_2\} \circ P$, where the Poisson bracket on $\mathfrak{g}^*$ is given by the formula $\{h_1, h_2\}(\beta) = \beta(\{dh_1(\beta), dh_2(\beta)\})$, $\beta \in \mathfrak{g}^*$.

There exists a faithful representation of $\mathfrak{g}$ such that its associated bilinear form $\phi$ is negative definite on $\mathfrak{g}$. Let $\mathfrak{t}$ be the Lie algebra of the subgroup $K$ and $\pi$ the natural projection of $G$ onto $M$. Using $\Phi$ we can identify the space $\mathfrak{g}^*$ with $\mathfrak{g}$. The corresponding isomorphism denote by $\psi : \mathfrak{g}^* \to \mathfrak{g}$. Also identify $T^*\pi(e)M$ and $\mathfrak{m} \overset{\text{def}}{=} \{x \in \mathfrak{g} : \Phi(x, e) = 0\}$. It is evident that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. Under these identifications $P(x) = \Phi(x, \cdot) \in \mathfrak{g}^*$. Let $\mathfrak{g}^x = \{y \in \mathfrak{g} : [x, y] = 0\}$ and $\mathfrak{t}^x = \mathfrak{t} \cap \mathfrak{g}^x$, where $x \in \mathfrak{g}$.

2.2. Integrability. Let us show that the fulfillment of the condition

$$\dim(\mathfrak{g}^x/\mathfrak{t}^x) + \frac{1}{2} \dim(\mathfrak{g}/\mathfrak{g}^x) = \dim(\mathfrak{g}/\mathfrak{t}) - \varepsilon, \quad \varepsilon = \varepsilon(\mathfrak{g}, \mathfrak{t}) = 0 \text{ or } 1 \quad (1)$$

at any point $x$ from some neighborhood in $\mathfrak{m}$ is sufficient for the integrability of any Hamiltonian flow on $T^*M$ with $G$-invariant Hamiltonian function.

Let us consider the Zariski open subset $R(\mathfrak{m}) \subset \mathfrak{m}$ of points in general position:

$$R(\mathfrak{m}) = \{x \in \mathfrak{m} : \dim \mathfrak{g}^x \leq \dim \mathfrak{g}^y, \dim \mathfrak{t}^x \leq \dim \mathfrak{t}^y, \forall y \in \mathfrak{m}\}. \quad (2)$$

On $\mathfrak{g}^x \overset{\text{def}}{=} \mathfrak{g}$ there exists $s = \dim(\mathfrak{g}^x/\mathfrak{t}^x) + \frac{1}{2} \dim(\mathfrak{g}/\mathfrak{g}^x)$, $x \in R(\mathfrak{m})$, polynomial functions $h_1, h_2, \ldots, h_s$ such that the functions $h_1 \circ P, h_2 \circ P, \ldots, h_s \circ P$ are pairwise in involution on $T^*M$ and are independent at the point $x \in \mathfrak{m} = T^*\pi(e)M$ (the number $s$ is the maximal number of independent functions in involution of the form $h \circ P$ on $T^*M$) (see [My2]).

Let $W(x)$ be the tangent space to the orbit $G \cdot x \subset T^*M$ (of maximal dimension). Then $W(x)$ is generated by vectors $\xi(x)$, $\xi \in \mathfrak{g}$. Applying the slice-theorem to the action of the compact group $G$ on $T^*M$ (or to the $\Ad K$-action on $\mathfrak{m} = T^*\pi(e)M$) one deduces that the orthogonal complement $W(x)^\perp \overset{\text{def}}{=} \{X \in T_xT^*M : d\lambda(X, W(x)) = 0\}$ to $W(x)$ with respect to the symplectic structure $d\lambda$ is generated by Hamiltonian vector fields (at $x$) of $G$-invariant functions on $T^*M$. Let us show that codimension of $W^\perp(x) \cap W(x)$ in $W^\perp(x)$ is equal to $2\varepsilon$. Indeed, multiplying both sides of (1) by 2, we obtain after simple rearrangements

$$\dim(\mathfrak{g}^x/\mathfrak{t}^x) + \dim(\mathfrak{g}/\mathfrak{t}^x) = 2(\dim(\mathfrak{g}/\mathfrak{t}) - 2\varepsilon). \quad (3)$$

It follows from [My1] (see also [GS2], [PM]) that $\hat{\xi}(x) = 0$ iff $\xi \in \mathfrak{t}^x$ and consequently
Thus \( \dim \lambda \) is nondegenerate we obtain

\[
\dim W(x) + \dim W^\perp(x) = \dim T^*M = 2 \dim (g/\mathfrak{t}^e).
\]

Thus \( \dim W(x) - \dim (W(x) \cap W^\perp(x)) = 2\varepsilon \) and therefore if \( \varepsilon = 1 \) there is a \( G \)-invariant function \( F \) on \( T^*M \) which is independent of functions \( \{ h_i \circ P \}, i = 1, \ldots, s \) (the Hamiltonian vector fields of \( h_i \circ P \) are tangent to orbits of \( G \) in \( T^*M \)). The set \( \{ F, h_1 \circ P, \ldots, h_s \circ P \} \) is the maximal involutive set of independent functions: \( s + 1 = \frac{1}{2} \dim T^*M \). If the \( G \)-invariant function \( H \) is independent of \( \{ h_i \circ P \}, i = 1, \ldots, s \) then the set \( \{ F, h_1 \circ P, \ldots, h_s \circ P \} \) is a maximal involutive set; if \( H \) is dependent then we have the commutative set of integrals \( \{ F, h_1 \circ P, \ldots, h_s \circ P \} \). If \( \varepsilon = 0 \), then \( s = \frac{1}{2} \dim T^*M \); i.e., on the manifold \( T^*M \) there exists a collective completely integrable system and also any \( G \)-invariant flow is integrable. We proved

**PROPOSITION 1.** If condition (1) holds for all \( x \) from open subset of \( \mathfrak{m} \) then any Hamiltonian system on \( T^*M \) with a \( G \)-invariant Hamiltonian function \( H \) is integrable.

