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CLASSIFICATION OF ALMOST SPHERICAL PAIRS OF COMPACT SIMPLE LIE GROUPS

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Abstract. All homogeneous spaces G/K (G is a simple connected compact Lie group, K a connected closed subgroup) are enumerated for which arbitrary Hamiltonian flows on $T^*(G/K)$ with G-invariant Hamiltonians are integrable in the class of Noether integrals and G-invariant functions.

1. Introduction. Let G be a compact connected Lie group and K its closed connected subgroup. Denote by X a symplectic manifold on which G acts in a Hamiltonian fashion. Let $P : X \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, be the moment mapping. The functions of type $h \circ P$, for $h : \mathfrak{g}^* \to \mathbf{R}$, are called *collective*. Such $h \circ P$ are integrals for any flow on X with G-invariant Hamiltonian (Noether's theorem). A completely integrable system consisting of ($\dim X/2$ independent real-analytic commuting with respect to the Poisson bracket) functions of this type is called a *collective completely integrable system* [GS1]. All symmetric spaces G/K admit a collective completely integrable system on the phase space $T^*(G/K)$ ([Ti, Mi, My1, GS2] and [IW]). Moreover, the following conditions are equivalent [GS1, GS2, My2, PM]:

1) on the phase space $T^*(G/K)$ there exists a collective completely integrable system (and, consequently, every Hamiltonian system with a G-invariant Hamiltonian H is integrable);

2) the algebra of G-invariant functions on $T^*(G/K)$ is commutative;

3) the subgroup K of G is spherical; i.e., the quasiregular representation of G on

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the Hilbert space $L^2(G/K)$ (or on the space $\mathbf{C}[G/K]$ of regular functions on the affine algebraic variety G/K) has simple spectrum;

4) P separates (at least generically) the G-orbits, i.e. $P^{-1}(P(x)) \subset G \cdot x$.

The classification of spherical subgroups of *simple* compact (connected) Lie groups was obtained in [Kr] by M. Kramer (1979), of *semisimple* compact connected Lie groups in our paper [My2](see also [PM]) and by a different method in [Br] by M. Brion. In the case of a real noncompact Lie group G the generalization of these results was obtained by M. Chumak [Ch].

The spherical pairs of Lie groups were studied in many papers and have many beautiful features. But, since the Poisson structure on X is nondegenerate, then every G-invariant Hamiltonian H locally has the form $h \circ P$; i.e., for integrability we don't use the function H.

Let N_{max} be the maximal number of independent real-analytic commuting (with respect to the Poisson bracket) functions on $X = T^*(G/K)$ of type $h \circ P$. If $N_{max} = (\dim X/2) - 1$ we will call the corresponding system of functions an *almost collective completely integrable system* and the subgroup K an *almost spherical subgroup* of G. In this case every Hamiltonian system with a G-invariant Hamiltonian H, in particular, the geodesic flow, is also integrable: for the integrability we can use H or another G-invariant function.

In this paper we enumerate all such homogeneous spaces G/K with a simple compact Lie group G. Crucial ingredients in our classification are the dimensional criterion

$$\dim K \geq \frac{1}{2}(\dim G - \operatorname{rank} G) - 1,$$

Theorem 5, and inequality (9). Roughly speaking this criterion says that K has to be a "large" subgroup of G, Theorem 5 allows us to find only "maximal large" almost spherical subgroups K and to use the already known classification of spherical pairs of semisimple compact Lie groups. We show by inspection that simple groups G do not have many maximal large subgroups and there is only one pair (G, K), where the almost spherical subgroup K is maximal in G. With the exception of this one case, an almost spherical subgroup K of a simple group G is not maximal. Moreover, if $K \subset S \subset G$, where a connected subgroup S is a spherical maximal subgroup of G, then for the isotropy subgroup S^{x_2} of an arbitrary point $x_2 \in G/S$ in some neighborhood of the point $\{S\} \in G/S$ the following inequality holds:

$$\dim(S/K) \leq \frac{1}{2}(\dim S^{x_2} + \operatorname{rank} S^{x_2}) + 1.$$

A connected closed subgroup K of a compact simple group G is said to be of height at most n in G if there exist a chain of distinct connected (closed) subgroups $K \subset S^{(n)} \subset \ldots \subset S' \subset G$ and every chain of distinct connected (closed) subgroups between K and G is of length at most n. It is proved that if K is almost spherical, then K is of height at most n in G, where $n \leq 2$. There are eighteen types of such pairs (G, K), one of which is of height 0, four are of height 2, while the remaining are of height 1. In the latter case $K \subset S \subset G$, where G/S is a symmetric space.

2. Almost spherical pairs and the integrability of invariant flows

2.1. Moment map. Let G be a compact real Lie group, K its closed subgroup and $M = G/K$. The natural action of G on the quotient space M extends to an action of G on T^*M. This G-action on T^*M is symplectic since it preserves the canonical 1-form λ (the form "pdq"), and thus also the symplectic 2-form $d\lambda$. For each vector ξ belonging to the Lie algebra $\mathfrak{g}$ of G the 1-parameter subgroup $\exp t\xi$ induces the Hamiltonian vector field $\hat{\xi}$ on T^*M with the Hamiltonian function $f_\xi = \lambda(\hat{\xi}) : df_\xi = -\hat{\xi}\rfloor d\lambda$. The mapping $\xi \mapsto f_\xi$ of $\mathfrak{g}$ into the algebra $C^\infty(T^*M)$ (with the Poisson bracket) is an equivariant algebra homomorphism: $f(g^{-1}m) = f_{\operatorname{Ad} g(\xi)}(m)$ and hence the action of G on T^*M is Poisson [GS3]. This action defines the moment map $P : T^*M \to \mathfrak{g}^*$ from T^*M to the dual space of the Lie algebra $\mathfrak{g}$ by $P(x)(\xi) = f_\xi(x)$. For arbitrary smooth functions h_1 and h_2 on $\mathfrak{g}^*$ we have $\{h_1 \circ P, h_2 \circ P\} = \{h_1, h_2\} \circ P$, where the Poisson bracket on $\mathfrak{g}^*$ is given by the formula $\{h_1, h_2\}(\beta) = \beta([dh_1(\beta), dh_2(\beta)]), \beta \in \mathfrak{g}^*$.

