POISSON GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 51
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

SELF-SIMILARITY OF POISSON STRUCTURES ON TORI

KENTARO MIKAMI

Department of Computer Science and Engineering
Faculty of Engineering and Resource Science
Akita University, Akita, 010–8502, Japan
E-mail: mikami@math.akita-u.ac.jp

ALAN WEINSTEIN

Department of Mathematics, University of California
Berkeley, CA 94720, U.S.A.
E-mail: alanw@math.berkeley.edu

Abstract. We study the group of diffeomorphisms of a 3-dimensional Poisson torus which preserve the Poisson structure up to a constant multiplier, and the group of similarity ratios.

1. Introduction. A *scaling vector field* on a Poisson manifold (M, π) is a vector field X such that the Lie derivative $L_X\pi$ is a constant multiple of π. The set of all these vector fields forms a Lie algebra $\mathcal{L} = \mathcal{L}(M, \pi)$ containing the Poisson vector fields $\mathcal{L}_0$ as an ideal. The quotient $\mathcal{L}/\mathcal{L}_0$ has dimension 1 if π is exact and dimension 0 otherwise. If we think of $\mathcal{L}$ and $\mathcal{L}_0$ as Lie algebras of infinitesimal transformations, the corresponding groups of finite transformations are the *self-similarities* $\mathcal{G}$, i.e. the diffeomorphisms of M which preserve π up to a constant multiplier, and the normal subgroup $\mathcal{G}_0$ of Poisson automorphisms. The quotient $\mathcal{G}/\mathcal{G}_0$ is naturally isomorphic to the *scaling group* $\mathcal{S}(M, \pi)$, the subgroup of the nonzero real numbers consisting of the possible similarity ratios (i.e. multipliers). By convention, the similarity ratio for any self-similarity of $\pi = 0$ is declared to be 1.

The scaling group of $\mathbf{R}^{2n}$ with the standard symplectic structure is all of $\mathbf{R}^{\times}$. For a

2000 *Mathematics Subject Classification*: Primary 53D17.

Research of the first author partially supported by Grand-in-Aid for Scientific Research, The Priority Area of the infinite dimensional integrability systems (No. 08211207) and the Fundamental Research (C) (No. 08640084, 10640057), The ministry of Education, Science and Culture, Japan.

Research of the second author partially supported NSF Grant NSF DMS-9625122 and the Miller Institute for Basic Research in Science.

The paper is in final form and no version of it will be published elsewhere.

compact symplectic manifold, the scaling group is constrained by the finiteness of volume to be either $\{1\}$ or $\{-1, 1\}$. Iglésias [1] showed that the scaling groups are countable for nonexact symplectic manifolds. For general Poisson manifolds, scaling groups were introduced and discussed, with somewhat different names and notation, in Section 10 of [7] and Section 5 of [8] in connection with related quantum constructions. They are also related, though somewhat distantly, to the "von Neumann fundamental groups" of operator algebras (see the discussion in [5]).

In this paper, we study constant (i.e. translation-invariant) Poisson structures on the 3-dimensional torus and show that their scaling groups are finitely generated subgroups of $\mathbf{R}^\times$ having rank 0, 1, or 2.

As in Iglésias and Lachaud's study of singular "irrational tori" [2], we use homology theory to reduce the problem from geometry to linear algebra. We prove a result (Lemma 1 below) in elementary linear algebra (replacing some of the considerations in Section 2 of [2]), and then apply the Dirichlet unit theorem. We check and refine the results by direct calculation in several examples.

An alternative approach to finding centralizers in $SL_3(\mathbf{Z})$ would be to use results of Prasad and Raghunathan [4] on Cartan subgroups of algebraic groups. We do not pursue this approach here.

Acknowledgments. We dedicate this paper to the memory of our friend Stan Zakrzewski. We thank Ken Ribet, Klaus Schmidt and Dimitry Shlyakhtenko for helpful discussions. Mikami would also like to thank the University of California at Berkeley for hospitality while he was a visiting scholar from April 1997 till March 1998, and the Banach Center at Warsaw for hospitality and support during his two week stay there in August 1998.

