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SOME REMARKS ON DIRAC STRUCTURES AND POISSON REDUCTIONS

ZHANG-JU LIU

Department of Mathematics, Peking University
Beijing, 100871, China
E-mail: liuzj@pku.edu.cn

Abstract. Dirac structures are characterized in terms of their characteristic pairs defined in this note and then Poisson reductions are discussed from the point of view of Dirac structures.

1. Introduction. Dirac structures on manifolds include closed 2-forms, Poisson structures, and foliations which were introduced by Courant and Weinstein and thoroughly investigated in [2]. Their Lie algebroid version was developed by Weinstein, Xu and the author in [6] and [7].

In this note, we characterize a Dirac structure in terms of its characteristic pair (D, Ω), which consists of a subbundle and a bivector field. One can see that it becomes more convenient to check the integrability and to study the reduction of a Dirac structure described this way. In the Lie bialgebra case, such a description is given by Diatta and Menida in [3].

As natural generalizations of symplectic reductions, Poisson reductions admit many applications. There are already several versions and different approaches for doing it, e.g., Marsden and Ratiu's theorem in [10] and that in [13] given by Weinstein by means of the coisotropic calculus. In fact, it is also an original purpose of the construction of Dirac structures to describe Poisson reductions. Of course, the same conclusion can be obtained from all of other approaches. But, anyway, we wish to illuminate more geometric pictures and algebraic relations behind the Poisson reduction in order to construct a unified framework for its various generalizations (e.g., [1], [4], [8], [11] and [12]) by use of the theory of Dirac structure, which has been shown to be a powerful tool as well as a beautiful theory.

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2. Preliminaries. In this section, we recall some basic facts concerning Lie bialgebroids and Dirac structures, which will be used below.

A Lie bialgebroid is a pair of Lie algebroids $(A,\ A^*)$ satisfying the following compatibility condition (see [9], [5] for more details):

$$d_*[X,Y] = [d_*X,Y] + [X,d_*Y], \quad \forall X,Y \in \Gamma(A), \tag{1}$$

where the differential d_* on $\Gamma(\wedge^* A)$ comes from the Lie algebroid structure on A^*.

Given a Lie algebroid A over P with anchor a and a section $\Lambda \in \Gamma(\wedge^2 A)$, denote by $\Lambda^{\#}$ the bundle map $A^* \to A$ defined by $\Lambda^{\#}(\xi)(\eta) = \Lambda(\xi,\eta), \forall \xi,\eta \in \Gamma(A^*)$. Introduce a bracket on $\Gamma(A^*)$ by

$$[\xi,\eta]_\Lambda = L_{\Lambda^{\#}\xi}\eta - L_{\Lambda^{\#}\eta}\xi - d[\Lambda(\xi,\eta)]. \tag{2}$$

By a_* we denote the composition $a \circ \Lambda^{\#} : A^* \to TP$. Thus, A^* with the bracket and anchor a_* defined above becomes a Lie algebroid if

$$[\Lambda,\Lambda] = 0. \tag{3}$$

In this situation, the induced differential on $\Gamma(\wedge^* A)$ has the form $d_* = [\Lambda,\cdot]$, which satisfies condition (1) naturally. The Lie bialgebroid arising in this way is called a *triangular Lie bialgebroid* and denoted as (A, A^*, Λ). When P is reduced to a point, i.e., A is a Lie algebra, equation (3) is just the classical Yang-Baxter equation, i.e., Λ is a classical r-matrix. On the other hand, when A is the tangent bundle TP with the standard Lie algebroid structure, equation (3) is equivalent to Λ being a Poisson tensor.

For any vector bundle A with the dual bundle A^*, there exist two natural nondegenerate bilinear forms on the direct sum bundle $A \oplus A^*$, one symmetric and the other antisymmetric, which are defined as follows:

$$(X_1+\xi_1, X_2+\xi_2)_\pm = \frac{1}{2}(\langle \xi_1, X_2\rangle \pm \langle \xi_2, X_1\rangle). \tag{4}$$