There exists a faithful representation of $\mathfrak{g}$ such that its associated bilinear form Φ is negative definite on $\mathfrak{g}$. Let $\mathfrak{k}$ be the Lie algebra of the subgroup K and π the natural projection of G onto M. Using Φ we can identify the space $\mathfrak{g}^*$ with $\mathfrak{g}$, the corresponding isomorphism denote by $\psi : \mathfrak{g}^* \to \mathfrak{g}$. Also identify $T^*_{\pi(e)}M$ and $\mathfrak{m} \overset{\text{def}}{=} \{x \in \mathfrak{g} : \Phi(x, \mathfrak{k}) = 0\}$. It is evident that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Under these identifications $P(x) = \Phi(x, \cdot) \in \mathfrak{g}^*$. Let $\mathfrak{g}^x = \{y \in \mathfrak{g} : [x, y] = 0\}$ and $\mathfrak{k}^x = \mathfrak{k} \cap \mathfrak{g}^x$, where $x \in \mathfrak{g}$.

2.2. Integrability. Let us show that the fulfillment of the condition

$$\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^x) = \dim(\mathfrak{g}/\mathfrak{k}) - \varepsilon, \qquad \varepsilon = \varepsilon(\mathfrak{g}, \mathfrak{k}) = 0 \quad \text{or} \quad 1 \tag{1}$$

at any point x from some neighborhood in $\mathfrak{m}$ is sufficient for the integrability of any Hamiltonian flow on T^*M with G-invariant Hamiltonian function.

Let us consider the Zariski open subset $R(\mathfrak{m}) \subset \mathfrak{m}$ of points in general position:

$$R(\mathfrak{m}) = \{x \in \mathfrak{m} : \dim \mathfrak{g}^x \leq \dim \mathfrak{g}^y, \dim \mathfrak{k}^x \leq \dim \mathfrak{k}^y, \forall y \in \mathfrak{m}\}. \tag{2}$$

On $\mathfrak{g}^* \overset{\psi}{=} \mathfrak{g}$ there exists $s = \dim(\mathfrak{g}^x/\mathfrak{k}^x) + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^x)$, $x \in R(\mathfrak{m})$, polynomial functions $h_1, h_2, \ldots, h_s$ such that the functions $h_1 \circ P, h_2 \circ P, \ldots, h_s \circ P$ are pairwise in involution on T^*M and are independent at the point $x \in \mathfrak{m} = T^*_{\pi(e)}M$ (the number s is the maximal number of independent functions in involution of the form $h \circ P$ on T^*M) (see [My2]). Let $W(x)$ be the tangent space to the orbit $G \cdot x \subset T^*M$ (of maximal dimension). Then $W(x)$ is generated by vectors $\hat{\xi}(x)$, $\xi \in \mathfrak{g}$. Applying the slice-theorem to the action of the compact group G on T^*M (or to the $\operatorname{Ad} K$-action on $\mathfrak{m} = T^*_{\pi(e)}M$) one deduces that the orthogonal complement $W(x)^\perp \overset{\text{def}}{=} \{X \in T_xT^*M : d\lambda(X, W(x)) = 0\}$ to $W(x)$ with respect to the symplectic structure $d\lambda$ is generated by Hamiltonian vector fields (at x) of G-invariant functions on T^*M. Let us show that codimension of $W^\perp(x) \cap W(x)$ in $W^\perp(x)$ is equal to 2ε. Indeed, multiplying both sides of (1) by 2, we obtain after simple rearrangements

$$\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \dim(\mathfrak{g}/\mathfrak{k}^x) = 2\dim(\mathfrak{g}/\mathfrak{k}) - 2\varepsilon. \tag{3}$$

It follows from [My1] (see also [GS2], [PM]) that $\hat{\xi}(x) = 0$ iff $\xi \in \mathfrak{k}^x$ and consequently

$\dim W(x) = \dim G \cdot x = \dim(\mathfrak{g}/\mathfrak{k}^x)$. Because of the relations

$$\begin{aligned} d\lambda(\hat{\xi}(x), \hat{\eta}(x))(x) &= \{f_\eta, f_\xi\}(x) = f_{[\eta,\xi]}(x) = P(x)([\eta, \xi]) \\ &= \Phi(x, [\eta, \xi]) = \Phi([x, \eta], \xi) \quad \text{for any } \eta, \xi \in \mathfrak{g} \end{aligned}$$

we have that $W(x) \cap W^\perp(x) = \{\hat{\eta}(x), \eta \in \mathfrak{g}^x\}$ and $\dim W(x) \cap W^\perp(x) = \dim(\mathfrak{g}^x/\mathfrak{k}^x)$. Taking into account that the 2-form $d\lambda$ is nondegenerate we obtain

$$\dim W(x) + \dim W^\perp(x) = \dim T^*M = 2\dim(\mathfrak{g}/\mathfrak{k}).$$

Thus $\dim W(x)^\perp - \dim(W(x) \cap W^\perp(x)) = 2\varepsilon$ and therefore if $\varepsilon = 1$ there is a G-invariant function F on T^*M which is independent of functions $\{h_i \circ P\}, i = \overline{1, s}$ (the Hamiltonian vector fields of $h_i \circ P$ are tangent to orbits of G in T^*M). The set $\{F, h_1 \circ P, \dots, h_s \circ P\}$ is the maximal involutive set of independent functions: $s + 1 = \frac{1}{2}\dim T^*M$. If the G-invariant function H is independent of $\{h_i \circ P\}, i = \overline{1, s}$ then the set $\{H, h_1 \circ P, \dots, h_s \circ P\}$ is a maximal involutive set; if H is dependent then we have the commutative set of integrals $\{F, h_1 \circ P, \dots, h_s \circ P\}$. If $\varepsilon = 0$, then $s = \frac{1}{2}\dim T^*M$; i.e., on the manifold T^*M there exists a collective completely integrable system and also any G-invariant flow is integrable. We proved