2.3. Properties of spherical pairs of compact Lie algebras. Since \( \dim g = \dim \mathfrak{t} + \dim \mathfrak{m} \) equation (1) (the definition of \( \varepsilon(\mathfrak{g}, \mathfrak{t}) \)) is equivalent to

\[
\dim (g^e/\mathfrak{t}^e) + \dim (\mathfrak{t}/\mathfrak{t}^e) = \dim \mathfrak{m} - 2\varepsilon.
\]

By [My2, Prop. 1.1] (see also [Mi, GS2]) for any \( x \in R(\mathfrak{m}) \) the commutator \( [\mathfrak{g}^e, \mathfrak{t}^e] \) is contained in the algebra \( \mathfrak{t}^e = \mathfrak{t} \cap \mathfrak{g}^e \). Therefore for a semisimple element \( x \in R(\mathfrak{m}) \) we have \( \dim (g^e/\mathfrak{t}^e) = \text{rank } g - \text{rank } \mathfrak{t}^e \), and so (4) can be rewritten as

\[
(\text{rank } g - \text{rank } \mathfrak{t}^e) + \dim (\mathfrak{t}/\mathfrak{t}^e) = \dim \mathfrak{m} - 2\varepsilon.
\]

Moreover, it is evident that \( \dim (g^e/\mathfrak{t}^e) \leq \text{rank } g \leq \dim g^e \) and consequently (1) implies

\[
\dim \mathfrak{t} \geq \frac{1}{2} (\text{dim } g - \text{rank } g) - \varepsilon.
\]

For any \( x \in \mathfrak{m} \) put

\[
\mathfrak{m}(x) \overset{\text{def}}{=} \{ z \in \mathfrak{m} : [x, z] \in \mathfrak{m} \} = \{ z \in \mathfrak{m} : \Phi(z, \text{ad } x(t)) = 0 \}.
\]

**LEMMA 2.** For a pair \((\mathfrak{g}, \mathfrak{t})\) of compact Lie algebras and any point \( x \in \mathfrak{m} \) the following conditions are equivalent:

1. \( \{ \text{codim} \text{ad } x(\mathfrak{t}) \in \mathfrak{m} \} = \dim (g^e/\mathfrak{t}^e) + 2\varepsilon \); i.e., \( \dim \mathfrak{m}(x) = \dim (g^e/\mathfrak{t}^e) + 2\varepsilon \);
2. \( \{ \text{codim} g^e_{\mathfrak{m} \cap \mathfrak{m}(x)} \} = 2\varepsilon \), where \( (\cdot)_{\mathfrak{m}} \) is the projection onto \( \mathfrak{m} \) along \( \mathfrak{t} \) induced by the decomposition \( g = \mathfrak{t} \oplus \mathfrak{m} \);
3. \( \{ \text{codim} \text{ad } x(\mathfrak{m}(x)) \cap \text{ad } x(\mathfrak{t}) \} = 2\varepsilon \).

**PROOF.** It is sufficient to see that 1) \( g^e_{\mathfrak{m} \cap \mathfrak{m}(x)} \subset \mathfrak{m}(x) \); 2) \( \text{ad } x(\mathfrak{t}) \subset \mathfrak{m} \); 3) \( \mathfrak{m}(x) \oplus \text{ad } x(\mathfrak{t}) = \mathfrak{m} \) and if \( x' \in \mathfrak{m} \) then \( \text{ad } x(x') \in \text{ad } x(\mathfrak{t}) \Leftrightarrow x' \in g^e_{\mathfrak{m} \cap \mathfrak{m}(x)} \).
Definition 3. We say that a pair \((\mathfrak{g}, \mathfrak{t})\) of compact Lie algebras is an \textit{almost spherical pair} (resp. \textit{spherical pair}) and a subalgebra \(\mathfrak{t} \subset \mathfrak{g}\) is an \textit{almost spherical subalgebra} (resp. \textit{spherical subalgebra}) if for any \(x \in R(\mathfrak{m})\) the equivalent conditions of Lemma 2 or the equivalent equalities (1), (3)–(5) are satisfied, where \(\varepsilon = 1\) (resp., where \(\varepsilon = 0\)).

Remark 4. For a spherical pair \((\mathfrak{g}, \mathfrak{t})\) of compact Lie algebras the conditions (2), (3) of Lemma 2, \(\varepsilon = 0\) are equivalent to the conditions (2') \(m(x) = (g'')_m\) and (3') \((\text{ad}\,x)(m(x)) \subset \text{ad}\,x(\mathfrak{t}), x \in R(\mathfrak{m})\) [My2]. Thus all \textit{symmetric pairs} \((\mathfrak{g}, \mathfrak{t})\), i.e. those for which \(\mathfrak{t}\) is the algebra of fixed points of an involutive automorphism of the algebra \(\mathfrak{g}\), are spherical (see [Mi], [My2]). Indeed, \([\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}\) and consequently \(m(x) = g' \cap \mathfrak{m}\). All spherical subalgebras \(\mathfrak{t}\) of compact Lie algebras \(\mathfrak{g}\) are classified in [Kr] (for simple \(\mathfrak{g}\)) and in [My2], [Br] (the semisimple case).