When (A, A^*) is a Lie bialgebroid over P, with anchors a and a_* respectively, a bracket is introduced in [6], on the section space $\Gamma(A \oplus A^*)$, as follows:

$$\begin{array}{rcl} [e_1,e_2] & = & ([X_1,X_2] + L_{\xi_1}X_2 - L_{\xi_2}X_1 - d_*(e_1,e_2)_-) \\ & & +([\xi_1,\xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1,e_2)_-), \end{array} \tag{5}$$

where $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$. The bundle $A \oplus A^*$ together with this bracket and the bundle map $\rho := a + a_* : A \oplus A^* \to TP$ satisfies certain properties as listed in [6] and is called the *double* of Lie bialgebroid (A, A^*). Such an object is also named a *Courant algebroid* since if A is the tangent bundle Lie algebroid TP and $A^* = T^*P$ with zero bracket, then equation (5) takes the form:

$$[X_1+\xi_1, X_2+\xi_2] = [X_1,X_2] + \{L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1,e_2)_-\}, \tag{6}$$

which was first introduced by Courant in [2]. On the other hand, when (A, A^*) is a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the bracket above is reduced to the well known Lie bracket on the double $\mathfrak{g} \oplus \mathfrak{g}^*$.

A *Dirac structure* for a Lie bialgebroid $(A,\ A^*)$ is a subbundle $L \subset A \oplus A^*$, which is maximally isotropic with respect to symmetric bilinear form $(\cdot,\cdot)_+$ defined in (4) and integrable in the sense that $\Gamma(L)$ is closed with bracket operation (5). Two important classes of Dirac structures are given in the following two propositions, which are studied in [6].

First, let $\Omega \in \Gamma(\wedge^2 A)$ and $\Omega^\# : A^* \to A$, the corresponding bundle map. Then the graph of $\Omega^\#$,

$$L = graph(\Omega^\#) = \{\Omega^\# \xi + \xi | \forall \xi \in A^*\},$$

defines a maximal isotropic subbundle of $A \oplus A^*$.

Proposition 2.1. *With the notation above, L is a Dirac subbundle iff Ω satisfies the Maurer-Cartan type equation:*

$$d_* \Omega + \frac{1}{2}[\Omega, \Omega] = 0. \tag{7}$$

Such an Ω is called a *Hamiltonian operator*. Another class of Dirac structures related to Poisson reduction and dual pairs of Poisson manifolds, called *null* Dirac structures, is characterized in the following way.

Proposition 2.2. *Let $D \subseteq A$ be a smooth subbundle of A, and $D^\perp \subseteq A^*$ its conormal bundle. Then $L := D \oplus D^\perp \subset A \oplus A^*$ is a Dirac structure iff both D and $D^\perp$ are respectively Lie subalgebroids of A and A^*.*

3. Characteristic pairs of Dirac structures. It is seen that a null Dirac structure is characterized by a subbundle and a graph Dirac structure is characterized by a bivector field. For more general construction of Dirac structures, they can be characterized by their characteristic pairs defined below.

Let A be a vector bundle. Any pair (D, Ω) of a smooth subbundle $D \subseteq A$ and bivector field $\Omega \in \Gamma(\wedge^2 A)$ corresponds to a maximal isotropic subbundle of $A \oplus A^*$ with respect to the inner product $(\cdot,\cdot)_+$ defined in (4) as follows:

$$L = \{\, X + \Omega^\# \xi + \xi \,|\, \forall X \in D, \xi \in D^\perp\} = D \oplus graph(\Omega^\# | D^\perp). \tag{8}$$

where $D^\perp \subseteq A^*$ is the conormal bundle of D. We call such a pair the *characteristic pair* of L, where $D = L \cap A$ is called the characteristic subbundle of L.

Conversely, any maximal isotropic subbundle $L \subset A$ such that $L \cap A$ is of constant rank can be always described by such a characteristic pair. In general, the rank of $L \cap A$ is not necessarily constant. But, sometimes, L can also be described by its singular characteristic subbundle with a smooth bivector field. For simplicity, we consider the regular case only in this note.

