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THE THEORY OF DIFFERENTIAL INVARIANCE AND INFINITE DIMENSIONAL HAMILTONIAN EVOLUTIONS

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Dedicated to the memory of Staszek Zakrzewski

Abstract. In this paper we describe the close relationship between invariant evolutions of projective curves and the Hamiltonian evolutions of Adler, Gel'fand and Dikii. We also show how KdV evolutions are related as well to invariant evolutions of projective surfaces.

1. Introduction. Consider one parameter families of projective curves $\phi(t,x) : \mathbf{R}^2 \to \mathbf{RP}^{n-1}$. Consider the projective action of $\mathrm{SL}(n,\mathbf{R})$ on $\mathbf{RP}^{n-1}$. We ask ourselves the following question: what is the most general form of an evolution on t for ϕ of the form

$$\phi_t = F(\phi, \phi', \phi'', \ldots), \tag{1.1}$$

invariant under the $\mathrm{SL}(n,\mathbf{R})$ action? (that is, for which evolutions does the $\mathrm{SL}(n,\mathbf{R})$ action map solutions into solutions?). Here $' = \frac{d}{dx}$ and $_t = \frac{d}{dt}$. This question was answered successfully in [6] where a formula was found for such a general evolution, using the theory of differential invariance. In fact, one can easily show that any invariant evolution of projective curves of this kind is always of the form

$$\phi_t = \eta \mathcal{I}$$

where $\mathcal{I}$ is a vector of *differential invariants* for the action, and where η is certain nondegenerate matrix whose columns are *relative invariants*. The same question can be posed for surfaces on $\mathbf{RP}^{n-1}$, and one can take an identical approach to it.

At the beginning of this century a description of general differential invariants of projective curves under the $\mathrm{SL}(n,\mathbf{R})$ action was given by Wilczynski in [12]. An explicit formula for the matrix η was found in [6]. In the case of surfaces, a description of general differential invariants of projective surfaces under the $\mathrm{SL}(n,\mathbf{R})$ action was found in [9].

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These concepts belong to the theory of Klein geometries and differential and geometric invariants which had its high point towards the end of last century until the appearance of Cartan's approach to differential geometry. Differential invariants arise also in equivalence problems, where one faces the question of equivalence of two geometrical objects under the action of certain group. For example, given two curves on the plane, when are they equal up to an Euclidean motion?, or up to parametrization? One tries to answer those questions in terms of invariants, or expressions depending on the objects that do not change upon the action. If two objects are to be equivalent, their invariants ought to be the same. If, besides, these invariants are functions on some jet space (in the case of curves, if they depend on the curve and the derivatives with respect to the parameter x), then we call them differential invariants. If they classify the object up to the action of the group so that they will generate any other differential invariant, they are called *basic* differential invariants. In the case of curves on the plane under the action of the Euclidean group, the basic differential invariant is the Euclidean curvature. For an account on the modern state of this subject, see Olver's book [10].

A subject which seems unrelated to this theory is the study of Hamiltonian structures of partial differential equations, integrability and, in general, infinite dimensional Poisson Geometry. Is to this area that the so–called *Adler–Gel'fand–Dikii Hamiltonian structures* belong. They were defined originally by Adler ([1]) as a second Hamiltonian structure for the so–called higher dimensional KdV differential equations, to be used as a tool to prove their complete integrability. He could not prove Jacobi's identity for these brackets, but Gel'fand and Dikii proved it in [5]. Since then several definitions have been provided by different authors, aiming for a more intuitive and comprehensive way of defining these brackets than the original one. Equivalent definitions have been given by Kupershmidt and Wilson in [7] and by Drinfel'd and Sokolov in [2]. In this hierarchy of Poisson brackets the lowest dimensional case corresponds to the Lie–Poisson bracket on the dual of the Virasoro algebra, the only instance in which the bracket is linear.