Theorem 5. Let \(\mathfrak{g}\) be a compact Lie algebra with subalgebras \(\mathfrak{t} \subset \mathfrak{s}\). Let \((\mathfrak{g}, \mathfrak{t})\) be an \textit{almost spherical pair}. Then the pairs \((\mathfrak{g}, \mathfrak{s})\) and \((\mathfrak{s}, \mathfrak{t})\) are either \textit{almost spherical} or \textit{spherical}. Moreover, if \((\mathfrak{g}, \mathfrak{s})\) is \textit{almost spherical}, then the pair \((\mathfrak{s}, \mathfrak{t})\) is \textit{spherical}.

Proof. Let \(\mathfrak{m}_1\) (respectively \(\mathfrak{m}_2\)) be the orthogonal complement to the subalgebra \(\mathfrak{t}\) in \(\mathfrak{s}\) (respectively to the subalgebra \(\mathfrak{s}\) in \(\mathfrak{g}\)) with respect to the form \(\Phi\); i.e., \(\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2\). Fix an element \(x_1 + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)\) such that \(x_1 \in R(\mathfrak{m}_1)\) and \(x_2 \in R(\mathfrak{m}_2)\). Let \(V_1\) (respectively \(V_2\)) be the orthogonal complement to the subspace \((\mathfrak{s}^{x_1})_{\mathfrak{m}_1}\) in \(\mathfrak{m}_1(\mathfrak{x}_1)\) (respectively to the subspace \((\mathfrak{s}^{x_2})_{\mathfrak{m}_2}\) in \(\mathfrak{m}_2(\mathfrak{x}_2)\)). From the relation \((\mathfrak{s}^{x_1})_{\mathfrak{m}_1} = \{z \in \mathfrak{m}_1 : \text{ad}\,x_1(z) \in \text{ad}\,x_1(\mathfrak{t})\}\) it may be concluded that

\[
\text{ad}\,x_1(V_1) \cap \text{ad}\,x_1(\mathfrak{t}) = 0, \quad \text{dim} \text{ad}\,x_1(V_1) = \text{dim} V_1, \quad (8)
\]

If \(u_1 \in \mathfrak{m}_1(\mathfrak{x}_1)\) then by definition \([x_1, u_1] \in \mathfrak{m}_1\), hence \([x_1 + x_2, u_1] \in \mathfrak{m}\) and, consequently, \(\mathfrak{m}_1(x_1) \subset \mathfrak{m}(x_1 + x_2)\). Using the similar arguments we obtain that \(\mathfrak{m}_2(x_2) \subset \mathfrak{m}(x_1 + x_2)\).

The pair \((\mathfrak{s}, \mathfrak{t})\) is either spherical or almost spherical iff \(\text{dim} V_1 \leq 2\). Otherwise the space \(V_1\) is at least four-dimensional. From the relations \(V_1 \subset \mathfrak{m}_1(\mathfrak{x}_1) \subset \mathfrak{m}(\mathfrak{x}_1 + \mathfrak{x}_2)\) for the space \(V_1\) of the dimension \(\geq 4\) and condition (2) of Lemma 2: \(\text{dim} \mathfrak{m}(\mathfrak{x}_1 + \mathfrak{x}_2) - \text{dim}(\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}} = 2\), it follows that for some \(v_1 \in V_1, v_1 \neq 0: v_1 \in (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}}\). Thus \([x_1 + x_2, v_1] \in \text{ad}(x_1 + x_2)(\mathfrak{t})\). Therefore for some \(z \in \mathfrak{t}: [x_1, v_1] + [x_2, v_1] = [x_1, z] + [x_2, z]\) and consequently \([x_1, v_1] = [x_1, z] \in \mathfrak{m}_1\). This contradicts the first relation in (8) (by the second relation \([x_1, v_1] \neq 0\)) so that \(\text{dim} V_1 \leq 2\).