2. Invariant geometry on the 3-torus. As a basis for the constant vector fields on the torus $\mathbf{T}^3$ we take the coordinate vector fields ∂_1, ∂_2, and ∂_3; the naturally associated basis for the constant bivector fields is $\partial_i^* = \partial_{i+1} \wedge \partial_{i+2}$ (indices modulo 3). For a constant Poisson structure $\pi = \sum a_i \partial_i^*$, the symplectic leaves are the integral manifolds of the 1-form $\sum a_i dx_i$. They are 2-tori, cylinders, or planes according to whether the rational vector space $\mathbf{Q}a_1 + \mathbf{Q}a_2 + \mathbf{Q}a_3$ has dimension 1, 2, or 3. We will refer to this dimension as the *class* of the Poisson structure π.

For a structure of class 1, the symplectic leaves all have the same finite volume, from which it follows that $\mathcal{S}$ is contained in $\{-1, 1\}$. In fact, when the class is 1, some real multiple of (a_1, a_2, a_3) has relatively prime integer entries. This 3-vector can be taken as one column of a matrix in $GL_3(\mathbf{Z})$, which defines an automorphism of the torus transforming π into a multiple of $\partial_2 \wedge \partial_3$. The map $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ is then a self-similarity with similarity ratio -1. Thus the scaling group $\mathcal{S}$ is equal to $\{-1, 1\}$ for any structure of class 1. In particular, any such structure is *reversible* in the sense that -1 belongs to the scaling group. The group $\mathcal{G}$ itself is quite large; it is the semidirect product of the diffeomorphisms of the circle of symplectic leaves (all contained in $\mathcal{G}_0$) by the group of maps from the circle to the area preserving diffeomorphisms of a 2-torus (containing in particular the constant map to $(x_2, x_3) \mapsto (x_2, -x_3)$). It is an extension of $\mathcal{G}_0$ by $\mathbf{Z}_2$.

Having disposed of the class 1 case, we will henceforth restrict our attention to structures of classes 2 and 3, which are much more rigid by virtue of the density of their symplectic leaves. In fact, each such structure is *uniquely unimodular* in the sense that it admits an invariant measure which is unique up to a constant multiple. According to Section 9 of [6], a uniquely unimodular Poisson manifold (M, π) with finite invariant volume admits a well defined *normalized fundamental cycle* $[\pi]$ in $H_2(M, \mathbf{R})$. If ϕ is a self-similarity of (M, π) with similarity ratio ρ, then the induced map $[\phi]_2$ on the second homology satisfies $[\phi]_2([\pi]) = \rho[\pi]$, i.e. $[\pi]$ is an eigenvector of $[\phi]_2$ with eigenvalue ρ.

To apply the general ideas above to the case of the 3-torus, we will use the natural identifications (valid for any torus)

$$H_2 \cong (H^2)^* \cong (\text{constant 2-forms})^* \cong \text{constant bivector fields},$$

under which the fundamental class $[\pi]$ becomes identified with π itself. Also, $[\phi]_2$ is the second exterior power $\wedge^2[\phi]_1$ of the induced map on $H_1 \cong$ constant vector fields. Finally, we identify H_1 and H_2 with $\mathbf{R}^3$, so that $[\phi]_1$ and $[\phi]_2$ are represented by 3×3 matrices B and A respectively, which belong to $GL_3(\mathbf{Z})$. In fact, in dimension 3, $A = \wedge^2 B$ is simply the classical adjoint $(\det A)({}^tA)^{-1}$, which belongs to $SL_3(\mathbf{Z})$. (In other words, the second exterior power of a 3-dimensional vector space carries a natural orientation!) We also note that the map $A \mapsto (\det A)({}^tA)^{-1}$ is surjective with kernel $\{-I, I\}$.

As a consequence of this analysis, we have the following reduction of our problem for classes 2 and 3.

Definition 1. The *projective stabilizer* of a vector $\mathbf{a} \in \mathbf{R}^n$ is the subgroup of $SL_n(\mathbf{Z})$ having $\mathbf{a}$ as an eigenvector; i.e. it is the stabilizer $SL_n(\mathbf{Z})^{<\mathbf{a}>}$ of the line $< \mathbf{a} >$ through $\mathbf{a}$ for the action of $SL_n(\mathbf{Z})$ on $\mathbf{RP}^{n-1}$. The group of possible eigenvalues (naturally isomorphic to the quotient of the projective stabilizer by the stabilizer $SL_n(\mathbf{Z})^{\mathbf{a}}$ of $\mathbf{a}$ itself) is the *scaling group* $\mathcal{S}(\mathbf{a}) \subseteq \mathbf{R}^\times$ of $\mathbf{a}$.