Notice that the restricted bundle map $\Omega^\# | D^\perp$ is equivalent to a bivector field on quotient bundleA/D. Thus, taking $pr : A \to A/D$ as the natural projection, we see that two pairs (D_1, Ω_1) and (D_2, Ω_2) define the same subbundle by equation (8) iff

$$D_1 = D_2 \quad \text{and} \quad pr(\Omega_1) = pr(\Omega_2),\ i.e.,\ \Omega_1 - \Omega_2 \equiv 0 (mod\, D),$$

where pr denotes also its induced map $\Gamma(\wedge^* A) \to \Gamma(\wedge^* A/D)$. In the above equation as well as in the sequel, a section $\Omega \in \Gamma(\wedge^* A)$ is said to equal zero modulo D, denoted as $\Omega \equiv 0(mod\, D)$, if its projection under pr vanishes in $\Gamma(\wedge^* A/D)$.

Even though L is related only to $pr(\Omega) \in \Gamma(\wedge^2 A/D)$ instead of Ω itself, it is convenient to characterize the integrability conditions of L in terms of Ω since sections of $\wedge^* A$ admit nice operations such as the exterior derivative and the Schouten bracket.

THEOREM 3.1. *Let (A, A^*) be a Lie bialgebroid, $L \subset A \oplus A^*$ a maximal isotropic subbundle defined by a characteristic pair (D, Ω). Then L is a Dirac structure if and only if the following three conditions hold:*

1. *D is a Lie subalgebroid.*
2. *Ω satisfies the Maurer-Cartan type equation (mod D), i.e.,*
$$d_*\Omega + \frac{1}{2}[\Omega, \Omega] \equiv 0(modD). \tag{9}$$
3. *$D^\perp$ is integrable for the sum bracket $[\cdot,\cdot] + [\cdot,\cdot]_\Omega$ where $[\cdot,\cdot]_\Omega$ is given by expression (2), i.e.,*
$$[\xi, \eta] + [\xi, \eta]_\Omega \in \Gamma(D^\perp), \qquad \forall \xi, \eta \in \Gamma(D^\perp). \tag{10}$$

PROOF. By definition, L is a Dirac structure iff L is closed with respect to bracket (5). First, for $\forall \xi, \eta \in \Gamma(D^\perp)$, by straightforward calculation, we can get the following formula:
$$\begin{array}{rcl} [\Omega^\# \xi + \xi, \Omega^\# \eta + \eta] & = & -(d_*\Omega + \frac{1}{2}[\Omega, \Omega])^\#(\xi, \eta) \\ & & +\Omega^\#([\xi, \eta] + [\xi, \eta]_\Omega) + ([\xi, \eta] + [\xi, \eta]_\Omega). \end{array} \tag{11}$$
Comparing with (8), the definition of L, we have $[\Omega^\# \xi + \xi, \Omega^\# \eta + \eta] \in \Gamma(L)$ iff
$$(d_*\Omega + \frac{1}{2}[\Omega, \Omega])^\#(\xi, \eta) \in \Gamma(D) \quad \text{and} \quad [\xi, \eta] + [\xi, \eta]_\Omega \in \Gamma(D^\perp),$$
which are just conditions (9) and (10). Moreover, for $\forall X \in \Gamma(D)$ and $\xi \in \Gamma(D^\perp)$, we have
$$\begin{array}{rcl} [X, \Omega^\# \xi + \xi] & = & [X, \Omega^\# \xi] + [X, \xi] \\ & = & [X, \Omega^\# \xi] + L_X \xi - L_\xi X + \frac{1}{2}(d_* - d) < X, \xi > \\ & = & [X, \Omega^\# \xi] - L_\xi X - \Omega^\#(L_X \xi) \\ & & +\Omega^\#(L_X \xi) + L_X \xi \end{array} \tag{12}$$
It is easy to see that $L_X \xi \in \Gamma(D^\perp)$, if D is integrable. Thus, $[X, \Omega^\# \xi + \xi] \in \Gamma(L)$ iff
$$[X, \Omega^\# \xi] - L_\xi X - \Omega^\#(L_X \xi) \in \Gamma(D). \tag{13}$$
Now, we take any $\eta \in \Gamma(D^\perp)$ and calculate the following expression, which means that condition (13) is equivalent to condition (10).
$$\begin{array}{rcl} < [\xi, \eta] + [\xi, \eta]_\Omega, X > & = & < [\xi, \eta], X > + < L_{\Omega^\# \xi} \eta, X > \\ & & - < L_{\Omega^\# \eta} \xi, > -X < \Omega^\# \xi, \eta > \\ & = & - < \eta, L_\xi X > + < \eta, [X, \Omega^\# \xi] > - < \eta, \Omega^\#(L_X \xi) > \\ & = & < \eta\,, [X, \Omega^\# \xi] - L_\xi X - \Omega^\#(L_X \xi) > . \end{array}$$
That is, $[\Gamma(D), \Gamma(graph(\Omega^\# | D^\perp))] \subseteq \Gamma(L)$ is equivalent to condition (10) and the integrability of D. This concludes the proof. ∎