To obtain the contradiction, suppose that the pair \((\mathfrak{g}, \mathfrak{s})\) is neither almost spherical nor spherical. In this case the space \(V_2 = V_2(\mathfrak{x}_2) \subset \mathfrak{m}(\mathfrak{x}_2)\) is at least four-dimensional. But the pair \((\mathfrak{g}, \mathfrak{t})\) is almost spherical and \(V_2 \subset \mathfrak{m}(\mathfrak{x}_2) \subset \mathfrak{m}(\mathfrak{x}_1 + \mathfrak{x}_2)\) so that the intersection \(V_2 \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}}\) is at least two-dimensional. Since for \(x \in R(\mathfrak{m})\) dimension \(\text{dim}(\mathfrak{g}^x)_{\mathfrak{m}} = \text{dim}(\mathfrak{g}^x/\mathfrak{t}^x)\) is constant, we can define the Zariski open subset \(Q_1 = Q_1(\mathfrak{x}_2)\) of \(\mathfrak{m}_1\) of all elements \(x_1\) such that (1) \(x_1' \in R(\mathfrak{m}_1)\) and \(x_1' + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)\); (2) the space \(V_2 \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}} \equiv \text{V}_2(x_1')\) has the minimal possible dimension \(l\) (from what has already been showed \(l \geq 2\)). Since the set \(Q_1\) is not empty and \(0 \in Q_1\), the space \(V_2 \cap (\mathfrak{g}^{x_2})_{\mathfrak{m}}\) is at least \(l\)-dimensional (the Grassmann manifold of all \(l\) planes in \(V_2\) is compact, \([\mathfrak{t}, \mathfrak{m}_1] \subset \mathfrak{m}_1\)). This contradicts the definition of \(V_2\) because \((\mathfrak{g}^{x_2})_{\mathfrak{m}} \cap \mathfrak{m}_2 \subset (\mathfrak{g}^{x_2})_{\mathfrak{m}_2}\) so that \(\text{dim} V_2 \leq 2\).
It remains to prove that if \( \dim V_2 = 2 \) then \( V_1 = 0 \). Otherwise, assume that \( \dim V_1 = 2 \). Since \( V_1 \oplus V_2 \subset m_1(x_1) \oplus m_2(x_2) \subset m(x_1 + x_2) \), by condition (2) of Lemma 2, the space \( V' = (V_1 \oplus V_2) \cap (g^{x_1+x_2})_m \) has dimension \( \geq 2 \). Therefore, for any non-zero \( v_1 + v_2 \in V' \subset V_1 \oplus V_2 \) there exists an element \( z \in \mathfrak{k} \) such that \( [x_1 + x_2, v_1 + v_2] = [x_1 + x_2, z] \). But \( [\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i, i = 1, 2 \) so that \( [x_1, v_1] = [x_1, z] \) which contradicts (8) if \( v_1 \neq 0 \). Thus \( V' = V_2 \), \( (\dim V' \geq 2) \) and \( V_2 \subset (g^{x_1+x_2})_m \); i.e., for any non-zero \( v_2 \in V_2 \) we have \( [x_1 + x_2, v_2] \in \text{ad}(x_1 + x_2)(\mathfrak{k}) \) and consequently \( [x_1 + x_2, v_2] \in \text{ad} x_2(\mathfrak{k}) \). The latter holds for any \( x_1' \in \mathfrak{m} \) from some neighborhood of \( x_1 \), hence \( [x_2, v_2] \in \text{ad} x_2(\mathfrak{k}) \subset \text{ad} x_2(\mathfrak{s}) \). Therefore \( v_2 \in (g^{x_2})_{m_2} \), which is impossible, a contradiction. 

In the notation of the proof of the theorem if a pair \((\mathfrak{g}, \mathfrak{k})\) is almost spherical and \((\mathfrak{g}, \mathfrak{s})\) is spherical or almost spherical then
\[
\text{rank} \mathfrak{g} - \text{rank} \mathfrak{t}^{x_1+x_2} + \dim(\mathfrak{t}/\mathfrak{t}^{x_1+x_2}) = \dim(\mathfrak{g}/\mathfrak{k}) - 2
\]
and
\[
\text{rank} \mathfrak{g} - \text{rank} \mathfrak{s}^{x_2} + \dim(\mathfrak{s}/\mathfrak{s}^{x_2}) = \dim(\mathfrak{g}/\mathfrak{s}) - 2\varepsilon_2, \quad \text{where} \quad \varepsilon_2 = 0, 1.
\]
Hence
\[
2\dim(\mathfrak{s}/\mathfrak{k}) = (\dim \mathfrak{s}^{x_2} + \text{rank} \mathfrak{s}^{x_2}) - (\dim \mathfrak{t}^{x_1+x_2} + \text{rank} \mathfrak{t}^{x_1+x_2}) + (2 - 2\varepsilon_2)
\]
and
\[
\dim(\mathfrak{s}/\mathfrak{k}) \leq \frac{1}{2}(\dim \mathfrak{s}^{x_2} + \text{rank} \mathfrak{s}^{x_2}) + 1 - \varepsilon_2.
\]

Remark 6. Let \( \mathfrak{g} \) be a semisimple compact Lie algebra and let \( \mathfrak{a} \oplus \mathfrak{j} \) be its subalgebra with one-dimensional center \( \mathfrak{j} \). If the pair \((\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{j})\) is spherical then the pair \((\mathfrak{g}, \mathfrak{a})\) is either spherical or almost spherical. To prove this fact it is sufficient to see that \( \mathfrak{m} \overset{\text{def}}{=} (\mathfrak{a} \oplus \mathfrak{j})^+ \subset \mathfrak{g} \) is the subspace of \( \mathfrak{a}^+ = \mathfrak{m} \oplus \mathfrak{j} \) and \( R(\mathfrak{m}) \subset R(\mathfrak{m} \oplus \mathfrak{j}) \) [My2, Prop.2.2].

3. Almost spherical subalgebras of simple Lie algebras

3.1. Preliminary remarks. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra with a compact real form \( \mathfrak{g}_0 \) and let \( \Phi \) be the Killing form of \( \mathfrak{g} \). Let \( \mathfrak{k} \) be a reductive subalgebra of \( \mathfrak{g} \) such that \( \mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \) is the real form of \( \mathfrak{k} \) with respect to \( \Phi \). It is evident that \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) and \( \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{i} \mathfrak{m}_0 \), where \( \mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0 \). Using (2) define the Zariski open subset \( R'(\mathfrak{m}) \) (over \( \mathbb{C} \)) of \( \mathfrak{m} \). Let \( R(\mathfrak{m}) \) denote a set of all \( x \in R'(\mathfrak{m}) \) which are semisimple elements of Lie algebra \( \mathfrak{g} \). Then \( R(\mathfrak{m}) \) is a Zariski open subset of \( \mathfrak{m} \) (see [My2]). It is clear that \( R(\mathfrak{m}_0) \subset R(\mathfrak{m}) \). We say that a pair \((\mathfrak{g}, \mathfrak{k})\) is almost spherical (resp. spherical) if the pair \((\mathfrak{g}_0, \mathfrak{k}_0)\) of compact Lie algebras is almost spherical (resp. spherical); i.e., if for any \( x \in R(\mathfrak{m}) \) the equivalent conditions like (1), (3)–(5) hold. To verify these conditions we use the results of [El], where for all simple complex Lie algebras \( \mathfrak{k} \) all their representations \( \pi_\mathfrak{k} \) and types of corresponding isotropic subalgebras \( \mathfrak{t}^x \) (of elements in general position) if \( \mathfrak{t}^x \neq 0 \) are enumerated.