Proposition 1. *Let $\pi = \sum a_i \partial_i^*$ be a Poisson structure of class 2 or 3 on $\mathbf{T}^3$. Then $\mathcal{S}(\mathbf{T}^3, \pi) = \mathcal{S}(a_1, a_2, a_3)$.*

The question of reversibility is rather easily disposed of.

Proposition 2. *A Poisson structure is reversible if and only if it is of class 1 or 2.*

Proof. We have already seen that a structure of class 1 is reversible. Supposing now that the structure is of class 2 or 3, we will apply Proposition 1. If -1 is an eigenvalue of $A \in SL_3(\mathbf{Z})$ with eigenvalue -1, then the kernel of the integer matrix $A + I$ has a basis of rational eigenvectors. If -1 is a simple eigenvalue, any eigenvector is a multiple of a rational vector, and the corresponding Poisson structure is of class 1. If -1 is a double eigenvalue, then the third eigenvalue must be 1 since A is unimodular. There is an eigenvector for the eigenvalue 1 with rational entries, and hence one with relatively prime integer entries. It follows that A is conjugate in $SL_3(\mathbf{Z})$ to a matrix of the form

$$A = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}.$$

Since -1 is still a double eigenvalue, the matrix must in fact have the form

$$A = \begin{bmatrix} 1 & a & b \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now we may assume (using the torus automorphism corresponding to the conjugating matrix) that A had this triangular form to begin with. If (x, y, z) is an eigenvector with eigenvalue -1, then $x = -\frac{1}{2}(ay + bz)$; since a and b are integers, the corresponding Poisson structure is of class 1 or 2.

For the converse, assuming that the Poisson structure has class 2, its cylindrical symplectic leaf contains a circle which can be transformed by an element of $SL_3(\mathbf{Z})$ to the standard "first coordinate circle." This means that $\mathbf{a} = (0, a_2, a_3)$, which is reversed by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \ \blacksquare$$

3. An algebraic lemma. The analysis of class 2 structures in the proof above implies that the scaling group of such a structure is equal to the scaling group of a vector in $\mathbf{R}^2$ for the action of $GL_2(\mathbf{Z})$. Thus the structures of class 2 and class 3 can be studied in a very similar way. The following lemma will be useful in both cases. Although it could be derived from the results in Section 2 of [2], we prefer to give a direct, elementary proof.

LEMMA 1. *Assume that the characteristic polynomial of $A \in GL_n(\mathbf{Z})$ is irreducible over $\mathbf{Q}$. If $B \in GL_n(\mathbf{Z})$ has at least one eigenvector in common with A, then B has the same eigenvectors as A and is a rational polynomial in A, in particular, $AB = BA$.*

Furthermore, if B has a rational eigenvalue, then B is a scalar matrix, and hence, $B = kI$ for some $k \in \{\pm 1\}$.

PROOF. If λ is an eigenvalue of A, the last row of $A - \lambda I$ is a linear combination of the others, so to find an eigenvector

$${}^t[x_1 \cdots x_n]$$

it suffices to solve the system

$$\begin{bmatrix} a_{11} - \lambda & \cdots & a_{1n'} \\ \vdots & \ddots & \vdots \\ a_{n'1} & \cdots & a_{n'n'} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n'} \end{bmatrix} = -x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{n'n} \end{bmatrix} \tag{1}$$

where $n' = n - 1$.

Since the characteristic polynomial of A is irreducible, λ is not a root of any polynomial of degree n', so

$$D := \begin{vmatrix} a_{11} - \lambda & \cdots & a_{1n'} \\ \vdots & \ddots & \vdots \\ a_{n'1} & \cdots & a_{n'n'} - \lambda \end{vmatrix} \neq 0.$$

Taking $x_n = D$ in (1), we get by Cramer's rule a unique solution vector whose entries are polynomials of degree $n-1$ in λ and the matrix entries a_{ij}. (The last entry of the solution is $x_n = D$.) It follows that we get an eigenvector of the form

$$^t\left[p_1(\lambda)\cdots p_{n'}(\lambda)p_n(\lambda)\right]$$

where the same polynomials $p_j \in \mathbf{Z}[x]$ work for all the eigenvalues λ_i of A. Thus, an eigenvector for each λ_i is given by

$$\mathbf{e}_i = {}^t\left[p_1(\lambda_i)\cdots p_n(\lambda_i)\right].$$

Now let $B \in GL_n(\mathbf{Z})$ have $\mathbf{e}_i$ as an eigenvector for some i. The eigenvector condition can be written as $(B\mathbf{e}_i)\wedge \mathbf{e}_i = 0$, which when written in terms of the p_j's is a set of $\binom{n}{2}$ polynomial equations with integer coefficients, of degree $(2n-2)$, all satisfied by λ_i. Since any polynomial with integer coefficients having λ_i as a root is divisible by the (irreducible) characteristic polynomial of A, such a polynomial has *all* of the $\lambda_j{}'$s as roots; hence $B\mathbf{e}_j \wedge \mathbf{e}_j = 0$ for each j, and each $\mathbf{e}_j$ is an eigenvector for B. Thus B commutes with A.

