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CONTACT HAMILTONIANS DISTINGUISHING LOCALLY CERTAIN GOURSAT SYSTEMS

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Abstract. For the first time in dimension 9, the Goursat distributions are not locally smoothly classified by their small growth vector at a point. As shown in [M1], in dimension 9 of the underlying manifold 93 different local behaviours are possible and four irregular pairs of them have coinciding small growth vectors. In the present paper we distinguish *geometrically* objects in three of those pairs. Smooth functions in three variables – *contact hamiltonians* in the terminology of Arnold, [A] – help to do that. One pair of models, however, resists this technique. Another example of similar resistance in dimension 10 is also given – through the exact classification in dimension 10 of one family of local pseudo-normal forms (with redundant real constants) for Goursat objects. The latter result is an harbinger of more general phenomena that will be treated in a subsequent paper.

1. Introduction. For any geometric distribution D (a subbundle of the tangent bundle) on a given smooth manifold M we set $D_1 = D,\ D_2 = D + [D, D], \ldots, D_{l+1} = D_l + [D, D_l];\quad D^{(0)} = D,\ D^{(1)} = D + [D, D], \ldots, D^{(l+1)} = D^{(l)} + [D^{(l)}, D^{(l)}]$. All these are, in general, modules of vector fields. By the *small growth vector* of D at p we understand the sequence $[n_1, n_2, n_3, \ldots]$ of dimensions at $p \in M$ of the flag $D_1 \subset D_2 \subset D_3 \subset \ldots$

Definition 1. Let D be a rank–2 distribution on an n-dimensional manifold. We say that D satisfies the *Goursat Condition* (GC for short in the sequel) when the members of the flag $D^{(0)} \subset D^{(1)} \subset D^{(2)} \subset \ldots$ have at every point linear dimensions $2, 3, \ldots, n-1, n$. The dual object $S = D^{\perp}$ we just call *a Goursat system.*

This condition is sometimes also called the Cartan–Goursat condition.

Every Goursat system S on an n-dimensional manifold admits locally around any fixed point certain *Kumpera–Ruiz pseudo-normal form* (*see* [KR] and, for newer presentations,

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[CM], [PR]), depending on that point and not unique (in general) even after fixing the point: the germ at $0 \in \mathbf{R}^n(x^1, x^2, \ldots, x^n)$ of $(\omega^1, \omega^2, \ldots, \omega^{n-2})$, where

$$\begin{aligned}
\omega^1 &= dx^{i_1} - x^3 dx^{j_1}, & (i_1, j_1) &= (2, 1)\\
\omega^2 &= dx^{i_2} - x^4 dx^{j_2}, & (i_2, j_2) &= (3, j_1)\\
\omega^3 &= dx^{i_3} - x^5 dx^{j_3}, & (i_3, j_3) &\in \{(4, j_2),\ (j_2, 4)\}\\
\omega^4 &= dx^{i_4} - X^6 dx^{j_4}, & (i_4, j_4) &\in \{(5, j_3),\ (j_3, 5)\}\\
& \bullet & & \bullet\\
\omega^{n-2} &= dx^{i_{n-2}} - X^n dx^{j_{n-2}}, & (i_{n-2}, j_{n-2}) &\in \{(n-1, j_{n-3}),\ (j_{n-3}, n-1)\},
\end{aligned}$$

and where for $6 \le l \le n$, $X^l = x^l$ if $(i_{l-2}, j_{l-2}) = (j_{l-3}, l-1)$ and $X^l = x^l + c^l$ in the opposite case of $(i_{l-2}, j_{l-2}) = (l-1, j_{l-3})$. The $c^6, c^7, \ldots, c^n$ are certain real constants.[1]

Moreover, one gets locally the successive derived systems of S by always removing the last (bottommost) Pfaffian equation.

(Throughout this paper, in contrast to [KR] and [CM], we write the forms ω^l with the $-$ signs instead of $+$. The same convention was used in [G].)

We code the KR pseudo-normal forms in dimensions $n \ge 5$ as in [CM], assigning to a Pfaffian equation $\omega^l = 0$, $l = 3, 4, \ldots, n-2$:

1, when the first alternative for ω^l holds and $c^{l+2} = 0$,

2, in the case of the first alternative, but $c^{l+2} \neq 0$,

3, in the case of the second alternative for ω^l,

and writing those integers, when l runs through the set $\{3, 4, \ldots, n-2\}$, from left to right, separated by dots. Specifically, we write **2** instead of *2* when the relevant constant is 1, and **2**$-$ when it is -1.

EXAMPLE 1. The germ at $0 \in \mathbf{R}^9(x^1, x^2, \ldots, x^9)$ of $(\omega^1, \omega^2, \ldots, \omega^7)$, where $\omega^1 = dx^2 - x^3 dx^1$, $\omega^2 = dx^3 - x^4 dx^1$, $\omega^3 = dx^1 - x^5 dx^4$, $\omega^4 = dx^5 - (1 + x^6) dx^4$, $\omega^5 = dx^4 - x^7 dx^6$, $\omega^6 = dx^6 - x^8 dx^7$, $\omega^7 = dx^8 - (c + x^9) dx^7$ with $c = 1$ (-1) is coded *3*.**2**.*3*.*3*.**2** (*3*.**2**.*3*.*3*.**2**$-$, respectively). These two local models form the exceptional pair $(***)$ listed in [M1], Main Theorem.