In this paper I give a brief description of the theory of differential invariance needed to describe the invariant evolution (1.1), and provide its explicit formula. I will also give a short definition of the Adler–Gel'fand–Dikii bracket, the original one of Adler. At the end of section 2 I will explain how, surprisingly enough, these two evolutions are essentially the same under a 1–to–1 (up to the action of $\mathrm{SL}(n, \mathbf{R})$) correspondance between projective curves and Lax operators. This result was conjectured in [6] and finally proved in [8]. Invariant evolutions were also linked to KdV evolutions in [11] and [13] from totally different points of view.

In the last section, section 3, I show how, in the same fashion as is done for curves, one can also describe invariant evolutions for projective surfaces, under the action of $\mathrm{SL}(n, \mathbf{R})$. We focus on the case of families of maps $s(t, x, y) : \mathbf{R}^3 \to \mathbf{RP}^{n-1}$, the straight generalization of the KdV case for projective curves. We prove how, as it happens for curves, invariant evolutions of these maps are also related to a family of Hamiltonian evolutions: one can obtain a Lie-Poisson bracket on the dual of the Virasoro algebra on each linear direction on the (x, y) plane. We also comment on the case of the KP equation, one of the best known examples of complete integrability in two independent variables.

2. Invariant evolutions of projective curves and KdV Hamiltonian evolutions. Before analyzing our case we will define briefly the situation in which we will be working. Let M me an m-dimensional manifold. Consider p-dimensional parametrized submanifolds, $u : \mathbf{R}^p \to M$. Let G be an r-dimensional Lie group acting smoothly on M (the action will not affect the parameter $x \in \mathbf{R}^p$). Let $\mathcal{J}^{(n)} = \mathcal{J}^{(n)}(\mathbf{R}^p, M)$ be the n^{th} order jet bundle (that is, the equivalence classes of submanifolds modulo n^{th} order contact). Since G preserves the order of contact, there exists an induced action of G on the jet bundle known as its n^{th} *prolongation*, which is defined as

$$\begin{gathered} G^{(n)} \times \mathcal{J}^{(n)} \to \mathcal{J}^{(n)} \\ (g, u_K) \to (gu)_K \end{gathered}$$

for any differential subindex K.

Definitions 1. Given any infinitesimal generator of the action, v, we define its n^{th} *order prolongation* as the infinitesimal generator of the prolongation action on $\mathcal{J}^{(n)}$. In fact, if $v = \sum_{i=1}^{n-1} \nu_i(\phi)\frac{\partial}{\partial \phi_i}$, then one can check that its n^{th} order prolongation, an element on the tangent of $\mathcal{J}^{(n)}$, is given by the formula

$$\text{pr}(v) = \sum_K \sum_{i=1}^{n-1} D_K \nu_i(\phi) \frac{\partial}{\partial \phi_i^K} \tag{2.1}$$

where the first sum is over all differential subindices K of order less or equal to n, and where D indicates the total derivative (see [10]).

2. An n^{th} *order differential invariant* is a function $I : \mathcal{J}^{(n)} \to \mathbf{R}$ which is invariant under the n^{th} prolongation action of G.

2.1. *Invariant evolutions of projective curves.* Consider one parameter families of projective curves $\phi(t, x) : \mathbf{R}^2 \to \mathbf{RP}^{n-1}$ such that the Wronskian of their derivatives does not vanish, that is, if $\phi = (\phi_1, \phi_2, \ldots, \phi_{n-1})$, then

$$W(\phi_1', \phi_2', \ldots, \phi_{n-1}') = \begin{vmatrix} \phi_1' & \phi_2' & \ldots & \phi_{n-1}' \\ \vdots & \vdots & \ldots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \ldots & \phi_{n-1}^{(n-1)} \end{vmatrix} \neq 0.$$

The need for this condition will be clear later on. As we indicated in the introduction, we want to find a formula for the most general evolution of these families of the form (1.1), which is invariant under the $\mathrm{SL}(n, \mathbf{R})$ projective action on $\mathbf{RP}^{n-1}$.