In the Lie bialgebra case, D becomes a Lie subalgebra and the other two conditions (9) and (10) are reduced to those given by Diatta and Menida in [3]. For the triangular Lie bialgebroids, we have an alternative statement of the theorem above, which is particularly helpful to discuss Poisson reductions since to any Poisson manifold there corresponds naturally a triangular Lie bialgebroid. First, note that, for a triangular Lie bialgebroid (A, A^*, Λ), there are the following formulas:

$$d_*\Omega + \frac{1}{2}[\Omega, \Omega] = [\Lambda, \Omega] + \frac{1}{2}[\Omega, \Omega] = \frac{1}{2}[\Lambda + \Omega, \Lambda + \Omega]$$

and

$$[\cdot, \cdot]_\Lambda + [\cdot, \cdot]_\Omega = [\cdot, \cdot]_{\Lambda+\Omega}.$$

COROLLARY 3.2. *Let (A, A^*, Λ) be a triangular Lie bialgebroid, $L \subset A \oplus A^*$ a maximal isotropic subbundle with a fixed characteristic pair (D, Ω). Then L is a Dirac structure iff the following three conditions hold:*

1. *D is a Lie subalgebroid.*
2. *$[\Lambda + \Omega, \Lambda + \Omega] \equiv 0(mod\, D)$.*
3. *$\Lambda + \Omega$ is D-invariant $(mod\, D)$, i.e., $[X, \Lambda + \Omega] \equiv 0(mod\, D)$, $\forall X \in \Gamma(D)$.*

PROOF. From the analysis above, we need only show that $\Lambda + \Omega$ being D-invariant $(mod\, D)$ is equivalent to $D^\perp \subset A^*$ being closed under the bracket operation $[\cdot, \cdot]_{\Lambda+\Omega}$. This fact can be checked directly. ■

EXAMPLE. An integrable Jacobi structure on a manifold P is a pair (X, Ω) of a vector field and a bivector field such that

$$[\Omega, \Omega] = 2X \wedge \Omega \quad \text{and} \quad [X, \Omega] = 0.$$

Suppose that X is nondegenerate everywhere and, by D, denote the 1-dimensional distribution of TP generalized via X, which is certainly integrable. Then, $[\Omega, \Omega] = 2X \wedge \Omega$ means that $[\Omega, \Omega] \equiv 0(mod\, D)$ Thus, by Corollary 3.2, (D, Ω) is the characteristic pair of a Dirac structure for the ordinary Lie bialgebroid $(TP, T^*P, 0)$.

REMARK 3.3. Let (P, Λ) be a Poisson manifold. Then (TP, T^*P, Λ) is a triangular Lie bialgebroid. For a characteristic pair (D, Ω), D is called *reducible* if, for the induced foliation $\mathcal{F}$, the quotient space $P/\mathcal{F}$ is a nice manifold and the projection is a submersion. Then, in this case, it is clear that $\Lambda + \Omega$ being D-invariant $(mod\, D)$ is equivalent to that $\Lambda + \Omega$ can be reduced to a bivector field on $P/\mathcal{F}$ and the condition $[\Lambda + \Omega, \Lambda + \Omega] \equiv 0(mod\, D)$ is equivalent to the fact that the reduced bivector field is a Poisson tensor on $P/\mathcal{F}$.

To construct the Poisson reduction in terms of Dirac structures, we need to study the pullback Dirac structrues. In [7], with A. Weinstein and P. Xu, we studied such kind of Dirac structures by means of the Lie bialgebroid morphisms which are supposed to be surjective. Here we give a special kind of pullback Dirac structures with some alternative conditions, which will be used in the next section.