We are going to consider KR pseudo-normal forms in arbitrary dimension $n \ge 3$ not as germs at 0 but as representatives defined on the whole $\mathbf{R}^n$. By an *automorphism* of such a differential system S we mean any diffeomorphism $g : \mathbf{R}^n \hookleftarrow$ preserving S: $g^* S = S$. By an *infinitesimal automorphism* (i.a.) of S we mean any smooth vector field on $\mathbf{R}^n$ whose flow (at least for small $|t|$) preserves S. The set of all i.a.'s of S will be denoted – following the tradition established in [KR], [G] and [K] – by $L(S)$.

2. Contact hamiltonians parametrizing i.a.'s of KR pseudo-normal forms

2.1. Dimension 3. What are i.a.'s of the Darboux contact structure $\omega^1 = dx^2 - x^3 dx^1 = 0$ on $\mathbf{R}^3(x^1, x^2, x^3)$? The answer was given already by S. Lie [Li]. To reproduce

[1]The 'pseudo' refers to the fact that the constants are not, in general, invariants of the local classification.

it here, following the exposition in [Ly] (done for contact structures in arbitrary odd dimension, in fact), let us replace the coordinates x^1, x^2, x^3 by the coordinates q^1, u, p^1 in the bundle $J^1(\mathbf{R}^1)$ of the 1-jets of scalar functions $u(q^1)$ of one variable q^1, with $p^1 = \frac{du}{dq^1}$. The contact distribution $(dx^2 - x^3dx^1)^\perp$ becomes then the *Cartan distribution* $\mathcal{C}$ on $J^1(\mathbf{R}^1)$ [2] deciding which sections of this bundle come from genuine functions of one variable (those that are tangent to $\mathcal{C}$). All i. a.'s of $\mathcal{C}$ are of the form

$$-\frac{\partial f}{\partial p^1}\frac{\partial}{\partial q^1} + \left(f - p^1\frac{\partial f}{\partial p^1}\right)\frac{\partial}{\partial u} + \left(\frac{\partial f}{\partial q^1} + p^1\frac{\partial f}{\partial u}\right)\frac{\partial}{\partial p^1}\,,$$

where $f = f(q^1,\, u,\, p^1)$ is an arbitrary smooth function. Returning to our notation and writing ∂_k for $\partial/\partial x^k$, all i. a.'s of $(\mathbf{R}^3,\, \omega^1)$ are of the form

$$-f_3\,\partial_1 + (f - x^3f_3)\,\partial_2 + (f_1 + x^3f_2)\,\partial_3\,. \tag{1}$$

An arbitrary smooth function $f = f(x^1,\, x^2,\, x^3)$ in (1) is called a *contact hamiltonian* because of a natural (and deep) interpretation in terms of the symplectization of any given contact manifold – *see* [A], App. 4.

Expression (1) for i. a.'s of Darboux is used, omitting references, in [G] and [K].

2.2. Parametrization of dimension $n \geq 4$ by dimension 3. We recall in this section that when a KR pseudo-normal form S is given on $\mathbf{R}^n(x^1,\dots,\, x^n)$, its first derived system $S^{(1)}$ is defined on $\mathbf{R}^{n-1}(x^1,\dots,\, x^{n-1})$ and suspended only in the direction ∂_n, according to the information quoted in Sec. 1. In consequence, abusing notation a bit, members of $L(S^{(1)})$ are understood as vector fields on $\mathbf{R}^{n-1}$ only.

PROPOSITION 1 ([G], [K]). *(i) Every $X \in L(S)$ has all but the last coordinate functions depending only on $x^1,\dots,\, x^{n-1}$. In this way, denoting by $[X]_1$ the truncation of X by its last coordinate function, we have $[X]_1 \in L(S^{(1)})$.*
(ii) Every $Y \in L(S^{(1)})$ has a unique prolongation $[Y]^1 \in L(S)$ s. t. $\left[[Y]^1\right]_1 = Y$.
(iii) In the situation of (i), by the uniqueness in (ii), $\left[[X]_1\right]^1 = X$.

PROOF. (i). The key factor is Cor. 3.1 of [M1].[3] Because of that, the flow φ^t_X of X can be written as

$$\varphi^t_X(x^1,\, x^2,\dots,\, x^n) = (\, f^1(t, x^1,\dots, x^{n-1}),\dots,\, f^{n-1}(t, x^1,\dots, x^{n-1}),\, f^n(t, x^1,\dots, x^n)\,)\,.$$

Therefore, writing horizontally the coordinates of a vector field,

$$X(x^1,\, x^2,\dots,\, x^n) = \left[\frac{d}{dt}f^1\,|_{t=0},\dots,\, \frac{d}{dt}f^{n-1}\,|_{t=0},\, \frac{d}{dt}f^n\,|_{t=0}\right].$$

This shows that all but last coordinate functions of X depend only on t, $x^1,\dots,\, x^{n-1}$, hence only on $x^1,\dots,\, x^{n-1}$, X being time-independent.

(ii). We seek prolongation(s) of Y, $Y + F\partial_n$ preserving (infinitesimally) the new Pfaffian equation $\omega^{n-2} = 0$ modulo $(\omega^1,\dots,\, \omega^{n-3})$:

$$L_{Y+F\partial_n}(\omega^{n-2}) \in (\,\omega^1,\dots,\, \omega^{n-2}\,)\,. \tag{2}$$

[2] Lychagin calls $du - \sum_{i=1}^k p^i dq^i$ a *classifying element* on $J^1(\mathbf{R}^k)$.