Following [10], equation (1.1) is invariant if and only if the following vector equality holds

$$\text{pr}(v)(\phi_t) = \text{pr}(v)(F)$$

for all infinitesimal generators v of the $\mathrm{SL}(n, \mathbf{R})$ action. One can easily check that, if $v = \sum_{i=1}^{n-1} \nu_i(\phi)\frac{\partial}{\partial \phi_i}$, then

$$\text{pr}(v)(\phi_t) = D_t \nu_{\phi_t = F} = \frac{\partial \nu}{\partial \phi} F, \tag{2.2}$$

and, therefore, F must satisfy the relationship

$$\text{pr}(v)(F) = \frac{\partial \nu}{\partial \phi} F \tag{2.3}$$

for all infinitesimal generators of the action v. In such a case we say that F is a *relative differential invariant with weight* $\frac{\partial\nu}{\partial\phi}$. Thus, our problem is equivalent to that of finding relative invariants with certain weight. To simplify our task we apply the following Theorem.

THEOREM 2.1 ([10]). *Every relative differential invariant with weight* $\frac{\partial\nu}{\partial\phi}$ *must be of the form* $\eta\mathcal{I}$*, where* η *is a nondegenerate matrix whose columns are relative invariants with weight* $\frac{\partial\nu}{\partial\phi}$*, and where* $\mathcal{I}$ *is any general differential invariant vector.*

This way, the problem splits into two: first find the form of the most general differential invariant, and second find a formula for the nondegenerate matrix η.

The description of a general differential invariant of projective curves under the $\mathrm{SL}(n,\mathbf{R})$ action is due to Wilczynski ([12]) and can be summarized as follows. Let's lift ϕ uniquely to a family of curves on $\mathbf{R}^n$ with Wronskian equals 1. That is, $\xi:\mathbf{R}^2\to\mathbf{R}^n$ is a unique lift defined by

$$\xi = W(\phi_1',\ldots,\phi_{n-1})^{-\frac{1}{n}}(1,\phi).$$

Let the vector $u=(u_0,\ldots,u_{n-2})$ be defined through the relationship

$$y^{(n)}+u_{n-2}y^{(n-2)}+\ldots+u_1y'+u_0 = \begin{vmatrix} y^{(n)} & \xi_1^{(n)} & \ldots & \xi_n^{(n)} \\ y^{(n-1)} & \xi_1^{(n-1)} & \ldots & \xi_n^{(n-1)} \\ \vdots & \vdots & \ldots & \vdots \\ y' & \xi_1' & \ldots & \xi_n' \\ y & \xi_1 & \ldots & \xi_n \end{vmatrix} \tag{2.4}$$

that is, u is formed by the coefficients of the unique differential equation of the form (2.4) which has as solutions each one of the components of the lift ξ (this is normally refered to as ξ is a *solution curve* for the equation). Wilczynski's theorem states that u provides a set of basic differential invariants for the action.

THEOREM 2.2 ([12]). *Let* $\mathcal{I}$ *be a differential invariant of curves on* $\mathbf{RP}^{n-1}$ *for the projective action of* $\mathrm{SL}(n,\mathbf{R})$*. Then* $\mathcal{I}$ *must be a function of* u_i*,* $i=0,\ldots,n-2$ *and of their derivatives (of any order) with respect to the parameter* x*.*

This would solve the first part of the problem. To find the matrix η in Theorem 2.1 we need some definitions.

DEFINITIONS 1. Define $\omega_{i_1\ldots i_k}$ to be the following determinant

$$\omega_{i_1\ldots i_k} = \begin{vmatrix} \phi_1^{(i_1)} & \ldots & \phi_1^{(i_k)} \\ \vdots & \ldots & \vdots \\ \phi_k^{(i_1)} & \ldots & \phi_k^{(i_k)} \end{vmatrix}$$

2. Let q_k^r be defined as the quotient of determinants

$$q_k^r = \frac{\omega_{1\ldots\hat{r}\ldots k}}{\omega_{1\ldots(k-1)}}$$

where $\hat{r}$ indicates that the index r has been removed.

