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ON LIOUVILLE FORMS

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This work is dedicated to the memory of Stanisław Zakrzewski

Abstract. We give different notions of Liouville forms, generalized Liouville forms and vertical Liouville forms with respect to a locally trivial fibration $\pi : E \to M$. These notions are linked with those of semi-basic forms and vertical forms. We study the infinitesimal automorphisms of these forms; we also investigate the relations with momentum maps.

0. Introduction. The source of this paper is the article due to D. Alekseevsky, J. Grabowski, G. Marmo and P. W. Michor [A.G.M] on "Poisson structures on the cotangent bundle of a Lie group or a principal bundle and their reductions".

Our purpose is to recover and complete some of their results by utilizing the tools introduced in the book [L.M], mainly the semi-basic forms with respect to a locally trivial fibration $\pi : E \to M$. Such a form η on E is the pull-back of the natural Liouville form θ_M on M by a morphism $f : E \to T^*M$. This morphism associated with a semi-basic form is underlying the paper [A.G.M].

The Liouville forms and generalized Liouville forms are semi-basic forms η such that $d\eta$ is symplectic on E. So the fibers are Lagrangian, or at least isotropic (in the generalized case). We study the infinitesimal automorphisms of these forms.

When dealing with generalized Liouville forms we prescribe the extra condition: the foliation defined by the fibration $\pi : E \to M$ is symplectically complete. Then we have a single local model if the dimension of E is given. In the case of a Liouville form, we have a Lagrangian foliation and the condition is satisfied.

We introduce the "vertical Liouville forms". In the case of a principal G-bundle $\pi : P \to M$, they are linked with the momentum map $J : T^*P \to \mathcal{G}^*$ associated with the Hamiltonian action of G on T^*P. In particular the submanifold $J^{-1}(0)$ of T^*P is the vector subbundle $\widetilde{P} = P \times_M T^*M$ of T^*P whose sections are the semi-basic forms. The reduced manifold $J^{-1}(0)/G$ is symplectically diffeomorphic to T^*M. We compare our

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results with those of Kummer [K] and Weinstein [W]. We thank P. Urbański for his counter-example.

Notations. All manifolds and maps are supposed to be C^∞. The objects which do not depend upon a connection (semi-basic forms, vertical forms) will be called natural. The horizontal objects depend upon a connection.

The projections $TN \to N$ and $T^*N \to N$ will be denoted by p and q. The Lie derivative with respect to a vector field X will be denoted $\mathcal{L}(X)$. We shall distinguish

$\mathbf{R}_*$ = multiplicative group of non-zero real numbers,

$\mathbf{R}^*$ = dual of the 1-dimensional vector space $\mathbf{R}$.

1. Some facts about Pfaffian semi-basic and vertical forms [L.M]

1.1. Let $\pi : E \to M$ be a locally trivial fibration, $VE = \ker T\pi$ be the vertical bundle.

We have the exact sequence of vector bundles with base E:

$$0 \to VE \xrightarrow{\iota} TE \to TE/VE = E \times_M TM \to 0\,;$$

by duality we obtain

$$0 \to E \times_M T^*M = \pi^* T^*M \xrightarrow{\psi} T^*E \xrightarrow{j} V^*E \to 0,$$

where ψ is defined by $\psi(y, \varphi) = T_y^*\pi(\varphi)$; so $E \times_M T^*M$, kernel of the projection j, is the annihilator $(VE)^0$ of the vertical bundle VE. Let η be a Pfaffian form on E. We have proved in [L.M, ch. II] that the following properties are equivalent:

1) For every vertical vector field X on E, i.e. for every section of $VE \to E$,

$$i(X)\eta = 0.$$

2) For every $y \in E$, there exists a unique form φ in $T^*_{\pi(y)}M$ such that

$$\eta_y = T_y^*\pi(\varphi).$$

3) η is a section of the vector fibration

$$\mathrm{pr}_1 : E \times_M T^*M \to E.$$

A Pfaffian form η on E is said to be *semi-basic* if it has any one of these three properties.