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). Write \( R(\Lambda) \) for the irreducible representation of \( \mathfrak{g} \) with highest weight \( \Lambda \), and \( R'(\Lambda) \) for its contragredient representation. Let \( \eta \) denote the one-dimensional trivial representation. If \( \{\alpha_i\} \) is a basis for the root system of \( \mathfrak{g} \) relative \( \mathfrak{h} \), and \( \{\varphi_i\} \) are the corresponding fundamental weights, then \( \Lambda = \sum \Lambda_i \varphi_i \), where \( \Lambda_i \in \mathbb{Z}^+ \). We shall index the roots of a basis for the root system of a simple Lie algebra in the order given in [Bo2, Tables I-IX of Chapter VI].
Let \( \rho \) be a faithful linear representation of a simple complex Lie algebra \( \mathfrak{g} \) in a space of the smallest possible dimension. We associate the embedding \( j_0 \) of a subalgebra \( \mathfrak{t}_0 \) in \( \mathfrak{g}_0 \) and the embedding \( j \) of the natural complex extension \( \mathfrak{t} = \mathfrak{t}_0^C \) in \( \mathfrak{g} \) with the linear (complex) representation \( \hat{\rho} \) obtained by restricting \( \rho \circ j \) to its semisimple part. We denote such a subalgebra \( \mathfrak{t}_0 \subset \mathfrak{g}_0 (\mathfrak{t} \subset \mathfrak{g}) \) by a pair \((, \hat{\rho})\), where for the first entry we put the type of an algebra \( \mathfrak{t}_0 \) or \( \mathfrak{t} \).

### 3.2. Almost spherical maximal subalgebras of classical Lie algebras.

Let \( V \) be a linear space of dimension \( n \) over \( \mathbb{C} \). For the rest of this subsection \( \mathfrak{g} \) will denote a classical complex Lie algebra, i.e., one of \( \mathfrak{sl}(n) \), \( \mathfrak{so}(n) \), or \( \mathfrak{sp}(n) \) (for \( n \) even), with \( \mathfrak{t} \) a reductive subalgebra. Since \( \mathfrak{t} \) is reductive, \( V \) is a semisimple \( \mathfrak{t} \)-module. If \( V \) is a simple \( \mathfrak{t} \)-module we shall say that \( \mathfrak{t} \) is irreducible.

**Proposition 7.** Suppose that \( \mathfrak{t} \) is an almost spherical subalgebra of a simple classical Lie algebra \( \mathfrak{g} \) and \( \mathfrak{t} \) is maximal in \( \mathfrak{g} \). Then \( \mathfrak{g} \cong \mathfrak{b}_2 \) and \( \mathfrak{t} \) is a unique (up to inner automorphisms) principal \( \mathfrak{s}l_2 \)-subalgebra of \( \mathfrak{g} \); i.e., \((\mathfrak{g}, \mathfrak{t}) \cong (\mathfrak{sp}(4), (A_1, R(3\varphi))) \cong (\mathfrak{so}(5), (A_1, 4\varphi))\). Moreover, any almost spherical subalgebra \( \mathfrak{t}_1 \) of \( \mathfrak{so}(5) \) (\( \mathfrak{sp}(4) \)) such that \( \mathfrak{t}_1 \subset \mathfrak{t} \) coincides with \( \mathfrak{t} \).

**Proof.** Since the subalgebra \( \mathfrak{t} \) is reductive, the \( \mathfrak{t} \)-module \( V \) is semisimple. Suppose that \( V \) is a nonsimple \( \mathfrak{t} \)-module; i.e., \( V = V_1 \oplus V_2 \) is a direct sum of two nonzero semisimple \( \mathfrak{t} \)-modules \( V_1 \) and \( V_2 \). But the subalgebra \( \mathfrak{s} = \mathfrak{g}(V_1, V_2) = \{ x \in \mathfrak{g} : x(V_1) \subset V_1, x(V_2) \subset V_2 \} \) (which contains \( \mathfrak{t} \)) of the classical Lie algebra \( \mathfrak{g} \) is maximal because the pair \((\mathfrak{g}, \mathfrak{s})\) is symmetric [He]. Therefore \( \mathfrak{t} \neq \mathfrak{s} \) and \( V \) is a simple \( \mathfrak{t} \)-module.

