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POISSON STRUCTURES ON $\mathbb{R}^{2N}$ HAVING ONLY TWO SYMPLECTIC LEAVES: THE ORIGIN AND THE REST

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Abstract. Examples of Poisson structures with isolated non-symplectic points are constructed from classical r-matrices.

1. Introduction (*by A. Weinstein*)**.** Several years ago, when I described the modular vector field of a Poisson structure [1] to Dusa McDuff and showed her the example of the structure $\{x, y\} = x^2 + y^2$, for which the modular flow consists of rotations, she asked me whether I knew of similar examples in higher dimensions. In fact, I knew of no Poisson structure in dimension bigger than 2 with an isolated non-symplectic point, and I raised the question of finding such structures in the original version of the survey article [2]. In November, 1997, Stan Zakrzewski visited Berkeley, and I showed him this manuscript. Within a few days, he showed me how his general construction [3] involving r-matrices could be used to produce many examples of such structures.

The examples given by formulas (7) and (8) in the note below turn out to be simply the images of the standard symplectic structures on $\mathbb{R}^{2n}$ under inversion through the unit sphere. This allows one to transplant Stan's examples to the $2n$-dimensional spheres. I leave it as an exercise for the reader to compute the modular vector fields of these examples.

It seems very appropriate to include Stan's short note in the proceedings of this symposium dedicated to his memory. It is presented below with only very minor editing, including updated references.

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The paper is in final form and no version of it will be published elsewhere.

2. Examples of Poisson structures derived from r-matrices. In a recent survey on Poisson geometry [2], Alan Weinstein posed the question of the existence of a Poisson structure on $\mathbb{R}^{2n}$, $n \geq 2$, for which the complement of the origin is a symplectic leaf. In this note, we construct such Poisson structures for each $n \geq 2$.

Clearly, in order to construct an example it is natural to look for Poisson structures with some kind of symmetry. One could think for example about structures which are invariant with respect to the action of $SO(2n)$. However, as is easily seen, the only $SO(N)$-invariant bivector field on $\mathbb{R}^N$ for $N > 3$ is zero.

An entirely different situation arises when we search for $SU(n)$-invariant Poisson structures on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. As discussed already in [3] (cf. formula (48) therein), smooth $SU(n)$-invariant bivector fields on $\mathbb{C}^n$ are exactly those of the form

$$\Delta = \frac{1}{2} a\pi_0 + \frac{1}{2} bz \wedge Jz \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \tag{1}$$

where J is the multiplication by the imaginary unit i,

$$\pi_0 = 2i \sum_k \partial_k \wedge \overline{\partial}_k$$

is the canonical constant bi-vector on $\mathbb{C}^n = \mathbb{R}^{2n}$, and $a = a(\| z \|^2)$, $b = b(\| z \|^2)$ are arbitrary real smooth functions of $\|z\|^2 := |z_1|^2 + \ldots + |z_n|^2$. From the proof of Lemma 7.2 in [3], it is easily seen that Δ is a Poisson structure if and only if

$$aa' + b(a - a't) = 0 \tag{2}$$

(this replaces (49) of [3]). Here $t \equiv \|z\|^2$ and prime indicates differentiation with respect to the variable t. The corresponding Poisson brackets are then given by

$$\{z_j, z_k\} = 0 \tag{3}$$

$$\{z_j, \overline{z}_k\} = i(a\delta_{jk} - bz_j\overline{z}_k) \tag{4}$$

(δ_{jk} is the Kronecker delta). For $z \neq 0$, the determinant of the matrix

$$a\delta_{jk} - bz_j\overline{z}_k = -bt\left(\frac{z_j\overline{z}_k}{\|z\|^2} - \frac{a}{bt}\delta_{jk}\right)$$

is not zero if and only if

$$\frac{a}{bt} \notin \{0, 1\} \tag{5}$$

(the spectrum of a projection). The simplest way to guarantee (5) is to require

$$a = \gamma bt,$$

where γ is a constant different from 0 and 1. In this case, (2) has the following solution

$$b = t^m, \qquad a = \frac{m}{m+1} t^{m+1} \qquad (m = 1, 2, \ldots). \tag{6}$$

For $m = 1$ we obtain the following Poisson structure:

$$\{z_j, z_k\} = 0 \tag{7}$$

$$\{z_j, \overline{z}_k\} = i\, \|z\|^2 \left(\frac{\|z\|^2}{2}\delta_{jk} - z_j\overline{z}_k\right). \tag{8}$$

This is the solution of the original problem. Our simplest example is polynomial of order 4. (Taking arbitrary natural m in (6) we obtain a polynomial structure of order $2(m+1)$.)

3. Remarks

1. Another (more complicated) source of examples is the family of Poisson structures on $\mathbb{C}^n$ such that the action of $SU(n)$ is a Poisson action (for the standard Poisson structure on $SU(n)$). In this case we deal with equation (49) and Poisson brackets (52)-(54) of [3] (unfortunately, there is a misprint in (54) of [3]; it should read $\{z^j, \overline{z}^j\} = i\sum_k \operatorname{sgn}(j-k)\cdot|z^k|^2 + ia - ib|z^j|^2$). For $n = 2$, one of the simplest Poisson structures which solves the problem this way is given by

$$\{z_1, z_2\} = iz_1z_2 \tag{9}$$

$$\{\overline{z}_1, z_2\} = ib\overline{z}_1z_2 \tag{10}$$

$$\{\overline{z}_1, z_1\} = i[b\overline{z}_1z_1 + \overline{z}_2z_2 - a] \tag{11}$$

$$\{\overline{z}_2, z_2\} = i[b\overline{z}_2z_2 - \overline{z}_1z_1 - a], \tag{12}$$

where

$$a(t) := t + t^2, \qquad b(t) := 3 + 2t. \tag{13}$$

2. Can one construct a quantum analogue of our simplest example?

References

[1] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. 23 (1997) 379–394.

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[3] S. Zakrzewski, *Phase spaces related to standard classical r-matrices*, J. Phys. A: Math. Gen. 29 (1996), 1841–1857.