PROPOSITION 3.4. *Let A be a Lie algebroid, (B, B^*, Λ) a triangular Lie bialgebroid and $\Phi : A \to B$ a Lie algebroid morphism with constant rank, which covers a surjective map between the bases. Then the following two statements are equivalent.*

1. *There exists a Dirac structure for the Lie bialgebroid $(A, A^*, 0)$ such that its characteristic pair is $(ker\Phi, \bar{\Lambda})$, where $\Phi(\bar{\Lambda}) = \Lambda$, i.e., $\bar{\Lambda}$ and Λ are Φ-related.*

2.

$$Im(\Lambda^{\#}) \subseteq Im(\Phi). \tag{14}$$

PROOF. It is clear that $ker\Phi \subseteq A$ is an integrable subbundle and $ker\Phi^{\perp} = Im(\Phi^*) \subseteq A^*$ can be identified as the dual bundle of $A/ker\Phi$. Note that, for $\alpha, \beta \in \Gamma(ker\Phi^{\perp})$, there exist $\xi, \eta \in \Gamma(B^*)$ such that $\alpha = \Phi^*\xi$ and $\beta = \Phi^*\eta$. By definition, if there is some $\bar{\Lambda} \in \Gamma(\wedge^2 A/ker\Phi)$ which is Φ-related to Λ, it should be defined as

$$\bar{\Lambda}(\alpha, \beta) = \Lambda(\xi, \eta) =< \Lambda^{\#}\xi, \eta >, \quad \forall \alpha, \beta \in \Gamma(ker\Phi^{\perp}).$$

It is easy to see that such a $\bar{\Lambda}$ is well-defined iff $ker\Phi^* \subseteq ker\Lambda^{\#}$, i.e., $Im(\Lambda^{\#}) \subseteq Im(\Phi)$. In this case, $\bar{\Lambda}$ is a section of the quotient bundle $\wedge^2(A/ker\Phi)$. We take the same symbol to denote an arbitrary representative of $\bar{\Lambda}$ in $\Gamma(\wedge^2 A)$. Moreover, we have

$$\Phi[\bar{\Lambda}, \bar{\Lambda}] = [\Phi\bar{\Lambda}, \Phi\bar{\Lambda}] = [\Lambda, \Lambda] = 0 \quad \Longleftrightarrow \quad [\bar{\Lambda}, \bar{\Lambda}] \equiv 0 (mod\ ker\Phi)$$

since Φ is Lie algebroid morphism and Λ is triangular. Morover, for any $X \in \Gamma(\ker \Phi)$ we have

$$\Phi[X, \bar{\Lambda}] = [\Phi X, \Phi\bar{\Lambda}] = [0, \Phi\bar{\Lambda}] = 0 \quad \Longleftrightarrow \quad [X, \bar{\Lambda}] \equiv 0 (mod\ ker\Phi).$$

Thus, by Corollary 3.2, $(ker\Phi,\ \bar{\Lambda})$ corresponds to a Dirac structure for the Lie bialgebroid $(A, A^*, 0)$. ■

4. Poisson reductions. In this section we consider the Poisson reductions in terms of Dirac structures as an application of the characteristic pairs. As mentioned at the beginning of this note, the conclusion is essentially the same with all the others, e.g., in [10] and [13]. But we wish to construct a general framework for the Poisson reduction and its various generalizations by use of the theory of Dirac structure.

Let (P, Λ) be a Poisson manifold, $N \subset P$ a submanifold and $i : N \to P$ the inclusion. Suppose that $D \subset TP$ is a smooth subbundle and both D and $D_0 := D \cap TN$ are reducible (see Remark 3.3) with their induced foliations $\mathcal{F}$ on P and $\mathcal{F}_0$ on N respectively. This implies that we have the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{i} & P \\ {\scriptstyle\rho}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\ N/\mathcal{F}_0 & \xrightarrow[\varphi]{} & P/\mathcal{F} \end{array} \tag{15}$$

where ρ and π are the projections. Notice that any leaf of $\mathcal{F}_0$, i.e., a point in $N/\mathcal{F}_0$, is just a connected component of the intersection between N and some leaf of $\mathcal{F}$, so that under some clean intersection condition, we can always suppose $\varphi : N/\mathcal{F}_0 \to P/\mathcal{F}$ is an immersion (locally injective).