Proposition 1. *If condition (1) holds for all x from open subset of $\mathfrak{m}$ then any Hamiltonian system on T^*M with a G-invariant Hamiltonian function H is integrable.*

2.3. Properties of spherical pairs of compact Lie algebras. Since $\dim \mathfrak{g} = \dim \mathfrak{k} + \dim \mathfrak{m}$ equation (1) (the definition of $\varepsilon(\mathfrak{g}, \mathfrak{k})$) is equivalent to

$$\dim(\mathfrak{g}^x/\mathfrak{k}^x) + \dim(\mathfrak{k}/\mathfrak{k}^x) = \dim \mathfrak{m} - 2\varepsilon. \tag{4}$$

By [My2, Prop. 1.1] (see also [Mi, GS2]) for any $x \in R(\mathfrak{m})$ the commutator $[\mathfrak{g}^x, \mathfrak{g}^x]$ is contained in the algebra $\mathfrak{k}^x = \mathfrak{k} \cap \mathfrak{g}^x$. Therefore for a semisimple element $x \in R(\mathfrak{m})$ we have $\dim(\mathfrak{g}^x/\mathfrak{k}^x) = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}^x$, and so (4) can be rewritten as

$$(\operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}^x) + \dim(\mathfrak{k}/\mathfrak{k}^x) = \dim \mathfrak{m} - 2\varepsilon. \tag{5}$$

Moreover, it is evident that $\dim(\mathfrak{g}^x/\mathfrak{k}^x) \le \operatorname{rank} \mathfrak{g} \le \dim \mathfrak{g}^x$ and consequently (1) implies

$$\dim \mathfrak{k} \ge \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}) - \varepsilon. \tag{6}$$

For any $x \in \mathfrak{m}$ put

$$\mathfrak{m}(x) \stackrel{\text{def}}{=} \{z \in \mathfrak{m} : [x, z] \in \mathfrak{m}\} = \{z \in \mathfrak{m} : \Phi(z, \operatorname{ad} x(\mathfrak{k})) = 0\}. \tag{7}$$

Lemma 2. *For a pair $(\mathfrak{g}, \mathfrak{k})$ of compact Lie algebras and any point $x \in \mathfrak{m}$ the following conditions are equivalent:*

(1) $\{\operatorname{codim} \operatorname{ad} x(\mathfrak{k})$ *in* $\mathfrak{m}\} = \dim(\mathfrak{g}^x/\mathfrak{k}^x) + 2\varepsilon$*; i.e.,* $\dim \mathfrak{m}(x) = \dim(\mathfrak{g}^x/\mathfrak{k}^x) + 2\varepsilon$*;*

(2) $\{\operatorname{codim}(\mathfrak{g}^x)_\mathfrak{m}$ *in* $\mathfrak{m}(x)\} = 2\varepsilon$*, where* $(\cdot)_\mathfrak{m}$ *is the projection onto* $\mathfrak{m}$ *along* $\mathfrak{k}$ *induced by the decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$*;*

(3) $\{\operatorname{codim}[\operatorname{ad} x(\mathfrak{m}(x)) \cap \operatorname{ad} x(\mathfrak{k})]$ *in* $\operatorname{ad} x(\mathfrak{m}(x))\} = 2\varepsilon$.

Proof. It is sufficient to see that 1) $(\mathfrak{g}^x)_\mathfrak{m} \subset \mathfrak{m}(x)$; 2) $\operatorname{ad} x(\mathfrak{k}) \subset \mathfrak{m}$; 3) $\mathfrak{m}(x) \oplus \operatorname{ad} x(\mathfrak{k}) = \mathfrak{m}$ and if $x' \in \mathfrak{m}$ then $\operatorname{ad} x(x') \in \operatorname{ad} x(\mathfrak{k}) \Leftrightarrow x' \in (\mathfrak{g}^x)_\mathfrak{m}$. ∎

DEFINITION 3. We say that a pair $(\mathfrak{g}, \mathfrak{k})$ of compact Lie algebras is an *almost spherical pair* (resp. *spherical pair*) and a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is an *almost spherical subalgebra* (resp. *spherical subalgebra*) if for any $x \in R(\mathfrak{m})$ the equivalent conditions of Lemma 2 or the equivalent equalities (1), (3)–(5) are satisfied, where $\varepsilon = 1$ (resp., where $\varepsilon = 0$).

REMARK 4. For a spherical pair $(\mathfrak{g}, \mathfrak{k})$ of compact Lie algebras the conditions (2), (3) of Lemma 2, $\varepsilon = 0$ are equivalent to the conditions (2') $\mathfrak{m}(x) = (\mathfrak{g}^x)_\mathfrak{m}$ and (3') $(\operatorname{ad} x)(\mathfrak{m}(x)) \subset \operatorname{ad} x(\mathfrak{k})$, $x \in R(\mathfrak{m})$ [My2]. Thus all *symmetric pairs* $(\mathfrak{g}, \mathfrak{k})$, i.e. those for which $\mathfrak{k}$ is the algebra of fixed points of an involutive automorphism of the algebra $\mathfrak{g}$, are spherical (see [Mi], [My2]). Indeed, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and consequently $\mathfrak{m}(x) = \mathfrak{g}^x \cap \mathfrak{m}$. All spherical subalgebras $\mathfrak{k}$ of compact Lie algebras $\mathfrak{g}$ are classified in [Kr] (for simple $\mathfrak{g}$) and in [My2], [Br] (the semisimple case).