The following theorem gives the solution as to the form of η.

Theorem 2.3 ([6]). *Let $\eta = \Phi(Id + A)$, where Id is the Identity matrix, Φ is the matrix*

$$\Phi = \begin{pmatrix} \phi_1' & \dots & \phi_1^{(n-1)} \\ \vdots & \dots & \vdots \\ \phi_{n-1}' & \dots & \phi_{n-1}^{(n-1)} \end{pmatrix}$$

and where A is strictly upper triangular, $A = (a_i^j)$, with $a_i^j = 0$ whenever $j \le i$ and

$$a_i^j = (-1)^{j-i} \frac{\begin{pmatrix} j \\ i \end{pmatrix}}{\begin{pmatrix} n \\ j-i \end{pmatrix}} q_n^{n-(j-i)}$$

whenever $i < j$. Then, each column of η is a relative differential invariant for the $\mathrm{SL}(n, \mathbf{R})$ *projective action with weight $\frac{\partial \nu}{\partial \phi}$, for all $v = \nu \frac{\partial}{\partial \phi}$ infinitesimal generators of the action. Obviously η is nondegenerate.*

As a consequence of these theorems, we obtain the final result of this subsection.

Corollary 2.4 ([6]). *If an evolution of projective curves of the form (1.1) is invariant under the* $\mathrm{SL}(n, \mathbf{R})$ *projective action, then the evolution must be of the form*

$$\phi_t = \Phi(Id + A)\mathcal{I} \tag{2.5}$$

where Φ and A are defined as in Theorem 2.3, and where $\mathcal{I}$ is a vector function of u_i, $i = 0, \dots, n-2$ as in (2.4) and their derivatives with respect to the parameter.

Example. In the case $n = 2$, Wilczynski's invariant is found as follows. We lift the curve $\phi : \mathbf{R} \to \mathbf{RP}^1$ uniquely to $\xi = (\xi_1, \xi_2) = ((\phi')^{-\frac{1}{2}}, (\phi')^{-\frac{1}{2}}\phi)$. Then we pose the differential equation

$$y'' + uy = \begin{vmatrix} y'' & (\xi_1)'' & (\xi_2)'' \\ y' & (\xi_1)' & (\xi_2)' \\ y & \xi_1 & \xi_2 \end{vmatrix} = y'' + \frac{1}{2} S(\phi) y$$

so that the differential invariant is the Schwarzian derivative of ϕ, $S(\phi) = \frac{\phi'''\phi' - \frac{3}{2}(\phi'')^2}{(\phi')^2}$. Now, the choices in (2.5) are $\Phi = \phi'$ and $A = 0$ so that the most general invariant evolution is $\phi_t = \phi' \mathcal{I}$, where $\mathcal{I}$ is any function depending on the Schwarzian derivative and its derivatives with respect to x. When $\mathcal{I} = S(\phi)$, we obtain the so–called Schwarzian KdV equation.

2.2. *The Adler–Gel'fand–Dikii Hamiltonian evolutions.* In this subsection I will give the definition of the Adler–Gel'fand–Dikii Poisson bracket. The definition below is the original one; even though it is not the most comprehensive, it is perhaps the shortest, making use of the formalism of Lax operators and pseudodifferential symbols. Definitions have also been given by Kupershmidt and Wilson ([7]) and by Drinfel'd and Sokolov ([2]), the latter being geometrically the most beautiful one.