Now let $L = D + D^{\perp}$ be a null Dirac structure for the triangular Lie bialgebroid $(TP, T^*P,\ \Lambda)$. So there is a reduced Poisson structure $\Lambda_r = \pi_*\Lambda$ on $P/\mathcal{F}$ such that π is Poisson map. By the regularity assumption , the map, $\pi \circ i : N \to P/\mathcal{F}$, has constant rank. Thus, its tangent map $\Phi := (\pi \circ i)_* : TN \to T(P/\mathcal{F})$ can be considered a Lie algebroid morphism with the constant rank and $ker\Phi = D_0$. Thus, by Proposition 3.4, there exists a Dirac structure for the Lie bialgebroid $(TN, T^*N, 0)$ with its characteristic pair, $(D_0, \bar{\Lambda}_r)$, and $\Phi(\bar{\Lambda}_r) = \Lambda_r$ iff $Im(\Lambda_r^{\#}) \subseteq Im(\Phi)$ holds on $T(P/\mathcal{F})$. Equivalently, on the submanifold $N \subset P$, the condition above is expressed as follows:

$$\Lambda^{\#}(D^{\perp}) \subseteq TN + D, \tag{16}$$

where TN is identified with $i_*(TN)$. Furthermore, from this pullback Dirac structure, we get a reduced Poisson structure Λ_0 on $N/\mathcal{F}_0$ such that $\Lambda_0 = \rho_* \bar{\Lambda}_r$ (see Remark 3.3). Notice that commutative diagram (15) means:

$$\Lambda_r = \Phi\bar{\Lambda}_r = (\pi \circ i)_* \bar{\Lambda}_r = (\varphi \circ \rho)_* \bar{\Lambda}_r = \varphi_* \Lambda_0.$$

This implies that $\varphi : N/\mathcal{F}_0 \to P/\mathcal{F}$ is a Poisson map. We summarize the result above in the following theorem.

Theorem 4.1. *With the notation above, let (P, Λ) be a Poisson manifold, $N \subset P$ a submanifold. $D \subset TP$ is a smooth subbundle such that $D + D^{\perp}$ is a null Dirac structure and both D and $D_0 = D \cap TN$ are reducible. Then the following two statements are equivalent.*

1. *There exists a Poisson structure Λ_0 on $N/\mathcal{F}_0$ such that*

$$\pi_* \Lambda = \varphi_* \Lambda_0. \tag{17}$$

2. $\Lambda^{\#}(D^{\perp}) \subseteq TN + D$ *holds on* N .

Here, two Poisson manifolds (P, Λ) (the phase space) and $(N/\mathcal{F}_0, \Lambda_0)$ (the reduced phase space) are connected by means of the Poisson manifold $(P/\mathcal{F}, \Lambda_r)$ (induced from the null Dirac structure) with two Poisson maps π and φ.

Remark 4.2. In fact, not only for a null Dirac structure but also for any Dirac structure $L \subset TP \oplus T^*P$ with its characteristic pair (D, Ω), we can get a twisted Poisson structure, $\Lambda_r + \Omega_r$ on $P/\mathcal{F}$. Thus, if condition (14) holds for $\Lambda_r + \Omega_r$ or, equivalently, (16) holds for $\Lambda + \Omega$, there is a pullback Dirac structure on N with its characteristic pair, $(D_0, \bar{\Lambda}_r + \bar{\Omega}_r)$, which is Φ-related to $\Lambda_r + \Omega_r$. Furthermore, there still is a reduced Poisson structure $\Lambda_0 + \Omega_0$ on $N/\mathcal{F}_0$. It is easy to see that $\varphi : N/\mathcal{F}_0 \to P/\mathcal{F}$ is always a Poisson map. But $\pi : P \to P/\mathcal{F}$ is a Poisson map iff $\Omega \equiv 0 (mod\, D)$, i.e., the corresponding Dirac structure is null.

When the Poisson tensor Λ is nondegenerate, i.e., (P, Λ^{-1}) is a symplectic manifold, it is not difficult to justify the following statement.