A *basic* form η is the pull back $\pi^*\mu$ of a Pfaffian form on M; a basic form is semi-basic, which is what justifies the terminology introduced by G. Reeb [R]. See also [G]. In [A.G.M] a semi-basic form is called horizontal according to the terminology introduced by Kobayashi-Nomizu.

To any semi-basic form η, there corresponds a fiber morphism $f = \mathrm{pr}_2 \circ \eta$ from E to T^*M (pr_2 being the projection $E \times_M T^*M \to T^*M$). Conversely if $f : E \to T^*M$ is given, the form η such that

$$\eta_y = T_y^*\pi\big(f(y)\big)$$

is semi-basic. In particular for the vector bundle $q : T^*M \to M$, the semi-basic form corresponding to the identity mapping of T^*M is the natural Liouville form θ_M, as was proved by G. Reeb [R]. In [L.M] it is also proved that the relation between η and f can be explained in the following way: *for any semi-basic form η on E, there exists a fiber* *morphism $f : E \to T^*M$ such that $\eta = f^*\theta_M$. Conversely if f is a morphism $E \to T^*M$, then the form $f^*\theta_M$ is semi-basic.*

By means of adapted coordinates $(x^1, \dots, x^n, y^1, \dots, y^k)$ on $\pi^{-1}(U)$ and $(x^1, \dots, x^n, p_1, \dots, p_n)$ on $q^{-1}(U)$, where U is an open set in M, the forms η and θ_M may be written

$$\eta = \sum_1^n a_i(x^1, \dots, x^n, y^1, \dots, y^k)\, dx^i, \qquad \theta_M = \sum_1^n p_i\, dx^i.$$

Then the morphism f is defined locally by

$$a_i = p_i.$$

It is also proved in [L.M] that *the forms induced on the manifold $E \times_M T^*M$ by the Liouville forms θ_E on T^*E and θ_M on T^*M coincide*. In other words we have

$$\psi^*\theta_E = \mathrm{pr}_2^*\,\theta_M.$$

This can be checked using local coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ on T^*M, $(x^1, \dots, x^n, y^1, \dots, y^k, p_1, \dots, p_n, r_1, \dots, r_k)$ on T^*E. The submanifold $E \times_M T^*M$ of T^*E is defined locally by $r_1 = \cdots = r_k = 0$. We have

$$\theta_E = \sum_1^n p_i\, dx^i + \sum_1^k r_\alpha\, dy^\alpha, \qquad \theta_M = \sum_1^n p_i\, dx^i,$$

hence

$$\psi^*\theta_E = \mathrm{pr}_2^*\,\theta_M = \sum_1^n p_i\, dx^i.$$

The form $\eta = \mathrm{pr}_2^*\,\theta_M$ is by definition basic with respect to the projection $E \times_M T^*M \to T^*M$. It could be called the *transverse* Liouville form.

1.2. A section μ of the vector bundle $V^*E \to E$ is called a *vertical* 1-form (or a 1-form along the fibers of $\pi : E \to M$). As V^*E is not a subbundle of T^*E, μ is *not* a differential form in the usual sense; μ acts only on vertical vectors.

The projection $j : T^*E \to V^*E$ associates a vertical 1-form μ with any 1-form on E. If η is semi-basic, then $\mu = 0$.

By means of local coordinates $(x^1, \dots, x^n, y^1, \dots, y^k)$ the vertical 1-form may be represented by

$$\mu = \sum b_i(x^1, \dots, x^n, y^1, \dots, y^k)\, dy^k.$$

Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be locally trivial fiber bundles. As a morphism $g : E_1 \to E_2$ maps fibers into fibers, Tg transforms vertical vectors into vertical vectors. We can thus define the *pull-back* $g^*\mu$ of a vertical form μ on E_2, as the vertical form on E_1 satisfying the condition: for every $y \in E_1$ and every $v \in V_yE_1$,

$$\big\langle (g^*\mu)_y, v \big\rangle = \big\langle \mu_{g(y)}, T_y g(v) \big\rangle.$$