A) Suppose that \( \mathfrak{t} \) is a simple irreducible subalgebra of \( \mathfrak{g} \) (by E.Cartan theorem any irreducible subalgebra of \( \mathfrak{g} \) is semisimple). If the pair \((\mathfrak{g}, \mathfrak{t})\) is almost spherical then inequality (6) is satisfied: \( \dim \mathfrak{g} \geq M(\mathfrak{g}) - 1 \), where \( M(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} - \text{rank } \mathfrak{g}) \). Therefore \( \dim \mathfrak{g} \geq \frac{1}{2}n(n-2) - 1 \), since \( \frac{1}{2}n(n-2) \leq M(\mathfrak{so}(n)) \leq M(\mathfrak{sp}(n)) \leq M(\mathfrak{sl}(n)) \). When \( \mathfrak{t} \) has type \( A_r \) (r \( \geq 1 \)), \( B_r \) (r \( \geq 2 \)), \( C_r \) (r \( \geq 3 \)), \( D_r \) (r \( \geq 4 \)), \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \), or \( G_2 \), it is not hard to verify that for the latter inequality to be satisfied it is necessary for \( n \) not to exceed 2r + 3, 4r, 4r - 2, 19, 25, 33, 17, or 9, respectively. The irreducible representations \( \rho \) of simple algebras \( \mathfrak{a} \) whose dimensions satisfy these restrictions can be found in [On, Lemma 3.2] with the exception of one case when \( \mathfrak{a} \cong A_r \) and \( n = 2r + 3 \). For an algebra \( \mathfrak{a} \) of type \( A_r \), the representations are the following: \( (r, \varphi_1 + r + 1, 1) \), \( (4, \varphi_4 + 1, 10, 0) \), \( (3, \varphi_2, 6, 1) \), \( (2, 2\varphi_1 + 2\varphi_2, 6, 0) \), \( (1, \varphi_1, 2, -1) \), \( (1, 2\varphi_1, 3, 1), (1, 3\varphi_1, 4, -1) \), (where the quadruple \((r, \Lambda, n(\Lambda), \varepsilon(\Lambda))\) we mean that \( r \) is the rank of \( \mathfrak{t} \), \( \Lambda \) and \( n(\Lambda) \) are the highest weight and dimension of \( \rho \) respectively; the symbol \( \varepsilon(\Lambda) \) is 1, -1, or 0 according as \( \rho \) is orthogonal, symplectic, or neither orthogonal nor symplectic, respectively. For a simple algebra \( \mathfrak{a} \) of type \( B_r \) we have the following representations: \( (r, \varphi_1 + 2r + 1, 1) \), \( (4, \varphi_4, 16, 1) \), \( (3, \varphi_3, 8, 1) \), \( (2, \varphi_2, 4, -1) \); type \( C_r \): \( (r, \varphi_1, 2r, -1) \); type \( D_r \): \( (r, \varphi_1, 2r, 1) \); type \( G_2 \): \( (2, \varphi_1, 7, 1) \); algebras of the remaining types have no such representations. Clearly \( \rho(\mathfrak{a}) \subset \mathfrak{so}(n) \) (\( \rho(\mathfrak{a}) \subset \mathfrak{sp}(n) \)) if the representation \( \rho \) is orthogonal (symplectic). It can be verified that inequality (6) is satisfied only for the following pairs \((\mathfrak{g}, \mathfrak{a})\) from among those found above: (a) \((\mathfrak{sl}(n), \mathfrak{so}(n))\), \( n \geq 3 \), \( n \neq 4 \); (b) \((\mathfrak{sl}(n), \mathfrak{sp}(n))\), \( n \geq 4 \); (c) \((\mathfrak{so}(8), (B_3, R(\varphi_3)))\) (spinor representation); (d) \((\mathfrak{so}(7), (G_2, R(\varphi_1)))\). All these
pairs (a)–(d) are spherical [Kr, My2]. Now assume that $n = 2r + 3$ for $\mathfrak{t} \simeq A_r$. Since in this case $\dim \mathfrak{t} < M(sl(2r + 3)) - 1$ for all $r \geq 1$ the representation $\rho$ of $\mathfrak{t}$ have to be either orthogonal or symplectic. If $r = 1$ then for the pair $(so(5), (A_1, R(4\varphi)))$ condition (6) is an equality. If $r \geq 2$ using the explicit formula for dimension of the representation $R(A)$ and properties of the root system $A_r$ (all roots have the same length) we obtain that a representation of $\mathfrak{t}$ which admits an invariant bilinear form and has minimal dimension is the representation $(r = 2, \varphi_1 + \varphi_r, r(2 + 1), 1)$, $(3, \varphi_2, 6, 1)$ or $(r = 2k + 1 \geq 5, \varphi_{k+1}, N_k \geq (k + 2)(k + 3), (−1)^{k+1})$, where $N_k = \frac{(2k+2)!}{(k+1)!(k+1)}$. Thus there are no such representations of Lie algebra $A_r$ of the dimension $2r + 3(r \geq 2)$.

B) Suppose that $\mathfrak{t}$ is an irreducible subalgebra of $\mathfrak{g}$ and $\mathfrak{t}$ is not simple (is semisimple). Then $\mathfrak{t} \in \{a\}$, where $\{a\}$ is a set of all maximal semisimple (not simple) subalgebras of $\mathfrak{g}$. The maximal subalgebra $\mathfrak{a}$ is isomorphic to the tensor product $sl(s) \otimes sl(t)$ $(st = n, 2 \leq s \leq t)$ if $\mathfrak{g} = sl(n)$; $sp(s) \otimes sp(t)$ $(st = n, 2 \leq s \leq t)$ or $so(s) \otimes so(t)$ $(st = n, 3 \leq s \leq t; s, t \neq 4)$ if $\mathfrak{g} = so(n)$; $sp(s) \otimes so(t)$ $(st = n, s \geq 2, t \geq 3, t \neq 4$ or $s = 2, t = 4)$ if $\mathfrak{g} = sp(n)$ ([Dy1, Theorems 1.3 and 1.4]). Inequality (6) holds only for two pairs $(\mathfrak{g}, a) = (sl(4), sl(2) \otimes sl(2))$ and $(so(8), sp(2) \otimes sp(4))$. These two pairs are spherical [Kr, My2].