Let $\mathcal{A}$ be the Fréchet manifold of *Lax operators* of the form

$$L = \frac{d^n}{dx^n} + u_{n-2} \frac{d^{n-2}}{dx^{n-2}} + \ldots + u_1 \frac{d}{dx} + u_0 \tag{2.6}$$

with smooth and periodic coefficients. Given a linear functional ℓ on $\mathcal{A}$, one can associate a symbol of the form $H = \sum_{i=1}^{n} h_i \partial^{-i}$, where h_i are all periodic, such that

$$\ell(L) = \ell_H(L) = \int_{S^1} \operatorname{res}(HL) dx,$$

where $\operatorname{res}(\sum a_i \partial^i) = a_{-1}$ is called the residue of the symbol. The AGD Poisson bracket is defined as

$$\{\ell_H, \ell_G\}(L) = \int_{S^1} \operatorname{res}(GV_H(L)) dx.$$

The Hamiltonian vector field $V_H(L)$ is defined as $V_H(L) = (LH)_+ L - L(HL)_+$, where $()_+$ selects the differential (positive) part of the operator. The coefficients h_i, $i = 1, \dots, n-1$ depend on the Fréchet differentiation of ℓ_H while h_n is determined once we ask V_H to be tangent to the manifold $\mathcal{A}$. This bracket determines a Hamiltonian evolution on $\mathcal{A}$ of the form

$$L_t = V_H(L) \tag{2.7}$$

which can indeed be written as a Hamiltonian evolution of the coefficients u_i, $i = 0, \dots, n-2$, of L. This evolution is normally referred to as the Adler–Gel'fand–Dikii evolution, or the second generalized KdV Hamiltonian evolution.

2.3. *The relationship between evolutions of curves and AGD.* Before describing how the two evolutions above are, up to $\mathrm{SL}(n, \mathbf{R})$, essentially the same evolution, I want to make a few comments that will put our minds to rest with respect to certain details.

Remark 1. If L as in (2.6) has periodic coefficients, any solution curve ξ has a monodromy $M \in \mathrm{SL}(n, \mathbf{R})$, that is, $\xi(x+T) = M\xi(x)$ for all x, where T is the period of L. (In fact, M is a conjugate of the transposed of the *Floquet matrix* associated to L.

Remark 2. If the initial condition $\phi(0,x)$ has monodromy $M \in \mathrm{SL}(n, \mathbf{R})$ and $\phi(t,x)$ is a unique solution of (2.5) with that initial condition, then ([6]) $\phi(t,x)$ has M as monodromy for all t.

Remark 3. If we impose on ξ the condition of having Wronskian equals one, then the correspondance between u, coefficients of the equation, and ξ solution curve is 1-to-1 up to an element of $\mathrm{SL}(n, \mathbf{R})$. It suffices to realize that ξ defines a fundamental matrix of solutions and any other solution curve will be ξ up to $\mathrm{SL}(n, \mathbf{R})$. This identically holds for the projectivization of ξ.

Theorem 2.3 ([8]). *Let $\phi(t,x) \in \mathbf{RP}^{n-1}$ be a solution of the invariant evolution (2.5)*

$$\phi_t = \Phi(\mathrm{Id} + A)\mathcal{I}$$

with monodromy $M \in \mathrm{SL}(n, \mathbf{R})$. Let $L(t,x)$ be a one parameter family of periodic Lax operators associated to $\phi(t,x)$ so that their coefficients are the projective curvatures of $\phi(t,x)$, that is, related to ϕ as in (2.4). Then, there exists an invariant matrix $\mathcal{M}$, such that, if

$$I = \mathcal{M} \begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}$$

and if $H = \sum_{i=1}^{n} h_i \partial^{-i}$, *then*

$$L_t = V_H(L),$$

so that, if $\mathcal{I}$ *satisfies certain integrability conditions, the evolution of* L *is the AGD Hamiltonian evolution with Hamiltonian symbol* H*.*

That is, there exists a matrix of invariants (thus depending on u and its derivatives) relating the invariant $\mathcal{I}$ and the Hamiltonian symbol H, and, under that relationship, both evolution are, essentially the same.