THEOREM 5. *Let $\mathfrak{g}$ be a compact Lie algebra with subalgebras $\mathfrak{k} \subset \mathfrak{s}$. Let $(\mathfrak{g}, \mathfrak{k})$ be an almost spherical pair. Then the pairs $(\mathfrak{g}, \mathfrak{s})$ and $(\mathfrak{s}, \mathfrak{k})$ are either almost spherical or spherical. Moreover, if $(\mathfrak{g}, \mathfrak{s})$ is almost spherical, then the pair $(\mathfrak{s}, \mathfrak{k})$ is spherical.*

PROOF. Let $\mathfrak{m}_1$ (respectively $\mathfrak{m}_2$) be the orthogonal complement to the subalgebra $\mathfrak{k}$ in $\mathfrak{s}$ (respectively to the subalgebra $\mathfrak{s}$ in $\mathfrak{g}$) with respect to the form Φ; i.e., $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Fix an element $x_1 + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ such that $x_1 \in R(\mathfrak{m}_1)$ and $x_2 \in R(\mathfrak{m}_2)$. Let V_1 (respectively V_2) be the orthogonal complement to the subspace $(\mathfrak{s}^{x_1})_{\mathfrak{m}_1}$ in $\mathfrak{m}_1(x_1)$ (respectively to the subspace $(\mathfrak{g}^{x_2})_{\mathfrak{m}_2}$ in $\mathfrak{m}_2(x_2)$). From the relation $(\mathfrak{s}^{x_1})_{\mathfrak{m}_1} = \{z \in \mathfrak{m}_1 : \operatorname{ad} x_1(z) \in \operatorname{ad} x_1(\mathfrak{k})\}$ it may be concluded that

$$\operatorname{ad} x_1(V_1) \cap \operatorname{ad} x_1(\mathfrak{k}) = 0, \qquad \dim \operatorname{ad} x_1(V_1) = \dim V_1. \tag{8}$$

If $u_1 \in \mathfrak{m}_1(x_1)$ then by definition $[x_1, u_1] \in \mathfrak{m}_1$, hence $[x_1 + x_2, u_1] \in \mathfrak{m}$ and, consequently, $\mathfrak{m}_1(x_1) \subset \mathfrak{m}(x_1 + x_2)$. Using the similar arguments we obtain that $\mathfrak{m}_2(x_2) \subset \mathfrak{m}(x_1 + x_2)$.

The pair $(\mathfrak{s}, \mathfrak{k})$ is either spherical or almost spherical iff $\dim V_1 \leq 2$. Otherwise the space V_1 is at least four-dimensional. From the relations $V_1 \subset \mathfrak{m}_1(x_1) \subset \mathfrak{m}(x_1 + x_2)$ for the space V_1 of the dimension ≥ 4 and condition (2) of Lemma 2: $\dim \mathfrak{m}(x_1 + x_2) - \dim(\mathfrak{g}^{x_1+x_2})_\mathfrak{m} = 2$, it follows that for some $v_1 \in V_1, v_1 \neq 0 : v_1 \in (\mathfrak{g}^{x_1+x_2})_\mathfrak{m}$. Thus $[x_1 + x_2, v_1] \in \operatorname{ad}(x_1 + x_2)(\mathfrak{k})$. Therefore for some $z \in \mathfrak{k}$: $[x_1, v_1] + [x_2, v_1] = [x_1, z] + [x_2, z]$ and consequently $[x_1, v_1] = [x_1, z] \in \mathfrak{m}_1$. This contradicts the first relation in (8) (by the second relation $[x_1, v_1] \neq 0$) so that $\dim V_1 \leq 2$.

To obtain the contradiction, suppose that the pair $(\mathfrak{g}, \mathfrak{s})$ is neither almost spherical nor spherical. In this case the space $V_2 = V_2(x_2) \subset \mathfrak{m}(x_2)$ is at least four-dimensional. But the pair $(\mathfrak{g}, \mathfrak{k})$ is almost spherical and $V_2 \subset \mathfrak{m}_2(x_2) \subset \mathfrak{m}(x_1 + x_2)$ so that the intersection $V_2 \cap (\mathfrak{g}^{x_1+x_2})_\mathfrak{m}$ is at least two-dimensional. Since for $x \in R(\mathfrak{m})$ dimension $\dim(\mathfrak{g}^x)_\mathfrak{m} = \dim(\mathfrak{g}^x/\mathfrak{k}^x)$ is constant, we can define the Zariski open subset $Q_1 = Q_1(x_2)$ of $\mathfrak{m}_1$ of all elements x_1' such that (1) $x_1' \in R(\mathfrak{m}_1)$ and $x_1' + x_2 \in R(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$; (2) the space $V_2 \cap (\mathfrak{g}^{x_1'+x_2})_\mathfrak{m} \stackrel{\text{def}}{=} V_2(x_1')$ has the minimal possible dimension l (from what has already been showed $l \geq 2$). Since the set Q_1 is not empty and $0 \in \overline{Q_1}$, the space $V_2 \cap (\mathfrak{g}^{x_2})_\mathfrak{m}$ is at least l-dimensional (the Grassmann manifold of all l planes in V_2 is compact, $[\mathfrak{k}, \mathfrak{m}_1] \subset \mathfrak{m}_1$). This contradicts the definition of V_2 because $(\mathfrak{g}^{x_2})_\mathfrak{m} \cap \mathfrak{m}_2 \subset (\mathfrak{g}^{x_2})_{\mathfrak{m}_2}$ so that $\dim V_2 \leq 2$.

It remains to prove that if $\dim V_2 = 2$ then $V_1 = 0$. Otherwise, assume that $\dim V_1 = 2$. Since $V_1 \oplus V_2 \subset \mathfrak{m}_1(x_1) \oplus \mathfrak{m}_2(x_2) \subset \mathfrak{m}(x_1 + x_2)$, by condition (2) of Lemma 2, the space $V' = (V_1 \oplus V_2) \cap (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}}$ has dimension ≥ 2. Therefore, for any non-zero $v_1 + v_2 \in V' \subset V_1 \oplus V_2$ there exists an element $z \in \mathfrak{k}$ such that $[x_1 + x_2, v_1 + v_2] = [x_1 + x_2, z]$. But $[\mathfrak{k}, \mathfrak{m}_i] \subset \mathfrak{m}_i, i = 1, 2$ so that $[x_1, v_1] = [x_1, z]$ which contradicts (8) if $v_1 \neq 0$. Thus $V' = V_2$, $(\dim V' \geq 2)$ and $V_2 \subset (\mathfrak{g}^{x_1+x_2})_{\mathfrak{m}}$; i.e., for any non-zero $v_2 \in V_2$ we have $[x_1 + x_2, v_2] \in \operatorname{ad}(x_1 + x_2)(\mathfrak{k})$ and consequently $[x_1 + x_2, v_2] \in \operatorname{ad} x_2(\mathfrak{k})$. The latter holds for any $x_1' \in \mathfrak{m}$ from some neighborhood of x_1, hence $[x_2, v_2] \in \operatorname{ad} x_2(\mathfrak{k}) \subset \operatorname{ad} x_2(\mathfrak{s})$. Therefore $v_2 \in (\mathfrak{g}^{x_2})_{\mathfrak{m}_2}$, which is impossible, a contradiction. ∎