2. On various notions of "Liouville forms"

2.1. In [L.M, chapter V] a Liouville structure on a manifold P is defined by a closed 2-form Ω without zeros which is homogeneous of degree 1 under the action h of the multiplicative group $\mathbf{R}_*$ i.e. $h_\lambda^*\Omega = \lambda\Omega$ for every $\lambda \in \mathbf{R}_*$. Then the vector field Z on P, opposite of the fundamental vector field satisfies the relation

$$\mathcal{L}(Z)\Omega = \Omega.$$

The form $\eta = i(Z)\Omega$ is called the *Liouville form* of the *Liouville structure*. This form satisfies the relations

$$d\eta = \Omega, \qquad \mathcal{L}(Z)\eta = \eta.$$

If Ω is symplectic, the Liouville structure is said to be symplectic.

When the action of $\mathbf{R}_*$ is free and regular, the projection π of P on the space M of orbits defines a $\mathbf{R}_*$-principal bundle (P, π, M). The fibers being the trajectories of the vector field Z, the Liouville form η is semi-basic with respect to the projection π. For instance let $\mathcal{E}$ be a Pfaffian equation on a manifold M i.e. $\mathcal{E}$ is a subbundle of rank 1 of T^*M; let us consider the complement $\mathcal{E}_0$ of the zero section of $\mathcal{E}$; then $(\mathcal{E}_0, q, M)$ is a principal bundle. The Liouville form η on $\mathcal{E}_0$ is the form induced by the natural Liouville form θ_M on T^*M. The form $d\eta$ is symplectic if and only if $\mathcal{E}$ is a contact structure i.e. if any nonzero section of $\mathcal{E}$ is a contact form; in this case the manifold $\mathcal{E}_0$ is called by Arnold [A] the *symplectification* of the contact manifold M.

2.2. In this paper, a semi-basic form η such that $d\eta$ is symplectic will be called a *generalized Liouville form* with respect to a locally trivial fibration $\pi : E \to M$.

Then the fibers are isotropic submanifolds of E; so each fiber is of dimension $k \leq n$, where $n = \dim M$.

For $k = 1$, we recover in particular the Liouville form, in the sense of 2.1, of a symplectic Liouville structure.

2.3. For $k = n$, the generalized Liouville form η will be called the *Liouville form* according to the terminology of [A.G.M]. This is the case of the natural Liouville form θ_M on T^*M.

When $\pi : E \to M$ is endowed with a Liouville form, then the rank of $\Omega = d\eta$ is $2n$ and the morphism $f : E \to T^*M$ such that $d\eta = f^*d\theta_M$ is *locally a diffeomorphism*. The fibration $\pi : E \to M$ could be called a *cotangent-like bundle*.

The fibers are Lagrangian submanifolds of E. It is known that the leaves of a Lagrangian foliation on a symplectic manifold are locally affine. In the case of a Liouville form, the proof is easy. Let U and U' be open subsets of M with local coordinates $(x^1, \dots, x^n)$ in U, $(y^1, \dots, y^n)$ in U'. If in an open subset $\mathcal{U}$ of $\pi^{-1}(U)$, η is written $\sum_{i=1}^n a_i\, dx^i$, then $d\eta = \sum_{i=1}^n da_i \wedge dx^i$ being of rank $2n$, the functions $(x^1, \dots, x^n, a_1, \dots, a_n)$ are independent and constitute a system of local coordinates. Similarly in an open subset $\mathcal{U}'$ of $\pi^{-1}(U')$, $d\eta$ is written $\sum_{j=1}^n db_j \wedge dy^j$, where $(y^1, \dots, y^n, b_1, \dots, b_n)$ are local coordinates. Suppose that $\mathcal{U} \cap \mathcal{U}'$ is not empty. Then in $U \cap U'$ we have $dy^j = \sum A_i^j\, dx^i$, hence $a_i = \sum_1^n A_i^j b_j$, where the A_i^j are functions in $U \cap U'$. In the connected components of the intersection of $\mathcal{U} \cap \mathcal{U}'$ with each fiber, the functions A_i^j are constant.