[3] Stated there for diffeos g preserving 0, but with the condition $g(0) = 0$ not used in the proof. That fact can be, essentially, retraced already in E. Cartan's work [C], p. 10.

Because the new variable (in the form of x^n or $X^n = c^n + x^n$) enters the new equation, while its derivative dx^n does not, equation (2) is always a linear equation for F with leading coefficient 1. A careful inspection of the KR pseudo-normal forms shows that the free term uses just the first derivatives of coordinate functions of Y. Instead of abstract writing, let us analyze in detail the initial cases $n = 4$ and 5. The general rules of prolongation do not differ much from them; as a consequence, the existence and uniqueness of F follow.

$n = 4$; we write $Y = A\partial_1 + B\partial_2 + C\partial_3$, and name $D\partial_4$ the term prolonging Y. Then (2) reads

$$dC - x^4 dA - D\,dx^1 = (C_2 - x^4 A_2)(dx^2 - x^3 dx^1) + (C_3 - x^4 A_3)(dx^3 - x^4 dx^1)\,,$$

implying $D = C_1 - x^4 A_1 + x^3(C_2 - x^4 A_2) + x^4(C_3 - x^4 A_3)$.

$n = 5$; $Y = A\partial_1 + B\partial_2 + C\partial_3 + D\partial_4$, the prolonging term denoted now by $E\partial_5$. If S is *1*, i.e., $\omega^3 = dx^4 - x^5 dx^1$, then (2) reads

$$\begin{aligned} dD - x^5 dA - E\,dx^1 = {} & (D_2 - x^5 A_2)(dx^2 - x^3 dx^1) + (D_3 - x^5 A_3)(dx^3 - x^4 dx^1) + \\ & D_4(dx^4 - x^5 dx^1)\,, \end{aligned}$$

or else $E = D_1 - x^5 A_1 + x^3(D_2 - x^5 A_2) + x^4(D_3 - x^5 A_3) + x^5 D_4$.

When S is *3*, that is, when $\omega^3 = dx^1 - x^5 dx^4$, (2) means

$$\begin{aligned} dA - x^5 dD - E dx^4 = {} & \lambda(dx^2 - x^3 dx^1) + \mu(dx^3 - x^4 dx^1) + \\ & (A_1 - x^5 D_1 + \lambda x^3 + \mu x^4)(dx^1 - x^5 dx^4)\,, \end{aligned}$$

where $\lambda = A_2 - x^5 D_2$, $\mu = A_3 - x^5 D_3$. Then $E = x^5(A_1 - x^5 D_1 + \underline{x^3(A_2 - x^5 D_2) +}$ $\underline{x^4(A_3 - x^5 D_3)} - D_4)$ (the underlined terms have been forgotten in [G], p. 72^5). ∎

Corollary 1. *Keeping a KR pseudo-normal form S fixed, every i. a. Y of the Darboux system $S^{(n-3)}$ has its unique prolongation $[Y]^{n-3} \in L(S)$ s. t. $\left[[Y]^{n-3}\right]_{n-3} = Y$. Each i. a. $X \in L(S)$ is obtained in this way: $\left[[X]_{n-3}\right]^{n-3} = X$ by the uniqueness of the prolongation.*

Therefore, the set $L(S)$ is parametrized – in the way depending on S – by the i. a.'s of Darboux, hence — by smooth functions on $\mathbf{R}^3$ (contact hamiltonians).[4]

Observation 1. $L(g^*S) = g_*^{-1} L(S)$ and $L(S)$ is a module over $\mathbf{R}$. Hence $\dim L(S)(0)$ is an invariant of the classification of germs at $0 \in \mathbf{R}^n$ of KR pseudo-normal forms S on $\mathbf{R}^n$. We call it the *symmetry dimension* of the germ of S at 0.

3. Symmetry dimension of members of the exceptional couples of models in dimension 9. Local classification of GC in dimension 9 (obtained by analytic considerations) consists of 93 pairwise non-equivalent models displaying altogether only 89

[4] $L(S)$ is even a Lie algebra; also the contact hamiltonians can be Poisson-multiplied because of their coming from hamiltonians on the symplectization of $(\mathbf{R}^3, \omega^1)$. Their product is called the *Lagrange bracket*, cf. for inst. [Ly], 1.4.4. In the present paper, however, we do not use those additional structures.

different small growth vectors at $0 \in \mathbf{R}^9$: four pairs are 'served' by growth vectors coinciding in those pairs ([M1], Main Theorem). We mean, sticking to the labels from [M1], the pairs:

(∗) $\mathit{1.3.1.}\mathbf{2}.\mathit{1}$ and $\mathit{1.3.1.}\mathbf{2.2}$,
(∗∗) $\mathit{3.1.3.}\mathbf{2}.\mathit{1}$ and $\mathit{3.1.3.}\mathbf{2.2}$,
(∗∗∗) $\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2}-$ and $\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2}$,
(∗∗∗∗) $\mathit{3.3.3.}\mathbf{2}.\mathit{1}$ and $\mathit{3.3.3.}\mathbf{2.2}$.