In the notation of the proof of the theorem if a pair $(\mathfrak{g}, \mathfrak{k})$ is almost spherical and $(\mathfrak{g}, \mathfrak{s})$ is spherical or almost spherical then

$$(\operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}^{x_1+x_2}) + \dim(\mathfrak{k}/\mathfrak{k}^{x_1+x_2}) = \dim(\mathfrak{g}/\mathfrak{k}) - 2$$

and

$$(\operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{s}^{s_2}) + \dim(\mathfrak{s}/\mathfrak{s}^{x_2}) = \dim(\mathfrak{g}/\mathfrak{s}) - 2\varepsilon_2, \text{ where } \varepsilon_2 = 0, 1.$$

Hence $2\dim(\mathfrak{s}/\mathfrak{k}) = (\dim \mathfrak{s}^{x_2} + \operatorname{rank} \mathfrak{s}^{x_2}) - (\dim \mathfrak{k}^{x_1+x_2} + \operatorname{rank} \mathfrak{k}^{x_1+x_2}) + (2 - 2\varepsilon_2)$ and

$$\dim(\mathfrak{s}/\mathfrak{k}) \leq \frac{1}{2}(\dim \mathfrak{s}^{x_2} + \operatorname{rank} \mathfrak{s}^{x_2}) + 1 - \varepsilon_2. \tag{9}$$

REMARK 6. Let $\mathfrak{g}$ be a semisimple compact Lie algebra and let $\mathfrak{a} \oplus \mathfrak{z}$ be its subalgebra with one-dimensional center $\mathfrak{z}$. If the pair $(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{z})$ is spherical then the pair $(\mathfrak{g}, \mathfrak{a})$ is either spherical or almost spherical. To prove this fact it is sufficient to see that $\mathfrak{m} \stackrel{\text{def}}{=} (\mathfrak{a} \oplus \mathfrak{z})^\perp \subset \mathfrak{g}$ is the subspace of $\mathfrak{a}^\perp = \mathfrak{m} \oplus \mathfrak{z}$ and $R(\mathfrak{m}) \in R(\mathfrak{m} \oplus \mathfrak{z})$ [My2, Prop.2.2].

3. Almost spherical subalgebras of simple Lie algebras

3.1. Preliminary remarks. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a compact real form $\mathfrak{g}_0$ and let Φ be the Killing form of $\mathfrak{g}$. Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$ such that $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ is the real form of $\mathfrak{k}$. Write $\mathfrak{m}$ for the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to Φ. It is evident that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{m} = \mathfrak{m}_0 \oplus i\mathfrak{m}_0$, where $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0$. Using (2) define the Zariski open subset $R'(\mathfrak{m})$ (over $\mathbf{C}$) of $\mathfrak{m}$. Let $R(\mathfrak{m})$ denote a set of all $x \in R'(\mathfrak{m})$ which are semisimple elements of Lie algebra $\mathfrak{g}$. Then $R(\mathfrak{m})$ is a Zariski open subset of $\mathfrak{m}$ (see [My2]). It is clear that $R(\mathfrak{m}_0) \subset R(\mathfrak{m})$. We say that a pair $(\mathfrak{g}, \mathfrak{k})$ is *almost spherical* (resp. *spherical*) if the pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ of compact Lie algebras is almost spherical (resp. spherical); i.e., if for any $x \in R(\mathfrak{m})$ the equivalent conditions like (1), (3)–(5) hold. To verify these conditions we use the results of [El], where for all simple complex Lie algebras $\mathfrak{k}$ all their representations $\pi_{\mathfrak{k}}$ and types of corresponding isotropic subalgebras $\mathfrak{k}^x$ (of elements in general position) if $\mathfrak{k}^x \neq 0$ are enumerated.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Write $R(\Lambda)$ for the irreducible representation of $\mathfrak{g}$ with highest weight Λ, and $R'(\Lambda)$ for its contragredient representation. Let η denote the one-dimensional trivial representation. If $\{\alpha_i\}$ is a basis for the root system of $\mathfrak{g}$ relative $\mathfrak{h}$, and $\{\varphi_i\}$ are the corresponding fundamental weights, then $\Lambda = \sum \Lambda_i \varphi_i$, where $\Lambda_i \in \mathbf{Z}^+$. We shall index the roots of a basis for the root system of a simple Lie algebra in the order given in [Bo2], Tables I-IX of Chapter VI.

Let ρ be a faithful linear representation of a simple complex Lie algebra $\mathfrak{g}$ in a space of the smallest possible dimension. We associate the embedding j_0 of a subalgebra $\mathfrak{k}_0$ in $\mathfrak{g}_0$ and the embedding j of the natural complex extension $\mathfrak{k} = \mathfrak{k}_0^{\mathbf{C}}$ in $\mathfrak{g}$ with the linear (complex) representation $\tilde{\rho}$ obtained by restricting $\rho \circ j$ to its semisimple part. We denote such a subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}_0 (\mathfrak{k} \subset \mathfrak{g})$ by a pair $(\cdot, \tilde{\rho})$, where for the first entry we put the type of an algebra $\mathfrak{k}_0$ or $\mathfrak{k}$.

3.2. Almost spherical maximal subalgebras of classical Lie algebras. Let V be a linear space of dimension n over $\mathbf{C}$. For the rest of this subsection $\mathfrak{g}$ will denote a classical complex Lie algebra, i.e. one of $sl(n)$, $so(n)$, or $sp(n)$ (for n even), with $\mathfrak{k}$ a reductive subalgebra. Since $\mathfrak{k}$ is reductive, V is a semisimple $\mathfrak{k}$-module. If V is a simple $\mathfrak{k}$-module we shall say that $\mathfrak{k}$ is *irreducible*.