Now it remains to prove that the pair $(\mathfrak{g}, \mathfrak{t}) = (so(5), (A_1, R(4\varphi)))$ is almost spherical. To compute the representation $\pi_\mathfrak{t}$ of the Lie algebra $\mathfrak{t}$ in $\mathfrak{m}$ consider the $sl_2$-triple $\{X_+, H, X_-\}$ in $\mathfrak{t} \simeq A_1$ [Bo3, Chapt. VIII, §1]. Then the eigenvalues of $H \in so(5) \subset sl(5)$ are the numbers $4, 2, 0, −2, −4$. Using the standard root system of $\mathfrak{g}$ with respect to the Cartan subalgebra (of diagonal matrices) $\mathfrak{h} \subset H$ of $\mathfrak{g}$ we obtain that $\alpha_i(H) = 2$ for every simple root $\alpha_i, i = 1, 2$ so that $\mathfrak{t}$ is principal $sl_2$-subalgebra of $\mathfrak{g}$ [Bo3, Chapt. VIII, §1,11] and $\pi_\mathfrak{t} = R(6\varphi)$. Thus $\mathfrak{t}^\mathfrak{g} = 0$ for any $\mathfrak{x} \in R(\mathfrak{m})$ [El] and consequently the pair $\mathfrak{g}, \mathfrak{t}$ is almost spherical. Since in this case (6) is equality $\mathfrak{t}$ does not contain proper almost spherical subalgebra of $\mathfrak{g}$. To prove that $(so(5), (A_1, R(4\varphi))) \simeq (sp(4), (A_1, R(3\varphi)))$ it is sufficient to make the following observation: $(A_1, R(3\varphi)) \subset sp(4)$ is principal $sl_2$-subalgebra of $sp(4)$ (the eigenvalues of $H \in sp(4) \subset sl(4)$ are $3, 1, −1, −3$) and all principal $sl_2$-subalgebras are conjugate. 

3.3. Almost spherical maximal subalgebras of exceptional Lie algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is regular if $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We say that $\mathfrak{a}$ is an $S$-subalgebra of $\mathfrak{g}$ if it is not contained in any proper regular subalgebra of $\mathfrak{g}$ [Dy2].

Let $\mathfrak{g}$ be an exceptional complex Lie algebra, $\mathfrak{s}$ its maximal reductive subalgebra. Let us find such subalgebras $\mathfrak{s}$ for which condition (6) holds: $\dim \mathfrak{s} \geq M(\mathfrak{g}) − 1 = (\dim \mathfrak{g} − \text{rank}\mathfrak{g})/2 − 1$.

A maximal reductive subalgebra $\mathfrak{s}$ of a simple Lie algebra $\mathfrak{g}$ is either regular or an $S$-subalgebra. The list of the types of all maximal $S$-subalgebras $\mathfrak{s}$ of the exceptional algebras $\mathfrak{g}$ is as follows: $G_2 = \{A_1\}$; $F_4 = \{A_1, A_1 \oplus G_2\}$; $E_6 = \{A_1, G_2, A_2 \oplus G_2, F_4, C_4\}$; $E_7 = \{A_1, A_1 \oplus A_2, G_2 \oplus C_3, A_1 \oplus F_4, A_1 \oplus G_2\}$; $E_8 = \{A_1, A_1 \oplus A_2, E_6, G_2 \oplus F_4\}$ [Dy2]. It is not hard to verify that necessary condition (6) for $(\mathfrak{g}, \mathfrak{s})$ is satisfied only for the pairs of types $(E_6, F_4)$ and $(E_6, C_4)$ which are symmetric [GG] so we can proceed to the case when $\mathfrak{s}$ is regular.
The list of the types of all maximal regular subalgebras \( s \) of the exceptional algebras \( g \) such that \((g,s)\) is not symmetric is as follows: \( G_2 - \{ A_2 \}; F_4 - \{ A_2 \oplus A_2 \}; E_6 - \{ A_2 \oplus A_2 \oplus A_2 \}; E_7 - \{ A_2 \oplus A_5 \}; E_8 - \{ A_8, A_1 \oplus A_1, A_2 \oplus E_6 \} \) [Dy2]. Condition (6) holds only for the pair of type \( \{ G_2, A_2 \} \) which is spherical. Thus we proved

**Proposition 8.** There is no almost spherical subalgebra \( \mathfrak{k} \) of a simple exceptional Lie algebra \( g \) such that \( \mathfrak{k} \) is maximal in \( g \).