EXAMPLE. If $n = 2$, we have already seen that the most invariant evolution of ϕ is given by $\phi_t = \phi'\mathcal{I}$, where $\mathcal{I}$ is any function of the Schwarzian derivative $S(\phi)$ ad its derivatives with respect to the parameter. A straightforward calculation shows that, if $\phi_t = \phi'\mathcal{I}$, then

$$(S(\phi))_t = \frac{DS(\phi)}{D\phi}\phi_t = (\frac{d^3}{dx^3} + 2S(\phi)\frac{d}{dx} + S(\phi)')I$$

which is exactly the coadjoint action of the Virasoro algebra on its dual, the second KdV Hamiltonian evolution as far as $\mathcal{I}$ is the Hamiltonian. In particular, if $\mathcal{I} = S(\phi) = u$, then we obtain the KdV equation $u_t = u_{xxx} + 3uu_x$.

3. Invariant evolutions of projective surfaces. The next natural question is, of course, whether or not this is a property intrinsically tied to projective curves, or will we find similar properties in, say, projective surfaces.

The situation described at the beginning of section 2 is valid for any parametrized submanifold, and therefore we can apply it equally to the case of surfaces. Thus, consider one parameter families of projective surfaces $s(t, x, y) : \mathbf{R}^3 \to \mathbf{RP}^{n-1}$. We want to find, again, an evolution for s, invariant under the projective action of $\mathrm{SL}(n, \mathbf{R})$ and of the form

$$s_t = \hat{F}(s, s_x, s_y, s_{xx}, s_{xy}, \ldots)$$

where $\hat{F}$ is a function of s and its derivatives with respect to the parameters (x, y). Again, a short calculation as in (2.2) shows that $\hat{F}$ must necessarily be a relative differential invariant of the action with weight $\frac{\partial\nu}{\partial s}$, for all infinitesimal generators $v = \nu\frac{\partial}{\partial s}$ of the $\mathrm{SL}(n, \mathbf{R})$ action. (Notice that we denote the coordinates in $\mathbf{RP}^{n-1}$ as $s = (s_1, \ldots, s_{n-1})$ instead of the notation $\phi = (\phi_1, \ldots, \phi_{n-1})$ of the previous section. Of course, this is simply to be coherent with the notation of the coordinates in the jet bundle of parametrized surfaces and, hopefully, will not lead to confusion.)

Here also, we can apply Theorem 2.1 to split the problem into two fronts, the search for invariants and the search for relative invariants with the Jacobian of the components of infinitesimal generators as weights. But, unlike in the case of curves, the invariants of projective parametrized surfaces have not been known until fairly recently. In fact, in [9] we found a set of generating differential invariants for projective surfaces in $\mathbf{RP}^{n-1}$, under the action of $\mathrm{SL}(n, \mathbf{R})$, for any n. We also found all the algebraic relationships among the derivatives of the invariants. These are normally called the *syzygies* of the invariants. There are no syzygies in the case of projective curves, but they naturally appear in higher dimensions. The method used to find these invariants is a modification of Cartan's method of moving frames, more practical in many instances, that has been recently developed by Olver and Fels in [3] and [4]. The method bypasses many of the complications, most notably the process of normalization, inherent in the traditional approach. From now on I will focus on the case $n = 3$, the straightforward generalization of the KdV (nongeneralized) case.