To show that $B \in \mathbf{Q}[A]$, we note first that at least one p_j is not identically zero and hence, since its degree is less than n, has no λ_i as a root. The eigenvalue equation $B\mathbf{e}_j = \rho_j \mathbf{e}_j$ implies that

$$\sum B_{jk}p_k(\lambda_i) = \rho_i p_j(\lambda_i),$$

so $\rho_i = \sum B_{jk}p_k(\lambda_i)/p_j(\lambda_i)$ is a rational function q/r of λ_i with integer coefficients independent of i. Hence $B = q(A)/r(A)$, and since $\mathbf{Q}[A]$ is a field (the minimal polynomial of A being irreducible), $B \in \mathbf{Q}[A]$.

Finally, if some ρ_i is rational, the polynomial $q - \rho_i r$ has rational coefficients and degree $n-1$, and has λ_i as a root, so $q - \rho_i r$ is identically zero, and hence q/r is constant, so all ρ_i are equal, and B is a scalar matrix. ∎

4. Structures of class 3. Let $\pi = \sum a_i \partial_i^*$ be a Poisson structure of class 3, and suppose that $\mathcal{S}(a_1, a_2, a_3)$ is nontrivial; i.e. $A\mathbf{a} = \rho\mathbf{a}$ for some $\mathbf{a} \in SL_3(\mathbf{Z})$, $\rho \neq 1$. If A had a rational eigenvalue, it would have to be ± 1, and then A would have class 1 or 2, so we can assume that the characteristic polynomial of A is irreducible over $\mathbf{Q}$. By Lemma 1, the projective stabilizer of A consists of those matrices in $SL_3(\mathbf{Z})$ which are rational polynomials in A. That is, if $B \in SL_3(\mathbf{Z})$ shares one of the eigenvectors of A, Lemma 1 implies that

$$B = (r_0 I + r_1 A + r_2 A^2)/r_3$$

for some integers r_0, r_1, r_2 and $r_3 \neq 0$, so that B belongs to the field $\mathbf{Q}[A]$ generated by A, which is an algebraic number field. Since we assume that $B \in M_3\mathbf{Z}$, its characteristic polynomial has integer coefficients, and hence B is an algebraic integer. From the definition (see [3]) of the norm operator N of the number field $\mathbf{Q}[A]$ as a product of conjugates, we have

$$N(B) = \prod_{j=1}^{3}(r_0 + r_1\lambda_j + r_2{\lambda_j}^2)/r_3\ .$$

Since A and B are simultaneously diagonalizable, $(r_0 + r_1\lambda_j + r_2{\lambda_j}^2)/r_3$ $(j = 1, 2, 3)$ are the eigenvalues of B, so $\prod_{j=1}^{3}(r_0+r_1\lambda_j+r_2{\lambda_j}^2)/r_3 = \det(B)$, and we get $N(B) = \det(B)$. Thus, our $B \in SL_3(\mathbf{Z})$ is a unit with norm 1.

According to Dirichlet's Theorem on the units in number fields [3], the group of units is $\mathcal{U} \times \mathbf{Z}^{r+s-1}$ where

$$r = \text{the number of real roots of } \chi_A, \quad 2s = \text{the number of complex roots of of } \chi_A$$

and $\mathcal{U}$ ={roots of 1 in the number field}. Since the degree of the characteristic polynomial is 3 (and so odd), the finite cyclic group $\mathcal{U}$ is just $\{\pm 1\}$. If we assume that the eigenvalues of A are all real, the rank of the group of units is 2. If a generator g of the free abelian group has norm -1, then $-g$ is a generator of norm 1. Thus, we can get a set of generators of norm 1. Since norm 1 is equivalent to determinant 1, the arrive at the following description of the scaling group of a Poisson structure of class 3.