Proposition 2. (i) *Symmetry dimension is an additional invariant that distinguishes* geometrically *local models in pairs* (∗), (∗∗), *and* (∗∗∗∗) *above.*

(ii) *Members of* (∗∗∗) *(*see Ex. 1*), however, are not discernible by symmetry dimension.*

Proof. (i). Having recursive formulas for the coordinate functions of i. a.'s of any KR pseudo-normal form (cf. proof of Prop. 1), we iterate those formulas backwards, always evaluating them at 0. In this way, we eventually express i. a.'s at 0 in terms of certain jets at $0 \in \mathbf{R}^3$ of contact hamiltonians. In the particular cases discussed in (i) the outcome is the following, writing horizontally the coordinates of v. f. and denoting, from now on, by $|0$ the evaluation at 0. For the pair (∗):

$$L(\mathit{1.3.1.}\mathbf{2}.\mathit{1})\,|0 = \{[-f_3, f, f_1, f_{11}, f_{111}, 0, 0, -3f_2 - 10f_{13}, 0]\,|0,\ f(x^1, x^2, x^3) \text{ smooth}\};$$
$$L(\mathit{1.3.1.}\mathbf{2.2})\,|0 = \{[-f_3, f, f_1, f_{11}, f_{111}, 0, 0, -3f_2 - 10f_{13}, -4f_2 - 13f_{13}]\,|0,\ f \text{ smooth}\}.$$

In view of the arbitrariness of f (Cor. 1), $L(\mathit{1.3.1.}\mathbf{2}.\mathit{1})\,|0 = (\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_8)$. On the other hand, because $\begin{vmatrix} -3 & -10 \\ -4 & -13 \end{vmatrix} \neq 0$, $L(\mathit{1.3.1.}\mathbf{2.2})\,|0 = (\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_8, \partial_9)$. Hence the symmetry dimensions are 6 and 7, respectively.

For the pair (∗∗):

$$L(\mathit{3.1.3.}\mathbf{2}.\mathit{1})\,|0 = \{[-f_3, f, f_1, f_{11}, 0, 0, 0, 5f_2 + 12f_{13}, 0]\,|0,\ f \text{ smooth}\};$$
$$L(\mathit{3.1.3.}\mathbf{2.2})\,|0 = \{[-f_3, f, f_1, f_{11}, 0, 0, 0, 5f_2 + 12f_{13}, 7f_2 + 17f_{13}]\,|0,\ f \text{ smooth}\}.$$

Here $\dim(L(\mathit{3.1.3.}\mathbf{2}.\mathit{1})\,|0) = 5$, whereas $\dim(L(\mathit{3.1.3.}\mathbf{2.2})\,|0) = 6$ (because $\begin{vmatrix} 5 & 12 \\ 7 & 17 \end{vmatrix} \neq 0$).

For the pair (∗∗∗∗):

$$L(\mathit{3.3.3.}\mathbf{2}.\mathit{1})\,|0 = \{[-f_3, f, f_1, f_{11}, 0, 0, 0, -5f_2 - 13f_{13}, 0]\,|0,\ f \text{ smooth}\};$$
$$L(\mathit{3.3.3.}\mathbf{2.2})\,|0 = \{[-f_3, f, f_1, f_{11}, 0, 0, 0, -5f_2 - 13f_{13}, -7f_2 - 18f_{13}]\,|0,\ f \text{ smooth}\}.$$

So, $\dim(L(\mathit{3.3.3.}\mathbf{2}.\mathit{1})\,|0) = 5$ and $\dim(L(\mathit{3.3.3.}\mathbf{2.2})\,|0) = 6$ (as $\begin{vmatrix} -5 & -13 \\ -7 & -18 \end{vmatrix} \neq 0$).

(ii). Proceeding exactly as in (i) one obtains

$$\dim(L(\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2}-)\,|0) = \dim(L(\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2})\,|0) = 6. \ \blacksquare$$

Open question. How to distinguish geometrically $\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2}-$ from $\mathit{3.}\mathbf{2}.\mathit{3.3.}\mathbf{2}$?

Remark 1. (a) It is visible that symmetry dimension is a geometric invariant (cf. Obs. 1) just complementary to small growth vector: it takes the same values 5 and 6 on pairs (∗∗) and (∗∗∗∗) differing by the small gr. vector. (And not only by it; (∗∗) and (∗∗∗∗) differ due to the very elementary reason of Cor. 3.2 of [M1].)

(b) It is to be underlined that this invariant has already been used for GC in [G]. One of the aims of Gaspar was to distinguish, in a different way than through so-called *reduced tensors* of KR, the five Kumpera–Ruiz local models in dimension 6 ([KR], p. 226–7).

That pioneering use occurred too early, in a sense,[5] if having already pointed to a limited discernment power of the symmetry dimension alone (taking on three values in that occurrence: *1.1* — 6, *1.3* — 5, *3.1* — 4, *3.***2** — 5, *3.3* — 4).