PROPOSITION 7. *Suppose that $\mathfrak{k}$ is an almost spherical subalgebra of a simple classical Lie algebra $\mathfrak{g}$ and $\mathfrak{k}$ is maximal in $\mathfrak{g}$. Then $\mathfrak{g} \simeq B_2$ and $\mathfrak{k}$ is a unique (up to inner automorphisms) principal sl_2-subalgebra of $\mathfrak{g}$; i.e., $(\mathfrak{g}, \mathfrak{k}) \simeq (sp(4), (A_1, R(3\varphi))) \simeq (so(5), (A_1, 4\varphi))$. Moreover, any almost spherical subalgebra $\mathfrak{k}_1$ of $so(5)$ ($sp(4)$) such that $\mathfrak{k}_1 \subset \mathfrak{k}$ coincides with $\mathfrak{k}$.*

PROOF. Since the subalgebra $\mathfrak{k}$ is reductive, the $\mathfrak{k}$-module V is semisimple. Suppose that V is a nonsimple $\mathfrak{k}$-module; i.e., $V = V_1 \oplus V_2$ is a direct sum of two nonzero semisimple $\mathfrak{k}$-modules V_1 and V_2. But the subalgebra $\mathfrak{s} = \mathfrak{g}(V_1, V_2) = \{x \in \mathfrak{g} : x(V_1) \subset V_1, x(V_2) \subset V_2\}$ (which contains $\mathfrak{k}$) of the classical Lie algebra $\mathfrak{g}$ is maximal because the pair $(\mathfrak{g}, \mathfrak{s})$ is symmetric [He]. Therefore $\mathfrak{k} \neq \mathfrak{s}$ and V is a simple $\mathfrak{k}$-module.

A) Suppose that $\mathfrak{k}$ is a simple irreducible subalgebra of $\mathfrak{g}$ (by E.Cartan theorem any irreducible subalgebra of $\mathfrak{g}$ is semisimple). If the pair $(\mathfrak{g}, \mathfrak{k})$ is almost spherical then inequality (6) is satisfied: $\dim \mathfrak{k} \geq M(\mathfrak{g}) - 1$, where $M(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$. Therefore $\dim \mathfrak{g} \geq \frac{1}{4} n(n-2) - 1$, since $\frac{1}{4} n(n-2) \leq M(so(n)) \leq M(sp(n)) \leq M(sl(n))$. When $\mathfrak{k}$ has type A_r $(r \geq 1)$, B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 4)$, E_6, E_7, E_8, F_4, or G_2, it is not hard to verify that for the latter inequality to be satisfied it is necessary for n not to exceed $2r+3$, $4r$, $4r$, $4r-2$, 19, 25, 33, 17, or 9, respectively. The irreducible representations ρ of simple algebras $\mathfrak{a}$ whose dimensions satisfy these restrictions can be found in [On, Lemma 3.2] with the exception of one case when $\mathfrak{a} \simeq A_r$ and $n = 2r+3$. For an algebra $\mathfrak{a}$ of type A_r the representations are the following: $(r \geq 2, \varphi_1(\varphi_r), r+1, 0)$, $(4, \varphi_2(\varphi_3), 10, 0)$, $(3, \varphi_2, 6, 1)$, $(2, 2\varphi_1(2\varphi_2), 6, 0)$, $(1, \varphi_1, 2, -1)$, $(1, 2\varphi_1, 3, 1)$, $(1, 3\varphi_1, 4, -1)$, where in the quadruple $(r, \Lambda, n(\Lambda), \varepsilon(\Lambda))$ we mean that r is the rank of $\mathfrak{k}$, Λ and $n(\Lambda)$ are the highest weight and dimension of ρ respectively; the symbol $\varepsilon(\Lambda)$ is 1, -1, or 0 according as ρ is orthogonal, symplectic, or neither orthogonal nor symplectic, respectively. For a simple algebra $\mathfrak{a}$ of type B_r we have the following representations: $(r, \varphi_1, 2r+1, 1)$, $(4, \varphi_4, 16, 1)$, $(3, \varphi_3, 8, 1)$, $(2, \varphi_2, 4, -1)$; type C_r: $(r, \varphi_1, 2r, -1)$; type D_r: $(r, \varphi_1, 2r, 1)$, $(5, \varphi_4(\varphi_5), 16, 0)$, $(4, \varphi_3(\varphi_4), 8, 1)$; type G_2: $(2, \varphi_1, 7, 1)$; algebras of the remaining types have no such representations. Clearly $\rho(\mathfrak{a}) \subset so(n)$ $(\rho(\mathfrak{a}) \subset sp(n))$ if the representation ρ is orthogonal (symplectic). It can be verified that inequality (6) is satisfied only for the following pairs $(\mathfrak{g}, \mathfrak{a})$ from among those found above: (a) $(sl(n), so(n))$, $n \geq 3$, $n \neq 4$; (b) $(sl(n), sp(n))$, $n \geq 4$; (c) $(so(8), (B_3, R(\varphi_3)))$ (spinor representation); (d) $(so(7), (G_2, R(\varphi_1)))$. All these pairs (a)–(d) are spherical [Kr, My2]. Now assume that $n = 2r+3$ for $\mathfrak{k} \simeq A_r$. Since in this case $\dim \mathfrak{k} < M(sl(2r+3)) - 1$ for all $r \geq 1$ the representation ρ of $\mathfrak{k}$ have to be either orthogonal or symplectic. If $r = 1$ then for the pair $(so(5), (A_1, R(4\varphi)))$ condition (6) is an equality. If $r \geq 2$ using the explicit formula for dimension of the representation $R(\Lambda)$ and properties of the root system A_r (all roots have the same length) we obtain that a representation of $\mathfrak{k}$ which admits an invariant bilinear form and has minimal dimension is the representation $(r \geq 2, \varphi_1 + \varphi_r, r(r+2), 1)$, $(3, \varphi_2, 6, 1)$ or $(r = 2k+1 \geq 5, \varphi_{k+1}, N_k \geq (k+2)(k+3), (-1)^{k+1})$, where $N_k = \frac{(2k+2)!}{(k+1)!(k+1)!}$. Thus there are no such representations of Lie algebra A_r of the dimension $2r+3 (r \geq 2)$.