2.4. Let $\pi : E \to M$ be endowed with a generalized Liouville form. Moreover suppose that M is parallelizable, i.e. there exists a diffeomorphism $\Phi : TM \to M \times V$ such that,

p_1 and p_2 being the projections of $M \times V$ on M and V, the following conditions are fulfilled.

a) $p_1 \circ \Phi = p$;
b) the restriction α_x to $T_x M$ of $\alpha = p_2 \circ \Phi$ is an isomorphism.

By duality we get a diffeomorphism $\Psi : T^*M \to M \times V^*$ such that, q_1 and q_2 being the projections of $M \times V^*$ on M and V^*,

a) $q_1 \circ \Psi = q$;
b) the restriction β_x to $T^*_x M$ of $\beta = q_2 \circ \Psi$ is the contragredient of α_x.

Then any morphism $f : E \to T^*M$ induces a mapping $g = \beta \circ f$ from E to V^*. Conversely a mapping $g : E \to V^*$ induces the morphism $f : E \to T^*M$ such that

$$f(z) = \Psi^{-1}\big(q(z), g(z)\big).$$

The corresponding semi-basic form $f^*\theta_M$ is the form Θ_g introduced in [A.G.M]. This form $f^*\theta_M$ is a Liouville form if and only if f is a local diffeomorphism.

For any manifold M the natural Liouville form θ_M is the form on T^*M which associates with any $v \in TT^*M$ the scalar

$$\big\langle p(v), Tq(v)\big\rangle.$$

When there exists a parallelism $\Phi : TM \to M \times V$, the form θ_M may be defined as the form associating with $v \in TT^*M$ the scalar

$$\big\langle \beta \circ p(v), \alpha \circ Tq(v)\big\rangle.$$

In particular this is the case when the base manifold M is a Lie group G, with Lie algebra $\mathcal{G}$. Then the tangent bundle TG and the cotangent bundle T^*G are endowed with two trivializations, corresponding to the liftings to TG and T^*G of the left and right translations on G.

Let TL_s and $\widehat{L}_s$ be the liftings to TG and T^*G of the left translation by $s \in G$. Then the left trivializations

$$\Phi_L : TG \to G \times \mathcal{G}_{,}, \qquad \Psi_L : T^*G \to G \times \mathcal{G}^*$$

are defined by the morphisms $\alpha_L : TG \to \mathcal{G}$ and $\beta_L : T^*G \to \mathcal{G}^*$ such that

$$\alpha_L(z) = TL_{q(z)^{-1}}(z), \qquad \beta_L(z) = \widehat{L}_{q(z)^{-1}}(z).$$

So for $z \in T^*G$ and $v \in T_z T^*G$, we obtain

$$\langle \theta_G|_z, v\rangle = \langle \widehat{L}_{q(z)^{-1}} z, TL_{q(z)^{-1}} \circ Tq(v)\rangle,$$

where $\theta_G|_z$ is the restriction of θ_G to $T_z T^*G$; if we consider θ_G as a section of the bundle $T^*T^*G \to T^*G$, then $\theta_G|_z$ may be written $\theta_G(z)$. We recover a formula of [L.M, ch.IV].

Similarly the Liouville form θ_G can be expressed in terms of right translations.

3. Infinitesimal automorphisms of a generalized Liouville form

3.1. Let Z be a vector field on a manifold N. A differential r-form μ on N (with $r \geq 0$) is said to be *homogeneous* of degree m with respect to Z if

$$(*) \qquad \mathcal{L}(Z)\mu = m\mu.$$

A vector field X on N is said to be homogeneous of degree s if

$$(**) \qquad [Z,X]=(s-1)X.$$

In particular, Z is homogeneous of degree 1.

In contrast to the definition in 2.1, Z is not necessarily complete.

Simple calculations show the following:

a) if μ is homogeneous of degree m, so is $d\mu$;

b) if μ is homogeneous of degree m and X homogeneous of degree s, then $i(X)\mu$ and $\mathcal{L}(X)\mu$ are homogeneous of degree $m+s-1$;

c) if X_1 and X_2 are homogeneous of degrees s_1 and s_2, then $[X_1,X_2]$ is homogeneous of degree s_1+s_2-1.