4. The first non-trivial branch of KR pseudo-normal forms in dimension 10. We want to supply some information concerning the true status of KR pseudo-normal forms beyond dimension 9. In dimension 10 and only among codes having just one *3* in the beginning (the simplest non-trivial pseudo-normal KR forms), the situation resembles the general situation with GC in dimension 9: small gr. vector is not the only invariant and the symmetry dimension is only partially complementary to it (cf. Prop. 2). In fact,

THEOREM 3. *In dimension* 10*, here is the list of all different local models among the KR pseudo-normal forms having a unique 3 at the beginning of the code. Small growth vectors at* $0 \in \mathbf{R}^{10}$ *are given in the second column, with the numbers of repetitions of an integer shown as subscripts. Symmetry dimensions at* $0 \in \mathbf{R}^{10}$ *are given (only when needed) in the third column.*

*3.***2***.1.1.1.1*	$[2, 3, 4, 5, 6, 7, 8, 9_2, 10]$	
*3.1.***2***.1.1.1*	$[2, 3, 4, 5, 6, 7, 8, 9_3, 10]$	
*3.1.1.***2***.1.***2**$-$	$[2, 3, 4, 5, 6, 7, 8, 9_4, 10]$	$\dim(L\,\vert\,0) = 7$
*3.1.1.***2***.1.1*	$[2, 3, 4, 5, 6, 7, 8, 9_4, 10]$	$\dim(L\,\vert\,0) = 6$
*3.1.1.***2***.1.***2**	$[2, 3, 4, 5, 6, 7, 8, 9_4, 10]$	$\dim(L\,\vert\,0) = 7$
*3.1.1.1.***2***.1*	$[2, 3, 4, 5, 6, 7, 8, 9_5, 10]$	
*3.1.1.1.1.***2**	$[2, 3, 4, 5, 6, 7, 8, 9_6, 10]$	
3.1.1.1.1.1	$[2, 3, 4, 5, 6, 7, 8, 9_7, 10]$	

PROOF. That the small growth vector at the origin depends – in the branch in question – only on the place of the first *2* in the code (i. e., on the index of the first non-zero constant in the pseudo-normal form), amounts to a simple calculation. When that first non-vanishing constant is NOT c^8, its normalization to 1, and reductions to 0 of the constants appearing after it, go in a way similar to [CM] (Thm. 27) and [M1] (Lemmas [32111], [31211]).

For $c^8 \neq 0$, however, it is otherwise. The subsequent constant c^9 can be annihilated by [M1], Lem. [31121], while the last constant c^{10} hides an obstacle. It can only be reduced, when keeping already $c^8 = 1$ and $c^9 = 0$, to exactly one of the values -1, 0, 1 (*see* the list of proposed local models in Thm. 3). We will start (1°) with the possibility of reduction (simple), then proceed (2°) to the non-equivalence of the three produced values of c^{10} (difficult).

As for the symmetry dimension in these exceptional cases, its computation is much similar to the calculus done in Sec. 3. The results are written in the theorem table – the

[5]Small gr. vector is the only local invariant of GC up to dimension 8 inclusive, as has later been shown in [CM].

symmetry dimension distinguishes 0 from -1 and 1, whereas the last two values are not discernible by this tool.

1^{o}. A reduction of a non-zero $c^{10} = c$ to $\mathrm{sgn}(c)$ is straightforward by passing to the bar variables:

$$x^{10} = |\,c\,|\,\overline{x^{10}}, \quad x^9 = |\,c\,|^{\frac12}\,\overline{x^9}, \quad x^4 = |\,c\,|^{-\frac12}\,\overline{x^4}, \quad x^8 = \overline{x^8}, \quad x^7 = |\,c\,|^{-\frac12}\,\overline{x^7}$$
$$x^6 = |\,c\,|^{-1}\,\overline{x^6}, \quad x^5 = |\,c\,|^{-\frac32}\,\overline{x^5}, \quad x^1 = |\,c\,|^{-2}\,\overline{x^1}, \quad x^3 = |\,c\,|^{-\frac52}\,\overline{x^3}, \quad x^2 = |\,c\,|^{-\frac92}\,\overline{x^2}\,.$$

This change of coordinates preserves all but the last Pfaffian equations of the KR writing of *3.1.1.***2***.1.2*, while that last equation assumes the form

$$dx^9 - (c + x^{10})dx^4 = \sqrt{|c|}d\overline{x^9} - \frac{c+\mid c\mid \overline{x^{10}}}{\sqrt{\mid c\mid}}d\overline{x^4} = \sqrt{|c|}\left(d\overline{x^9} - (\mathrm{sgn}(c) + \overline{x^{10}})d\overline{x^4}\right)\,.$$

2^{o}. The values -1, 0, and 1 of c^{10} are mutually non-equivalent.

In a rigid situation of keeping $c^8 = 1$ and $c^9 = 0$ fixed, we try to conjugate two a priori arbitrary values c and $\tilde c$ of the constant c^{10}.

The distributions D, $\widetilde D$ dual to the KR pseudo-normal forms in question are spanned by $(X,\, Y_c)$ and $(X,\, Y_{\tilde c})$, where $X = \partial_{10}$ and, in matrix notation (T – transpose),

$${Y_a}^{\mathrm{T}} = \left[\, x^5,\, x^3x^5,\, x^4x^5,\, 1,\, x^6,\, x^7,\, 1 + x^8,\, x^9,\, a + x^{10},\, 0\,\right]\,.$$