B) Suppose that $\mathfrak{k}$ is an irreducible subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$ is not simple (is semisimple). Then $\mathfrak{k} \in \{\mathfrak{a}\}$, where $\{\mathfrak{a}\}$ is a set of all maximal semisimple (not simple) subalgebras of $\mathfrak{g}$. The maximal subalgebra $\mathfrak{a}$ is isomorphic to the tensor product $sl(s) \otimes sl(t)$ $(st = n, 2 \leq s \leq t)$ if $\mathfrak{g} = sl(n)$; $sp(s) \otimes sp(t)$ $(st = n, 2 \leq s \leq t)$ or $so(s) \otimes so(t)$ $(st = n, 3 \leq s \leq t; s, t \neq 4)$ if $\mathfrak{g} = so(n)$; $sp(s) \otimes so(t)$ $(st = n, s \geq 2, t \geq 3, t \neq 4$ or $s = 2$, $t = 4)$ if $\mathfrak{g} = sp(n)$ ([Dy1, Theorems 1.3 and 1.4]). Inequality (6) holds only for two pairs $(\mathfrak{g}, \mathfrak{a})$: $(sl(4), sl(2) \otimes sl(2))$ and $(so(8), sp(2) \otimes sp(4))$. These two pairs are spherical [Kr, My2].

Now it remains to prove that the pair $(\mathfrak{g}, \mathfrak{k}) = (so(5), (A_1, R(4\varphi)))$ is almost spherical. To compute the representation $\pi_{\mathfrak{k}}$ of the Lie algebra $\mathfrak{k}$ in $\mathfrak{m}$ consider the sl_2-triple $\{X_+, H, X_-\}$ in $\mathfrak{k} \simeq A_1$ [Bo3, Chapt.VIII,§1]. Then the eigenvalues of $H \in so(5) \subset sl(5)$ are the numbers $4, 2, 0, -2, -4$. Using the standard root system of $\mathfrak{g}$ with respect to the Cartan subalgebra (of diagonal matrices) $\mathfrak{h} \ni H$ of $\mathfrak{g}$ we obtain that $\alpha_i(H) = 2$ for every simple root $\alpha_i, i = 1, 2$ so that $\mathfrak{k}$ is principal sl_2-subalgebra of $\mathfrak{g}$ [Bo3, Chapt.VIII,§1,11] and $\pi_{\mathfrak{k}} = R(6\varphi)$. Thus $\mathfrak{k}^x = 0$ for any $x \in R(\mathfrak{m})$ [El] and consequently the pair $\mathfrak{g}, \mathfrak{k}$ is almost spherical. Since in this case (6) is equality $\mathfrak{k}$ does not contain proper almost spherical subalgebra of $\mathfrak{g}$. To prove that $(so(5), (A_1, R(4\varphi))) \simeq (sp(4), (A_1, R(3\varphi)))$ it is sufficient to make the following observation: $(A_1, R(3\varphi))) \subset sp(4)$ is principal sl_2-subalgebra of $sp(4)$ (the eigenvalues of $H \in sp(4) \subset sl(4)$ are $3, 1, -1, -3$) and all principal sl_2-subalgebras are conjugate. ∎

3.3. Almost spherical maximal subalgebras of exceptional Lie algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is *regular* if $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We say that $\mathfrak{a}$ is an S-subalgebra of $\mathfrak{g}$ if it is not contained in any proper regular subalgebra of $\mathfrak{g}$ [Dy2].

Let $\mathfrak{g}$ be an exceptional complex Lie algebra, $\mathfrak{s}$ its maximal reductive subalgebra. Let us find such subalgebras $\mathfrak{s}$ for which condition (6) holds: $\dim \mathfrak{s} \geq M(\mathfrak{g}) - 1 = (\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})/2 - 1$.

A maximal reductive subalgebra $\mathfrak{s}$ of a simple Lie algebra $\mathfrak{g}$ is either regular or an S-subalgebra. The list of the types of all maximal S-subalgebras $\mathfrak{s}$ of the exceptional algebras $\mathfrak{g}$ is as follows: $G_2 - \{A_1\}$; $F_4 - \{A_1, A_1 \oplus G_2\}$; $E_6 - \{A_1, G_2, A_2 \oplus G_2, F_4, C_4\}$; $E_7 - \{A_1, A_1 \oplus A_1, A_2, G_2 \oplus C_3, A_1 \oplus F_4, A_1 \oplus G_2\}$; $E_8 - \{A_1, A_1 \oplus A_2, B_2, G_2 \oplus F_4\}$ [Dy2]. It is not hard to verify that necessary condition (6) for $(\mathfrak{g}, \mathfrak{s})$ is satisfied only for the pairs of types (E_6, F_4) and (E_6, C_4) which are symmetric [GG] so we can proceed to the case when $\mathfrak{s}$ is regular.

The list of the types of all maximal regular subalgebras $\mathfrak{s}$ of the exceptional algebras $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{s})$ is not symmetric is as follows: $G_2 - \{A_2\}$; $F_4 - \{A_2 \oplus A_2\}$; $E_6 - \{A_2 \oplus A_2 \oplus A_2\}$; $E_7 - \{A_2 \oplus A_5\}$; $E_8 - \{A_8, A_4 \oplus A_4, A_2 \oplus E_6\}$ [Dy2]. Condition (6) holds only for the pair of type $\{G_2, A_2\}$ which is spherical. Thus we proved

PROPOSITION 8. *There is no almost spherical subalgebra $\mathfrak{k}$ of a simple exceptional Lie algebra $\mathfrak{g}$ such that $\mathfrak{k}$ is maximal in $\mathfrak{g}$.*