In particular for forms and vector fields homogeneous of degree 1, $i(X)\mu$, $\mathcal{L}(X)\mu$, $[X_1,X_2]$ are homogeneous of degree 1.

From now on homogeneous will mean homogeneous of degree 1.

3.2. Let (N,Ω,Z) be a symplectic manifold such that Ω is homogeneous with respect to a vector field Z. Then the symplectic duality transforms homogeneous vector fields into homogeneous 1-forms. The converse is true. More precisely

LEMMA 1. *Let (N,Ω,Z) be a homogenous symplectic manifold, and X be a vector field on N such that $i(X)\Omega$ is homogeneous; then X is homogeneous.*

PROOF. From the formula

$$(*) \qquad i[X,Z]\Omega = i(X)\mathcal{L}(Z)\Omega - \mathcal{L}(Z)i(X)\Omega,$$

we deduce, when Ω and $i(X)\Omega$ are homogeneous,

$$i[X,Z]\Omega = i(X)\Omega - i(X)\Omega = 0\,;$$

as Ω is nondegenerate, the bracket $[X,Z]$ vanishes. ∎

On the manifold (N,Ω,Z) the form

$$\eta = i(Z)\Omega$$

is homogeneous and satisfies the relation

$$\Omega = d\eta.$$

Conversely if a Pfaffian form η on a manifold N has a symplectic differential $d\eta$, then η is homogeneous with respect to the vector field Z such that $\eta = i(Z)d\eta$.

An infinitesimal automorphism of the form η is a vector field X such that

$$\mathcal{L}(X)\eta = 0\,;$$

it follows that $\mathcal{L}(X)d\eta=0$.

Formula $(*)$ may also be written

$$(**) \qquad i[X,Z]\Omega = -i(Z)\mathcal{L}(X)\Omega + \mathcal{L}(X)i(Z)\Omega = -i(Z)\mathcal{L}(X)\Omega + \mathcal{L}(X)\eta.$$

We deduce

LEMMA 2. *An infinitesimal automorphism X of Ω is also an infinitesimal automorphism of η if and only if X is homogeneous.*

Moreover as

$$\mathcal{L}(X)\eta = i(X)d\eta + di(X)\eta,$$

if $\mathcal{L}(X)\eta = 0$, then X is the hamiltonian vector field X_h corresponding to the function $h = i(X)\eta$.

Conversely given a homogeneous function h, the hamiltonian vector field X_h such that $i(X_h)d\eta = -dh$ is homogeneous in view of lemma 1 and X_h is an infinitesimal automorphism of η. The homogeneous functions h and $i(X_h)\eta$ which have the same differential coincide.

The homogeneous symplectic structure on N induces by duality a homogeneous Poisson structure (N, Λ, Z). We have (see [D.L.M])

$$(***) \qquad \mathcal{L}(Z)\Lambda = -\Lambda,$$

which is equivalent to the relation

$$(****) \qquad \mathcal{L}(Z)\{h, g\} - \big\{\mathcal{L}(Z)h, g\big\} - \big\{h, \mathcal{L}(z)g\big\} = -\{h, g\},$$

where $\{h, g\}$ is the Poisson bracket of the functions h and g defined by

$$\{h, g\} = \Omega(X_h, X_g) = \Lambda(dh, dg).$$

This relation proves that the Poisson bracket of two homogeneous functions is homogeneous.

The preceding discussion may be summarized in the following theorem.

Theorem. *Let (N, Ω, Z) be a homogeneous symplectic manifold; set $\eta = i(Z)\Omega$.*

a. There exists an isomorphism Φ from the vector space L of the infinitesimal automorphisms of the form η onto the vector space L' of the real differentiable functions defined on N, that are homogeneous. This isomorphism is determined by

$$\Phi(X) = i(X)\eta.$$

Its inverse Φ^{-1} satisfies

$$\Phi^{-1}(h) = \Omega^\sharp dh, \quad i.e. \quad i(X)\Omega = -dh.$$

b. The bijection Φ is a Lie algebra isomorphism, with the usual bracket of vector fields as the bracket on L and the restriction of the Poisson bracket as the bracket on L'.