For any diffeo $g : (\mathbf{R}^{10},\, 0)\hookleftarrow$ sending D to $\widetilde D$, the linear subdistribution (X) common for D and $\widetilde D$ is preserved by g (a basic fact, used already in the proof of Prop. 1). Therefore, for certain functions $\bar f$ and $\bar h$, $\bar f\,|\,0 \neq 0$,

$$g_* Y_c = \bar f Y_{\tilde c} + \bar h X\,.$$

Taking this identity at $g(x)$ instead of x gives

$$Dg(x)Y_c(x) = f(x)\,Y_{\tilde c}(g(x)) + h(x)\,\partial_{10} \tag{3}$$

for $f = \bar f \circ g$, $h = \bar h \circ g$; $f\,|\,0 \neq 0$. At that, writing $g = (g^1,\, g^2, \dots,\, g^{10})$, g^1, g^2, g^3 depend only on x^1, x^2, x^3, and for $l \ge 4$ g^l depends on $x^1,\, x^2, \dots,\, x^l$ (cf. [M1], Cor. 3.1). Moreover,

$$g^5(x^1, \dots,\, x^5) = x^5 G(x^1, \dots,\, x^5) \tag{4}$$

for some function G, because the hypersurface $E = \{\, x^5 = 0\,\}$ can be characterized in invariant terms simultaneously for D and $\widetilde D$. In fact, a short calculation shows that both D and $\widetilde D$ have at points of E the small gr. vectors of the type $[2, 3, 4, 5, 6, 7, 8, 9, 9, \dots]$ and off E – just the small vector $[2, 3, 4, 5, 6, 7, 8, 9, 10]$. The preserving of E means $g^5(x^1,\, x^2,\, x^3,\, x^4,\, 0) = 0$ identically, or else (4).

In the sequel we shall write simply g^l_k for $\frac{\partial g^l}{\partial x^k}$. For instance, the inequality $\frac{\partial g^l}{\partial x^l}\,|\,0 \neq 0$ will henceforth be denoted $g^l_l\,|\,0 \neq 0$.

Equating the coefficients of ∂_9 in (3) (later we just say 'taking scalar equation "9" of (3)') and evaluating the result at 0, a cardinal relation linking c and $\tilde c$ reads

$$f^{-1}(g^9_4 + g^9_7 + c g^9_9)\,|\,0 = \tilde c\,.$$

Claim. $g^9_9\,|0 = f^{-1}\,|0$.

Proof. In view of (4), eq. "5" of (3) means

$$g^6 = f^{-1}\left((x^5)^2 G_1 + x^3(x^5)^2 G_2 + x^4(x^5)^2 G_3 + x^5 G_4 + x^6(G + x^5 G_5) \right),$$

implying

$$g^6 \in (x^5,\, x^6) \tag{$\dagger$}$$

($(x^5,\, x^6)$ means the *ideal* in $\mathcal{F}^{10}_0$ generated by the functions x^5 and x^6). In turn, eq. "6" of (3) gives g^7 in function of f and g^6. In fact, by (†),

$$g^7 \equiv f^{-1}(x^6 g^6_5 + x^7 g^6_6) \mod (x^5,\, x^6)\,,$$

yielding

$$g^7 \in (x^5,\, x^6,\, x^7)\,. \tag{$\dagger\dagger$}$$

Observe also that g^9 can be expressed by f and g^8 (eq. "8" of (3)), while g^8 gets expressed (eq. "7" of (3)) by g^7 and again f as follows:

$$1 + g^8 = f^{-1}\left(x^5 g^7_1 + x^3 x^5 g^7_2 + x^4 x^5 g^7_3 + g^7_4 + x^6 g^7_5 + x^7 g^7_6 + (1 + x^8) g^7_7\right)\,. \tag{5}$$

Evaluating this at 0, with (††) implying $g^7_4\,|0 = 0$, one gets an important piece of information

$$f^{-1} g^7_7\,|0 = 1\,. \tag{6}$$

As for f, it is explicitly given by eq. "4" of (3) and visibly depends only on $x^1, \dots, x^5$:

$$f = x^5 g^4_1 + x^3 x^5 g^4_2 + x^4 x^5 g^4_3 + g^4_4\,. \tag{7}$$

Now that we have (5) through (7), we can compute: $g^9_9\,|0 = f^{-1} g^8_8\,|0 = f^{-1}\cdot f^{-1} g^7_7\,|0 = f^{-1}\,|0$. Claim is proved.

In view of Claim, the relation linking c and $\tilde{c}$ assumes the form

$$f^{-1}(g^9_4 + g^9_7) + c f^{-2}\,|0 = \tilde{c}\,. \tag{8}$$

4.1. Basic Lemma. $g^9_4 + g^9_7\,|0 = 0$.

Proof. The fact that $c^8 = 1$ and $c^9 = 0$ are preserved by g means

$$g^7_4 + g^7_7\,|0 = f\,|0\,, \qquad g^8_4 + g^8_7\,|0 = 0\,. \tag{a}$$

On using (5) and (7) for computing $g^8_7\,|0$, and – additionally – the *first* identity in (a) in computing $g^8_4\,|0$, the *second* identity in (a) assumes the form

$$-\,f_4 + g^7_6 + 2g^7_{47}\,|0 = 0\,. \tag{b}$$

Continuing in this way, writing g^7 in terms of g^6 and f (eq. "6" of (3)), one arrives from (b), after a computation, at

$$-\,3f_4 + f^{-1} g^6_5 + 3 f^{-1} g^6_{46}\,|0 = 0\,. \tag{c}$$

What we eventually need is a relation between f and G. As g^6 can be written in terms of

f and g^5 (eq."5" of (3)), another calculation leads from (c) to the (important) identity

$$-\,3f_4 + 2f^{-2}G_4\,|0 = 0\,. \tag{d}$$

Now, putting $x^5 = x^6 = x^8 = x^9 = 0$ when computing the expression in Basic Lemma (i.e., working with $f^{-1}(g^8_4 + x^7 g^8_6 + g^8_7)$ instead of g^9),

$$g^9_4\,|0 = -\,f_4 f^{-2}(\underbrace{g^8_4 + g^8_7}_{=0 \text{ by (a)}}) + f^{-1}(g^8_{44} + g^8_{47})\,|0\,, \qquad g^9_7\,|0 = f^{-1}(g^8_{47} + g^8_6 + g^8_{77})\,|0\,,$$

so that we have so far

$$g^9_4 + g^9_7\,|0 = f^{-1}(g^8_6 + g^8_{44} + 2g^8_{47} + g^8_{77})\,|0\,. \tag{9}$$