### 3.4. Almost spherical subalgebras of simple Lie algebras

Let \( g \) be a complex simple Lie algebra. A reductive subalgebra \( \mathfrak{k} \) of \( g \) is said to be of height at most \( n \) in \( g \) if there exists a chain of distinct reductive subalgebras \( \mathfrak{k} \subset s^{(n)} \subset \ldots \subset s' \subset g \) and every chain of distinct reductive subalgebras between \( \mathfrak{k} \) and \( g \) is of length at most \( n \). From Propositions 7 and 8 it follows that 1) there is only one almost spherical pair \((g, \mathfrak{k})\) for which \( \mathfrak{k} \) is of height at most \( 0 \) in \( g \); 2) if an almost spherical subalgebra \( \mathfrak{k} \) is of height at most \( h \geq 1 \) in \( g \) then \( \mathfrak{k} \subset s \subset g \), where by Theorem 5 the subalgebra \( s \) is maximal spherical in \( g \) and \( \mathfrak{k} \) is either almost spherical or spherical (maximal if \( h = 1 \)) subalgebra of \( s \). But we already know all spherical subalgebras \( s \) of simple Lie algebras \( g \). If \( s \) is maximal then the pair \((g, s)\) is symmetric or is as in the following list: \((so(8), (B_3, R(\varphi_3)))\), \((so(8), sp(2) \oplus sp(4))\), \((so(7), (G_2, R(\varphi_1)))\), \((G_2, A_2)\) [Kr, My2]. Now applying the dimensional criterion (9) with \( \varepsilon_2 = 0 \) to such pairs \((g, s)\) (types of the isotropic subalgebras \( s^{\varepsilon_2} \) are enumerated in [Ar] for symmetric pairs and for remaining spherical pairs in [My2]) we find the set of such pairs \((g, \mathfrak{k})\) which contains all almost spherical pairs. It remains to establish that (5)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( g )</th>
<th>( \mathfrak{k} )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A_r, r \geq 2 )</td>
<td>( A_{r-2} \oplus 2C )</td>
<td>( R(\varphi_1) + 2\eta )</td>
</tr>
<tr>
<td>2*</td>
<td>( A_r, r \geq 4 )</td>
<td>( A_{r-2} \oplus C )</td>
<td>( R(\varphi_1) + 2\eta )</td>
</tr>
<tr>
<td>3</td>
<td>( A_{2r-1}, r \geq 1 )</td>
<td>( A_{r-1} \oplus A_{r-1} )</td>
<td>( R(\varphi_1) \oplus \eta + \eta \oplus R(\varphi_1) )</td>
</tr>
<tr>
<td>4</td>
<td>( B_r, r \geq 2 )</td>
<td>( A_{r-1} )</td>
<td>( R(\varphi_1) + R(\varphi_1) )</td>
</tr>
<tr>
<td>5</td>
<td>( C_r, r \geq 3 )</td>
<td>( B_{r-1} )</td>
<td>( R(\varphi_1) + 2\eta )</td>
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<tr>
<td>6</td>
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<td>( C_{r-1} )</td>
<td>( R(\varphi_1) + 2\eta )</td>
</tr>
<tr>
<td>7</td>
<td>( C_r, r \geq 3 )</td>
<td>( C_{r-2} \oplus A_1 \oplus A_1 )</td>
<td>( R(\varphi_1) \oplus \eta \oplus R(\varphi_1) )</td>
</tr>
<tr>
<td>8</td>
<td>( D_{2r}, r \geq 2 )</td>
<td>( A_{2r-1} )</td>
<td>( R(\varphi_1) + R(\varphi_1) )</td>
</tr>
<tr>
<td>9</td>
<td>( D_{2r}, r \geq 4 )</td>
<td>( D_{r-1} )</td>
<td>( R(\varphi_1) + 2\eta )</td>
</tr>
<tr>
<td>10</td>
<td>( A_5 )</td>
<td>( C_2 \oplus C_1 \oplus C )</td>
<td>( R(\varphi_1) \oplus \eta + \eta \oplus R(\varphi_1) )</td>
</tr>
<tr>
<td>11</td>
<td>( B_5 )</td>
<td>( B_3 \oplus A_1 )</td>
<td>( R(\varphi_3) \oplus \eta \oplus R(2\varphi_1) )</td>
</tr>
<tr>
<td>12</td>
<td>( B_5 )</td>
<td>( G_2 \oplus C )</td>
<td>( R(\varphi_1) + 2\eta )</td>
</tr>
<tr>
<td>13</td>
<td>( B_2 )</td>
<td>( A_1 )</td>
<td>( R(4\varphi_1) )</td>
</tr>
<tr>
<td>14</td>
<td>( D_5 )</td>
<td>( B_3 )</td>
<td>( R(\varphi_3) + 2\eta )</td>
</tr>
<tr>
<td>15</td>
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<td>( D_4 )</td>
<td>( D_4 \subset B_3 \subset F_4 )</td>
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<tr>
<td>16</td>
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<td>( B_4 \oplus C )</td>
<td>( B_4 \oplus C \subset D_5 \oplus C \subset E_6 )</td>
</tr>
<tr>
<td>17</td>
<td>( E_7 )</td>
<td>( E_6 )</td>
<td>( E_6 \subset E_6 \oplus C \subset E_7 )</td>
</tr>
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</table>
holds (or does not hold) at points \(x \in R(m)\) for all these pairs. For this it suffices to find the type of the centralizer \(k_x\), which is completely determined by the representation \(\pi : x \mapsto \text{ad}_m x\) of \(k\) in \(m\). An easy computation shows that often \(k_x \subset k_1\), where \(k_1\) is some simple ideal of \(k\), and, consequently, the algebra \(k_x\) is determined by the restriction \(\pi_{k_1}\) and its type is given in the tables in [El]. Thus using Theorem 5, Propositions 7,8 and dimensional criterion (9) we obtain

**Theorem 9.** Let \(g\) be a complex simple Lie algebra, \(k\) its reductive subalgebra. All almost spherical pairs \((g, k)\) are shown in Table 1, where the representations determining the embedding \(j : k \rightarrow g\) are also given.\(^1\) The almost spherical subalgebra \(k\) of the pair \((g, k)\) 14 is of height 0 (in \(g\)), four subalgebras \(k\) of pairs 2,4,7,15 are of height 2, while the remaining are of height 1.

**References**


\(^1\)In the case 2* the centralizer of \(k\) in \(g\) is the Lie algebra \(a_1 \oplus \mathfrak{z}_1 \cong A_1 \oplus \mathbb{C}\) and the 1-dimensional center of \(k\) is a diagonal subalgebra of \(h_1 \oplus \mathfrak{z}_1\), where \(h_1\) is the Cartan subalgebra of \(a_1\).