Proposition 3. *Let $A \in SL_3(\mathbf{Z})$ be diagonalizable with (real) irrational eigenvalues. The subgroup of $B \in SL_3(\mathbf{Z})$ which shares one fixed eigenvector $\mathbf{e}$ of A is written as $\{P^m Q^n \mid m, n \in \mathbf{Z}\}$, where $P, Q \in SL_3(\mathbf{Z})$ are given by rational polynomials of A as*

$$P = \frac{p_0 I + p_1 A + p_2 A^2}{p_3}, \qquad Q = \frac{q_0 I + q_1 A + q_2 A^2}{q_3}.$$

The set $\mathcal{S}$ of eigenvalues is equal to $\{{\rho_1}^m {\rho_2}^n \mid m, n \in \mathbf{Z}\}$ where

$$\rho_1 = \frac{p_0 \lambda + p_1 \lambda + p_2 \lambda^2}{p_3}, \qquad \rho_2 = \frac{q_0 \lambda + q_1 \lambda + q_2 \lambda^2}{q_3}.$$

Remark 1. We know of no general way to specify P and Q. The problem is close to that of determining which algebraic integers in $\mathbf{Q}[A]$ are matrices with integer entries. Here are two examples.

(1) Take the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. As generators of the group of units, we can take $g_1 = -3I + A$ and $g_2 = 2I - A$. Then we may set $P = g_1$ and $Q = g_2$. We see that $A = P^{-2}$.

(2) Take $A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -1 & -1 \\ -2 & -2 & -3 \end{bmatrix}$. As generators of the group of units, we can take $(A - I)/2$ and $(-A - I)/2$. They are not integer matrices, and we have to set

$$P = ((A - I)/2)\,((-A - I)/2)^2 = A \text{ and}$$
$$Q = ((A - I)/2)^4\,((-A - I)/2) = (3A^2 - 8A - I)/2.$$

We consider next the case where two eigenvalues of A are non-real. In this case, the group of units is free of rank 1, and we get a result similar to that in Proposition 3.

Proposition 4. *Let $A \in SL_3(\mathbf{Z})$ have one irrational and two non-real eigenvalues. Let $\mathbf{e}$ be a eigenvector for a non-real eigenvalue, and consider the Poisson structure π given by $\pi = \sqrt{-1}\mathbf{e} \wedge \overline{\mathbf{e}}$. The subgroup $\mathcal{G}$ of $B \in SL_3(\mathbf{Z})$ with the property*

$$\wedge^2(B)(\pi) = \rho\pi$$

is written as $\{P^m \mid m \in \mathbf{Z}\}$*, where* $P \in SL_3(\mathbf{Z})$ *is a rational polynomial in* A*:*

$$P = \frac{p_0 I + p_1 A + p_2 A^2}{p_3}.$$

The set $\mathcal{S}$ *of eigenvalues is equal to* $\{\rho^m \mid m \in \mathbf{Z}\}$ *where*

$$\rho = \frac{p_0\lambda + p_1\lambda + p_2\lambda^2}{p_3}$$

and λ *is the real eigenvalue of* A*.*

Remark 2. (1) Take the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$. As generator of the group of units, we can take A itself and $\mathcal{G} = \{A^m \mid m \in \mathbf{Z}\}$.

(2) Take $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -8 \\ 0 & 1 & 0 \end{bmatrix}$. As generator of the group of units, we can take $(A^2 - 2A + 2I)/5$, which is not a $\mathbf{Z}$-matrix, and $A = \big((A^2 - 2A + 2I)/5\big)^2$. Now we see that

$$\mathcal{G} = \{A^m \mid m \in \mathbf{Z}\}.$$

(3) If we start with A^2 for A in (2), then $\mathcal{G}$ is generated by A, not by A^2.

5. Structures of class 2. The analysis of structures of class 3 is similar to that for class 3, except that we only need to consider 2×2 matrices. The relevant number field is now quadratic, and our conclusion is the following.

Proposition 5. *If a Poisson structure of class 2 is a nontrivial eigenvector for a matrix* $A \in SL_3(\mathbf{Z})$*, then the scaling group of this structure has the form* $\{\pm(\rho^k)^m \mid m \in \mathbf{Z}\}$ *where* $\rho = (r_0 + r_1\lambda)/r_2$ *is the fundamental unit of the number field* $\mathbf{Q}[\lambda]$ *of* $\lambda^2 - (\mathrm{traceA} - \varepsilon)\lambda + \varepsilon = 0$ *and* k *is some integer.*

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