Given the summands to-be-computed in (9), we can put $x^5 = x^8 = 0$, i.e., replace g^8 by $-1 + f^{-1}(g^7_4 + x^6 g^7_5 + x^7 g^7_6 + g^7_7)$. In the course of computations we take into account certain side facts such as $g^7_{44}\,|0 = g^7_{444}\,|0 = g^7_{77} = 0$ (caused by (††) and g^7 being affine wrt x^7). As a result, $g^8_{77}\,|0 = 2f^{-1}g^7_{67}\,|0$, $g^8_6\,|0 = f^{-1}(g^7_{46} + g^7_5 + g^7_{67})\,|0$, and, using also the first identity in (a),

$$\begin{aligned} g^8_{44}\,|0 &= (f^{-1})_{44} f + 2(f^{-1})_4 g^7_{47} + f^{-1} g^7_{447}\,|0 \\ &= -\,f^{-1} f_{44} + 2f^{-2}(f_4)^2 - 2f^{-2} f_4 g^7_{47} + f^{-1} g^7_{447}\,|0\,, \end{aligned}$$

$$g^8_{47}\,|0 = -\,f^{-2} f_4 (g^7_{47} + g^7_6) + f^{-1}(g^7_{447} + g^7_{46})\,.$$

Summing up the above,

$$\begin{aligned} &g^8_6 + g^8_{44} + 2g^8_{47} + g^8_{77}\,|0 = \\ &\quad f^{-1}(3g^7_{46} + g^7_5 + 3g^7_{67} - f_{44} + 3g^7_{447} - 2f^{-1} f_4(-\,f_4 + g^7_6 + 2g^7_{47}))\,|0 = 0\,. \end{aligned}$$

Therefore, by (9) and (b)

$$g^9_4 + g^9_7\,|0 = f^{-2}(3g^7_{46} + g^7_5 + 3g^7_{67} - f_{44} + 3g^7_{447})\,|0\,. \tag{10}$$

One repeats, basically, the above procedure when reducing the RHS of (10) to g^6. Firstly, $g^7_7\,|0 = f^{-1} g^6_6\,|0$, or else, by (a) and (††),

$$g^6_6\,|0 = f^2\,|0\,. \tag{e}$$

In turn, $g^7_5\,|0 = f^{-1} g^6_{45}\,|0$, $g^7_{67}\,|0 = f^{-1} g^6_{66}\,|0 = 0$ (g^6 affine wrt x^6),

$$g^7_{46}\,|0 = -\,f^{-2} f_4 (g^6_{46} + g^6_5) + f^{-1}(g^6_{446} + g^6_{45})\,|0\,,$$

$$\begin{aligned} g^7_{447}\,|0 &= (-\,f^{-2} f_{44} + 2f^{-3}(f_4)^2) g^6_6 - 2f^{-2} f_4 g^6_{46} + f^{-1} g^6_{446}\,|0 \\ &\overset{\text{by (e)}}{=} -\,f_{44} + 2f^{-1}(f_4)^2 - 2f^{-2} f_4 g^6_{46} + f^{-1} g^6_{446}\,|0 \end{aligned}$$

Summing up,

$$\begin{aligned} &3g^7_{46} + g^7_5 + 3g^7_{67} - f_{44} + 3g^7_{447}\,|0 = \\ &\quad f^{-1}(4g^6_{45} + 6g^6_{446} - 4ff_{44} - 3(f_4)^2 - 3f_4(-\,3f_4 + f^{-1} g^6_5 + 3f^{-1} g^6_{46}))\,. \end{aligned}$$

Therefore, by (10) and (c),

$$g^9_4 + g^9_7\,|0 = f^{-3}(4g^6_{45} + 6g^6_{446} - 4ff_{44} - 3(f_4)^2)\,|0\,. \tag{11}$$

Clearly, the RHS of (11) can be reduced down to $g^5 = x^5 G$. Firstly $g^6_6\,|0 = f^{-1} G\,|0$, or else, by (e),

$$G\,|0 = f^3\,|0\,. \tag{f}$$

Secondly, $g^6_{45}\,|0 \;=\; (f^{-1}G_4)_4\,|0 \;=\; -\,f^{-2}f_4G_4 + f^{-1}G_{44}\,|0$. Thirdly,

$$g^6_{446}\,|0 \;=\; (f^{-1}G)_{44}\,|0 \;=\; (-\,f^{-2}f_{44} + 2f^{-3}(f_4)^2)\,G - 2f^{-2}f_4G_4 + f^{-1}G_{44}\,|0.$$

Taking (f) into account and summing up,

$$\begin{aligned}&4g^6_{45} + 6g^6_{446} - 4ff_{44} - 3(f_4)^2 =\\ &\quad f^{-1}(10G_{44} - 10f^2f_{44} + 9f(f_4)^2 - 24f(f_4)^2 - 8ff_4(-\,3f_4 + 2f^{-2}G_4))\,|0\,.\end{aligned}$$