This theorem has been formulated by Arnold [A] in the case of the symplectification of a contact manifold. It was proved in [L.M] in the case of a fibered Liouville structure in the sense of 2.1 which is not necessarily symplectic. The proof given here is a simplification.

3.3. In [L.M] we have proved that an infinitesimal automorphism X of a fibered Liouville structure is projectable onto the base manifold; in particular when we consider the symplectification of a contact structure, the projection Y of X is an infinitesimal automorphism of the contact structure; conversely any infinitesimal automorphism of the contact structure is lifted to an infinitesimal automorphism of the symplectic Liouville structure.

The proof utilizes the fact that an infinitesimal automorphism of the Liouville structure is an infinitesimal automorphism of Ω which is invariant under the action of $\mathbf{R}_*$;

this proof is valid even when the fibers are not connected; this is the case of a principal $\mathbf{R}_*$-bundle.

3.4. Let us consider the case of a Liouville form η relative to the fibration $\pi : E \to M$ (see 2.3). By means of adapted coordinates (x^i, a_i), the vector field Z may be written as $\sum_1^n a_i \partial/\partial a_i$. It is a vertical vector field which could be called the *Liouville vector field* associated with η.

Let X_h be a hamiltonian vector field on E. Its trajectories satisfy the Hamilton equations

$$\frac{dx^i}{dt} = \frac{\partial h}{\partial a_i}, \qquad \frac{da_i}{dt} = -\frac{\partial h}{\partial x^i}.$$

The function h will be called *strongly homogeneous* if $\partial^2 h/\partial a_i \partial a_j = 0$; then the $\partial h/\partial a_i$ are constant on the connected components of each fiber; this is independent of the choice of local adapted coordinates as the leaves are locally affine. So the vector field X_h is projectable along the leaves of the foliation onto a vector field Y tangent to M.

Conversely let Y be a vector field on M. According to [A.G.M], this vector field Y defines a function h on E and the corresponding hamiltonian vector field X_h is an infinitesimal automorphism of η. This can be proved as follows: the vector field Y defines a section $\widetilde{Y}$ of the bundle $E \times_M TM \to E$; on the other hand the semi-basic form η is a section of $E \times_M T^*M \to E$; so we may define the function $h = \langle \eta, Y \rangle$ by

$$\langle \eta, Y \rangle = \langle \eta, \widetilde{Y} \rangle.$$

From the local expression of $Y = \sum_{i=1}^n X^i \dfrac{\partial}{\partial x^i}$, we deduce that $h = \sum_{i=1}^n a_i X^i$; so h is strongly homogeneous. From the Hamilton equations

$$\frac{dx^i}{dt} = X^i,$$

it follows that Y is the projection of X_h. We remark that X_h is the only infinitesimal automorphism of η which is projectable onto Y. Otherwise we could have a nonzero vertical infinitesimal automorphism X of η; it is impossible because of the formula $\mathcal{L}(X)\eta = i(X)d\eta + di(X)\eta$, where η is semi-basic and $d\eta$ nondegenerate.

The preceding discussion may be summarized as follows.

Theorem. *Let η be a Liouville form relative to a locally trivial fibration $\pi : E \to M$. Any vector field Y on M may be lifted in a unique fashion to an infinitesimal automorphism of η. Conversely if the fibers are connected, any infinitesimal automorphism of η corresponding to a strongly homogeneous function is projectable on M.*