Corollary 4.3. *With the notation above, the reduced Poisson structure Λ_0 on $N/\mathcal{F}_0$ is nondegenerate iff $Im(\Lambda_r^{\#}) = Im(\Phi)$ holds in $T(P/\mathcal{F})$ or, equivalently, iff*

$$\Lambda^{\#}(D^{\perp}) + D = TN + D \qquad \Longleftrightarrow \qquad ker(\Lambda^{\#}|D^{\perp}) = TN^{\perp} \cap D^{\perp}$$

holds on $N \subset P$. In this situation, $(N/\mathcal{F}_0, \Lambda_0^{-1})$ is a symplectic manifold and its any connected component can be identified with a symplectic leaf of $(P/\mathcal{F}, \Lambda_r)$.

A well-known situation is as follows:

$$\begin{array}{ccc} G\times\{\mu\} & \xrightarrow{\;i\;} & T^*G \\ \rho\Big\downarrow & & \Big\downarrow\pi \\ G/G_\mu & \xrightarrow[\varphi]{} & \mathfrak{g}^* \end{array} \tag{18}$$

for a Lie group G and T^*G identified with $G\times\mathfrak{g}^*$ by left translation. This commutative diagram represents the famous Lie-Poisson reduction as a special situation of the Marsden-Weinstein reduction theorem. More general situations can be found in [8] for Poisson Lie group action and in [11] for symplectic groupoid action.

The Poisson reduction theorem above can be formulated in the dual form, i.e., by the Poisson bracket. For $g_0, h_0 \in C^\infty(P/\mathcal{F})$, we pull them back to P and $N/\mathcal{F}_0$ respectively and denote them by

$$g=\varphi^*g_0,\ h=\varphi^*h_0\in C^\infty(N/\mathcal{F}_0),\qquad G=\pi^*g_0,\ H=\pi^*h_0\in C^\infty(P).$$

By means of commutative diagram (15) and the fact that both π and φ are Poisson maps, we get the following relationship between the two Poisson manifolds:

$$\rho^*\{g,h\}_{N/\mathcal{F}_0}=i^*\{G,H\}_P. \tag{19}$$

Since φ is locally injective, any $g\in C^\infty(N/\mathcal{F}_0)$ can be considered locally (globally when φ is injective) as the pullback of some $g_0\in C^\infty(P/\mathcal{F})$ as above. Moreover, $G=\pi^*g_0$ is, in fact, a local extension of ρ^*g from N to P. It is easy to see that equations (17) and (19) are equivalent. Thus we have

Corollary 4.4. *With the same notations as in Theorem 4.1, the following statements are equivalent.*

1. *There exists a Poisson structure Λ_0 on $N/\mathcal{F}_0$ such that equation (19) holds for all $g,h\in C^\infty(N/\mathcal{F}_0)$ and their local extensions G,H.*
2. $\Lambda^\#(D^\perp)\subseteq TN+D$.

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References

[1] A. Alekseev and Y. Kosmann-Schwarzbach, *Manin pairs and moment maps*, preprint.
[2] T. J. Courant, *Dirac manifolds*, Trans. A.M.S. 319 (1990), 631–661.
[3] A. Diatta and A. Medina, *Poisson homogeneous spaces of a Poisson Lie group*, preprint.
[4] T. Kimura, *Generalized classical BRST cohomology and reduction of Poisson manifolds*, Commun. Math. Phys. 151 (1993), 155–182.
[5] Y. Kosmann-Schwarzbach, *Exact Gerstenhaber algebras and Lie bialgebroids*, Acta Appl. Math. 41 (1995), 153–165.
[6] Z.-J. Liu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Diff. Geom. 45 (1997), 547–574.

[7] Z.-J. Liu, A. Weinstein and P. Xu, *Dirac structures and Poisson homogeneous spaces*, Commun. Math. Phys. 192 (1998), 121–144.

[8] J.-H. Lu, *Momentum mappings and reductions of Poisson actions*, in: Symplectic Geometry, Groupoids and Integrable Systems, P. Dazord and A. Weinstein, eds., Springer-Verlag 1991, 209–226.

[9] K. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 18 (1994), 415–452.

[10] J. E. Marsden and T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. 11 (1986), 161–169.

[11] K. Mikami and A. Weinstein, *Moments and reduction for symplectic groupoid actions*, Publ. RIMS Kyoto Univ. 24 (1988), 121–140.

[12] J. Stasheff, *Homological reduction of constrained Poisson algebras*, J. Diff. Geom. 45 (1997), 221–240.

[13] A. Weinstein, *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan 40 (1988), 705–727.