Therefore, by (11) and (d),

$$g^9_4 + g^9_7\,|0 = f^{-4}(10G_{44} - 10f^2f_{44} - 15f(f_4)^2)\,|0\,. \tag{12}$$

Now G can be eliminated from the RHS of (12). fG is only affine wrt x^4, so $0 = (fG)_{44} = f_{44}G + 2f_4G_4 + fG_{44}$. This, (d), and (f) yield $G_{44}\,|0 = -\,f^2f_{44} - 3f(f_4)^2\,|0$. Substituting to (12),

$$g^9_4 + g^9_7\,|0 = -\,5f^{-3}(4ff_{44} + 9(f_4)^2)\,|0\,. \tag{13}$$

The RHS of (13) can be further simplified using (7): the derivatives of f entering (13) can be expressed by those of

$$g^4 = \frac{fG\,g^4}{fG} = \frac{A(x^1,\,x^2,\,x^3) + B(x^1,\,x^2,\,x^3)\,x^4}{C(x^1,\,x^2,\,x^3) + D(x^1,\,x^2,\,x^3)\,x^4}.$$

Remark 2. i). $A\,|0 = A + B\,x^4\,|0 = fG\,g^4\,|0 = 0\,.$
ii). $C\,|0 = fG\,|0 \neq 0\,.$
iii). $f\,|0 = g^4_4\,|0 = \dfrac{BC - AD}{C^2}\,|0 = \dfrac{B}{C}\,|0$ by i).
iv). $B\,|0 \neq 0$ by iii).
v). $f_4\,|0 = g^4_{44}\,|0 = -\,2\dfrac{BD}{C^2}\,|0\,.$
vi). $f_{44}\,|0 = g^4_{444}\,|0 = 6\dfrac{BD^2}{C^3}\,|0\,.$
vii). $G\,|0 = \dfrac{C^2}{B}\,|0$ (by ii) and iii)).

By Rem. 2 ((iii), (v), and (vi)) $4ff_{44}\,|0 \;=\; 24\dfrac{B^2D^2}{C^4}\,|0 \;=\; 6(f_4)^2\,|0$ and (13) boils down to

$$g^9_4 + g^9_7\,|0 = -\,75f^{-3}(f_4)^2\,|0\,. \tag{14}$$

On the other hand, $G = \dfrac{fG}{f} = \dfrac{C + D\,x^4}{g^4_4 + x^5(\star)}$ (*see* (7)). Hence, also by Rem. 2,

$$G_4\,|0 = \frac{D}{g4_4} - \frac{C\,g^4_{44}}{(g^4_4)^2}\,|0 = \frac{CD}{B} - C\left(-2\frac{BD}{C^2}\right)\left(\frac{C}{B}\right)^2\,|0 = 3\frac{CD}{B}\,|0\,.$$

Approaching the end, we are going to obtain a formula alternative to (d) for $G_4\,|0$. The identity (f) reads now (*see* (iii) and (vii) of Rem. 2) $\dfrac{C^2}{B}\,|0 = (\dfrac{B}{C})^3\,|0$, or else

$$B^4\,|0 = C^5\,|0\,. \tag{g}$$

Therefore,

$$3\frac{CD}{B}\,|\,0 = 3\left(\frac{B}{C}\right)^2\frac{BC^3D}{B^4}\,|\,0 \overset{\text{by (g)}}{=} 3\left(\frac{B}{C}\right)^2\frac{BC^3D}{C^5}\,|\,0 = -\frac{3}{2}\left(\frac{B}{C}\right)^2\left(-2\frac{BD}{C^2}\right)\,|\,0.$$

That is, by (iii) and (v) of Rem. 2,

$$G_4\,|\,0 = -\tfrac{3}{2}f^2 f_4\,|\,0\,. \tag{h}$$

Finally, (d) and (h) together give $f_4\,|\,0 = 0$ and Basic Lemma follows from (14). ∎

4.2. Concluding part of the proof of Thm. 3. In view of Basic Lemma, because of the factor $f^{-2}\,|\,0$ in (8), the values -1, 0, and 1 of c^{10} are mutually non-equivalent as asserted in 2°.

The assessment of classes of local equivalence among the KR pseudo-normal forms *3.1.1.***2**. *(1* or *2).(1* or *2)*, all covered by the small gr. vector [2, 3, 4, 5, 6, 7, 8, 9_4, 10], is now finished. Thm. 3 is proved. ∎

Addendum. A result obtained after the submission of the present paper underlines clearly a special role of the constant c^{10} in the whole branch of KR pseudo-normal forms *3.1.1.***2***.1.(1* or *2).(1* or *2)*... in dimensions from 11 onwards. (In this notation c^8 is already normalized to 1 and c^9 is annihilated – cf. the beginning of the proof of Thm. 3.) Namely, in every dimension $n > 10$, any such given KR local form is equivalent to exactly one of the following local models: $\mathit{3.1.1.}\mathbf{2}\mathit{.1.}\mathbf{2}-.\underbrace{\mathit{1.1\ldots 1}}_{n-10}$, $\mathit{3.1.1.}\mathbf{2}.\underbrace{\mathit{1.1\ldots 1}}_{n-8}$, $\mathit{3.1.1.}\mathbf{2}\mathit{.1.}\mathbf{2}.\underbrace{\mathit{1.1\ldots 1}}_{n-10}$. That is, all subsequent constants in this branch are reducible to 0, i.e. are not important. A proof will be included in a paper in preparation (cf. Section 9 of [M2]).

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