3.4. Almost spherical subalgebras of simple Lie algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra. A reductive subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is said to be of height at most n in $\mathfrak{g}$ if there exists a chain of distinct reductive subalgebras $\mathfrak{k} \subset \mathfrak{s}^{(n)} \subset \ldots \subset \mathfrak{s}' \subset \mathfrak{g}$ and every chain of distinct reductive subalgebras between $\mathfrak{k}$ and $\mathfrak{g}$ is of length at most n. From Propositions 7 and 8 it follows that 1) there is only one almost spherical pair $(\mathfrak{g}, \mathfrak{k})$ for which $\mathfrak{k}$ is of height at most 0 in $\mathfrak{g}$; 2) if an almost spherical subalgebra $\mathfrak{k}$ is of height at most $h \geq 1$ in $\mathfrak{g}$ then $\mathfrak{k} \subset \mathfrak{s} \subset \mathfrak{g}$, where by Theorem 5 the subalgebra $\mathfrak{s}$ is maximal spherical in $\mathfrak{g}$ and $\mathfrak{k}$ is either almost spherical or spherical (maximal if $h = 1$) subalgebra of $\mathfrak{s}$. But we already know all spherical subalgebras $\mathfrak{s}$ of simple Lie algebras $\mathfrak{g}$. If $\mathfrak{s}$ is maximal then the pair $(\mathfrak{g}, \mathfrak{s})$ is symmetric or is as in the following list: $(so(8), (B_3, R(\varphi_3)))$, $(so(8), sp(2) \otimes sp(4))$, $(so(7), (G_2, R(\varphi_1)))$, (G_2, A_2) [Kr, My2]. Now applying the dimensional criterion (9) with $\varepsilon_2 = 0$ to such pairs $(\mathfrak{g}, \mathfrak{s})$ (types of the isotropic subalgebras $\mathfrak{s}^{s_2}$ are enumerated in [Ar] for symmetric pairs and for remaining spherical pairs in [My2]) we find the set of such pairs $(\mathfrak{g}, \mathfrak{k})$ which contains all almost spherical pairs. It remains to establish that (5)

Table 1

N	$\mathfrak{g}$	$\mathfrak{k}$	j
1	$A_r, r \geq 2$	$A_{r-2} \oplus 2\mathbf{C}$	$R(\varphi_1) \dot{+} 2\eta$
2*	$A_r, r \geq 4$	$A_{r-2} \oplus \mathbf{C}$	$R(\varphi_1) \dot{+} 2\eta$
3	$A_{2r-1}, r \geq 1$	$A_{r-1} \oplus A_{r-1}$	$R(\varphi_1) \otimes \eta \dot{+} \eta \otimes R(\varphi_1)$
4	$B_r, r \geq 2$	A_{r-1}	$R(\varphi_1) \dot{+} R'(\varphi_1) \dot{+} \eta$
5	$B_r, r \geq 2$	B_{r-1}	$R(\varphi_1) \dot{+} 2\eta$
6	$C_r, r \geq 3$	A_{r-1}	$R(\varphi_1) \dot{+} R'(\varphi_1)$
7	$C_r, r \geq 3$	C_{r-1}	$R(\varphi_1) \dot{+} 2\eta$
8	$C_r, r \geq 3$	$C_{r-2} \oplus A_1 \oplus A_1$	$R(\varphi_1) \otimes \eta \otimes \eta \dot{+} \eta \otimes R(\varphi_1) \otimes \eta \dot{+} \eta \otimes \eta \otimes R(\varphi_1)$
9	$D_{2r}, r \geq 2$	A_{2r-1}	$R(\varphi_1) \dot{+} R'(\varphi_1)$
10	$D_r, r \geq 4$	D_{r-1}	$R(\varphi_1) \dot{+} 2\eta$
11	A_5	$C_2 \oplus C_1 \oplus \mathbf{C}$	$R(\varphi_1) \otimes \eta \dot{+} \eta \otimes R(\varphi_1)$
12	B_5	$B_3 \oplus A_1$	$R(\varphi_3) \otimes \eta \dot{+} \eta \otimes R(2\varphi_1)$
13	B_4	$G_2 \oplus \mathbf{C}$	$R(\varphi_1) \dot{+} 2\eta$
14	B_2	A_1	$R(4\varphi_1)$
15	D_5	B_3	$R(\varphi_3) \dot{+} 2\eta$
16	F_4	D_4	$D_4 \subset B_4 \subset F_4$
17	E_6	$B_4 \oplus \mathbf{C}$	$B_4 \oplus \mathbf{C} \subset D_5 \oplus \mathbf{C} \subset E_6$
18	E_7	E_6	$E_6 \subset E_6 \oplus \mathbf{C} \subset E_7$

holds (or does not hold) at points $x \in R(\mathfrak{m})$ for all these pairs. For this it suffices to find the type of the centralizer $\mathfrak{k}^x$, which is completely determined by the representation $\pi : x \mapsto \mathrm{ad}_{\mathfrak{m}}\, x$ of $\mathfrak{k}$ in $\mathfrak{m}$. An easy computation shows that often $\mathfrak{k}^x \subset \mathfrak{k}_1$, where $\mathfrak{k}_1$ is some simple ideal of $\mathfrak{k}$, and, consequently, the algebra $\mathfrak{k}^x$ is determined by the restriction $\pi_{\mathfrak{k}_1}$ and its type is given in the tables in [El]. Thus using Theorem 5, Propositions 7,8 and dimensional criterion (9) we obtain

THEOREM 9. *Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{k}$ its reductive subalgebra. All almost spherical pairs $(\mathfrak{g}, \mathfrak{k})$ are shown in Table 1, where the representations determining the embedding $j : \mathfrak{k} \to \mathfrak{g}$ are also given* [1]. *The almost spherical subalgebra $\mathfrak{k}$ of the pair $(\mathfrak{g}, \mathfrak{k})$ 14 is of height 0 (in $\mathfrak{g}$), four subalgebras $\mathfrak{k}$ of pairs 2,4,7,15 are of height 2, while the remaining are of height 1.*

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[1] In the case 2^* the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$ is the Lie algebra $\mathfrak{a}_1 \oplus \mathfrak{z}_1 \simeq A_1 \oplus \mathbf{C}$ and the 1-dimensional center of $\mathfrak{k}$ is a diagonal subalgebra of $\mathfrak{h}_1 \oplus \mathfrak{z}_1$, where $\mathfrak{h}_1$ is the Cartan subalgebra of $\mathfrak{a}_1$.

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