Remark. If h is homogeneous but not strongly homogeneous, X_h may not be projectable. P. Urbański has exhibited the following counter-example. Let $\theta = a_1\,dx^1 + a_2\,dx^2$ and $Z = a_1 \partial/\partial a_1 + a_2 \partial/\partial a_2$ be the Liouville form and the dilation vector field in $\mathbf{R}^2 \times \mathbf{R}^2$. The function $h = (a_1)^2/a_2$ is not defined at the common zeros of θ and Z. In the open set $U = \{(x^1, x^2, a_1, a_2) \mid a_2 > 0\}$ the fibers are connected and h is homogeneous with respect to Z. Indeed

$$dh = \frac{2a_1}{a_2}\,da_1 - \left(\frac{a_1}{a_2}\right)^2 da_2 \quad \text{and} \quad \mathcal{L}(Z)h = i(Z)\,dh = h.$$

The vector field

$$X_h = 2\frac{a_1}{a_2}\,\frac{\partial}{\partial x^1} - \left(\frac{a_1}{a_2}\right)^2 \frac{\partial}{\partial x_2}$$

such that $i(X_h)\,d\theta = -dh$ is an infinitesimal automorphism of θ. From Lemma 2 of 3.2 we deduce that X_h is an infinitesimal automorphism of θ but X_h is not projectable on $\mathbf{R}^2$.

3.5. We now consider a generalized Liouville form η relative to a fibration $\pi : E \to M$ such that the dimension of the fibers is strictly inferior to the dimension of M. To simplify we shall assume that we have a simple foliation $\mathcal{F}$ (i.e. the fibers are connected).

A particular case of symplectic foliated manifold is discussed in [L1]. The foliation is said to be *symplectically complete* if the Poisson bracket (induced by the dual Λ of Ω) of two first integrals of $\mathcal{F}$ is also a first integral of $\mathcal{F}$ (eventually a constant). Moreover the foliation is *symplectically regular* if the rank of the form induced on the leaves by Ω is constant.

Let F be the subbundle of TE tangent to the foliation $\mathcal{F}$. We have proved [L1] that $\mathcal{F}$ is symplectically complete if and only if the orthogonal distribution $\operatorname{orth} F$ is completely integrable defining a foliation $\mathcal{F}^{\perp}$ which also is symplectically complete.

In the case of a generalized Liouville form, the leaves are isotropic hence the foliation is symplectically regular; F is the vertical subbundle VE and $\operatorname{orth}(VE)$ contains VE.

For a symplectically complete foliation there exists on the base M a unique Poisson structure $\overline{\Lambda}$ such that $\pi : E \to M$ is a Poisson morphism. Indeed the first integrals of the foliation being the pull-backs of the functions on M, the bracket $\{\overline{h}, \overline{g}\}$ of two functions on M is the unique function such that $\pi^*\{\overline{h}, \overline{g}\} = \{\pi^*\overline{h}, \pi^*\overline{g}\}$. For any first integral $h = \pi^*\overline{h}$, the hamiltonian vector field X_h is projectable on M onto a vector field $\overline{X}_h$ such that $\mathcal{L}(\overline{X}_{\overline{h}})\overline{g} = \mathcal{L}(X_h)g$.

Moreover we shall assume that the foliation $\mathcal{F}$ is homogeneous in the sense of [L2], i.e. the vector field Z (such that $i(Z)\Omega = \eta$) satisfies the condition: $\mathcal{L}(Z)h$ is a first integral whenever h is a first integral. It follows that Z is projectable onto a vector field $\overline{Z}$ and the Poisson structure $\overline{\Lambda}$ is homogeneous with respect to $\overline{Z}$. This property is a consequence of formula $(****)$ of 3.2. When h and g are first integrals, by projection we obtain

$$\mathcal{L}(\overline{Z})\{\overline{h}, \overline{g}\} - \{\mathcal{L}(\overline{Z})\overline{h}, \overline{g}\} - \{\overline{h}, \mathcal{L}(\overline{Z})\overline{g}\} = -\{\overline{h}, \overline{g}\}.$$

From formula $(****)$ we deduce also that if Z is tangent to the foliation (or equivalently $\overline{Z} = 0$), then $\{h, g\} = 0$ for any pair of first integrals and the foliation is co-isotropic. As this foliation was assumed to be isotropic, it is lagrangian and we are in the situation investigated in 3.4.

For a *Liouville structure* as defined in 2.1, the vector field Z generates the fibers; so the foliation defined by the fibration is *homogeneous* but *not symplectically complete*.