If $n = 3$, assume that, as in the case of curves (for $n = 2$ we had the condition $\phi' \neq 0$ imposed on ϕ), we have the nondegeneracy condition

$$D = \begin{vmatrix} s_x^1 & s_y^1 \\ s_x^2 & s_y^2 \end{vmatrix} \neq 0,$$

where $s = (s^1, s^2) \in \mathbf{RP}^2$. The problem of finding a basic set of generating invariants is thus solved in [9]. We find 4 basic invariants given by

$$I_1 = \frac{1}{D}\begin{vmatrix} s_x^1 & s_{xx}^1 \\ s_x^2 & s_{xx}^2 \end{vmatrix} \qquad I_2 = \frac{1}{D}\begin{vmatrix} s_y^1 & s_{yy}^1 \\ s_y^2 & s_{yy}^2 \end{vmatrix} \tag{3.1}$$

$$I_3 = \frac{1}{D}\left(\begin{vmatrix} s_{xy}^1 & s_y^1 \\ s_{xy}^2 & s_y^2 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} s_x^1 & s_{yy}^1 \\ s_x^2 & s_{yy}^2 \end{vmatrix}\right) \qquad I_4 = \frac{1}{D}\left(\begin{vmatrix} s_x^1 & s_{xy}^1 \\ s_x^2 & s_{xy}^2 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} s_{xx}^1 & s_y^1 \\ s_{xx}^2 & s_y^2 \end{vmatrix}\right),$$

which again solves the first part of our problem. These basic differential invariants carry two syzygies, which will generate any other algebraic relationship among themselves and their derivatives. These are

$$\begin{aligned} 3(I_2)_{xx} - 2(I_3)_{yy} - 4(I_4)_{xy} &= 4I_4(I_3)_y + 6I_2(I_3)_x + 4(I_4^2)_x \\ &\quad + 6I_3(I_2)_x - 3I_2(I_1)_y - 3(I_1I_2)_y \\ 3(I_1)_{yy} - 2(I_4)_{xx} - 4(I_3)_{xy} &= 4I_3(I_4)_x + 6I_1(I_4)_y + 4(I_3^2)_y \\ &\quad + 6I_4(I_1)_y - 3I_1(I_2)_x - 3(I_1I_2)_x. \end{aligned}$$

Finding the matrix of relative invariants is also easy in this case.

PROPOSITION 3.1. *Let*

$$\mu = \begin{pmatrix} s_x^1 & s_y^1 \\ s_x^2 & s_y^2 \end{pmatrix}.$$

Then μ *is a nondegenerate matrix whose columns are relative differential invariants for the action of* $\mathrm{SL}(3, \mathbf{R})$ *on parametrized surfaces on* $\mathbf{RP}^2$*, with weight* $\frac{\partial \nu}{\partial s}$*, for all* $v = \nu\frac{\partial}{\partial s}$ *infinitesimal generators of the action.*

PROOF. Consider the generating set of infinitesimal generators of the action

$$v_i = \frac{\partial}{\partial s_i} \qquad v_i^j = s_i\frac{\partial}{\partial s_j} \qquad w_j = s_j(\sum_{i=1}^{n-1}\frac{\partial}{\partial s_i})$$

with $i, j = 1, 2$. Apply them to the columns of μ to obtain the relationship (2.3). ∎

And so, we obtain again that the most general invariant evolution of these surfaces (which perhaps is better to call maps, since they lie on $\mathbf{RP}^2$) is of the form

$$s_t = \begin{pmatrix} s_x^1 & s_y^1 \\ s_x^2 & s_y^2 \end{pmatrix}\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \tag{3.2}$$

for any J_1, J_2 functions of I_i, $i = 1, 2, 3, 4$ as in (3.1) and their derivatives with respect to the parameters (x, y).

If we are to identically follow the process of the previous section we ought to write the evolution for the invariants (3.1) induced upon them by the evolution in s (3.2). But soon one realizes that one can never get a Hamiltonian evolution that way. For once we only have a 2-dimensional vector of invariants (J_1, J_2), versus 4 invariants. And even if we choose only two of them we can never find an independent Hamiltonian evolution: the invariants depend on the second jet of the surfaces, rather than on the third jet, or higher which is the case of curves. In view of these problems we need to find other invariants (third order) that will evolve following a Hamiltonian system. The following three invariants will generate an entire family of Hamiltonian evolutions on the dual of the Virasoro algebra, one per direction on the (x, y)-plane. Consider the following three invariants:

$$\begin{aligned} I &= 3(I_1)_y - 2(I_3)_x - 2I_3^2 - 3I_1I_4, \\ J &= 3(I_2)_y - 2(I_4)_x - 2I_4^2 - 3I_2I_3, \\ K &= 4(I_4)_x + 4(I_3)_y + 2I_3I_4 - \frac{9}{2}I_1I_2. \end{aligned} \tag{3.3}$$

We can now prove the following theorem

THEOREM 3.2 *Given any combination $\partial_z = \alpha\partial_x + \beta\partial_y$, $\alpha, \beta \in \mathbf{R}$, there exists an invariant functional $E_{\alpha,\beta}$ such that, if $J_1 = \alpha h$ and $J_2 = \beta h$, and if*

$$s_t = \begin{pmatrix} s_x^1 & s_y^1 \\ s_x^2 & s_y^2 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

then $E_{\alpha,\beta}$ evolves following the KdV Hamiltonian evolution

$$(E_{\alpha,\beta})_t = h_{zzz} + 2E_{\alpha,\beta}h_z + (E_{\alpha,\beta})_z h.$$

PROOF. It suffices to choose $E_{\alpha,\beta} = \alpha^2 I + \alpha\beta K + \beta^2 J$, where I, J, K are given as in (3.3), to obtain the result. The rest is a straightforward calculation. ∎

In this sense $E_{\alpha,\beta}$ would be generalizations of the traditional Schwarzian derivative.

Of course, no one working in infinite Hamiltonian systems in two independent variables would resist the search for classical Hamiltonian systems such as the KP equation. In that search one finds one unavoidable complication: the nonlocal character of either the Hamiltonian functional or of the Poisson structure itself.

THEOREM 3.3. *Consider the invariant I defined in (3.1). Then there exists $h(x, y)$: $\mathbf{R}^2 \to \mathbf{R}$, analytic on (x, y), such that, if $J_1 = I + h$ and $J_2 = h$, then whenever*

$$s_t = \begin{pmatrix} s_x^1 & s_y^1 \\ s_x^2 & s_y^2 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \tag{3.4}$$

I will satisfy the KP equation

$$(I_t)_x = (I_{xxx} + 3II_x)_x + \frac{3}{2}k^2 I_{yy}. \tag{3.5}$$

PROOF. We can find straightforwardly that the evolution of I induced by (3.2) is

$$I_t = [(J_1)_{xxx} + 2I(J_1)_x + IJ_1] + [(J_2)_{xxy} + K(J_2)_x + I_yJ_2 - L_1(J_2)]$$

where $L_1 = \frac{3}{2}I_1\partial_y^2 - I_4\partial_x^2 - 2I_3\partial_{xy}^2$ and K is as in (3.3). If we consider $J_1 = I + h$ and $J_2 = h$, and we impose upon I to be a solution of the KP equation (3.5), we obtain an equation for h. This equation can be written in vectorial form as

$$g_x = G(g, g_y) \tag{3.6}$$

where $g = (h, h_y, h_x, h_{xy}, h_{xx}, h_{xxy}, h_{xxx})$, and where G is an analytic function. The standard theorem of existence of Cauchy and Kovalewsky gives us a unique and analytic solution of (3.6) for any choice of analytic initial condition along the x axis. An appropriate choice of constant initial condition will result in the proof of the theorem. ∎

Of course, h in the Theorem will never be a local functional, and, therefore, we will never be able to define it as a functional on the jet space. We can't talk about the differential invariance of h, since it is not well defined for such purposes (a differential invariant is always a functional on the jet space). That is, a more general theory would be necessary to include these nonlocal systems in the framework we have presented in this paper. The higher order cases, and the cases of higher dimension parametrized submanifolds are still unresolved.

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