3.6. As was done in the preceding sections except 3.5 we shall use the same letter for a function on the base and its pull-back to E.

On a symplectic manifold (E, Ω) endowed with a symplectically regular foliation, the Cartan-Darboux theorem (see [L1], [L.M] chapter III under the name of "second version of Cartan's theorem") gives a local expression of Ω in terms of first integrals of the distributions $\mathcal{F}$ and $\mathcal{F}^{\perp}$. This theorem is utilized in [L2] to give a local expression of $\eta = i(Z)\Omega$ when the foliation is moreover homogeneous. The results of [L2] applied to the case of an isotropic foliation may be stated as follows.

THEOREM . *Let η be a generalized Liouville form relative to a locally trivial fibration $\pi : E \to M$. If the foliation $\mathcal{F}$ defined by the fibration is symplectically regular and homogeneous, then in the neighbourhood of every $y \in E$, there exist local coordinates $x^1, \ldots, x^{2s}, x^{2s+1}, \ldots, x^n, z^1, \ldots, z^k$ (with $n = \dim M$, $n + k = \dim E$, $2s = n - k$) such that η may be written $\eta = \eta_1 + \eta_2$ with*

$$\eta_1 = x^1\, dx^2 + \cdots + x^{2s-1}\, dx^{2s},$$
$$\eta_2 = z^1\, dx^{2s+1} + \cdots + z^k\, dx^n,$$

where $x^1, \ldots, x^{2s}, x^{2s+1}, \ldots, x^n$ are first integrals of $\mathcal{F}$, $x^{2s+1}, \ldots, x^n$ being also first integrals of the foliation $\mathcal{F}^{\perp}$.

The form η_1 is basic with respect of the fibration $\pi : E \to M$. It represents the form induced by the form η on each fiber of the foliation $\mathcal{F}^{\perp}$. If this foliation is simple, i.e. there exists a fibration $\widetilde{\pi} : E \to N = E/\mathcal{F}^{\perp}$, then η_1 represents a vertical form (in the sense of 1.2) and η_2 is a Liouville form with respect of the fibration $\widetilde{\pi} : E \to N$.

The vector field Z such that $i(Z)\Omega = \eta$ is written $Z = Z_1 + Z_2$ with

$$Z_1 = x^1 \frac{\partial}{\partial x^1} + \cdots + x^{2s-1} \frac{\partial}{\partial x^{2s-1}},$$
$$Z_2 = z^1 \frac{\partial}{\partial z^1} + \cdots + z^k \frac{\partial}{\partial z^k}.$$

The vector field Z_1 represents the projection of Z onto M and Z_2 is vertical.

A generalized Liouville form with $k = 1$ occurs in Mechanics. Let N be a manifold such that T^*N is endowed with a time-dependent Hamiltonian H_t, i.e. there exists a function $H : T^*N \times \mathbf{R} \to \mathbf{R}$. The Poincaré-Cartan integral invariant

$$\omega = \theta_N - H\, dt$$

is a contact form on $M = T^*N \times \mathbf{R}$ if the function $A = i(Z_N)dH - H$ has no zeros; here Z_N means the dilations vector field on T^*N. Let us consider the manifold $Q = N \times \mathbf{R}$ (configuration space-time). For its cotangent bundle T^*Q we get

$$T^*Q = T^*N \times T^*\mathbf{R} = T^*N \times \mathbf{R} \times \mathbf{R}^* = M \times \mathbf{R}^*\,;$$

the natural Liouville form θ_Q is written:

$$\theta_Q = \theta_N + p_0\, dt$$

where p_0 and t are coordinates on $\mathbf{R}^*$ and $\mathbf{R}$.

For the trivial fibration $\pi : T^*Q \to M$, the form θ_Q is a generalized Liouville form η with $\eta_1 = \theta_N$ and $\eta_2 = p_0\, dt$. The vector field $Z = Z_n + p_0\, \partial/\partial p_0$.

Let $\Phi : T^*N \times \mathbf{R} \to T^*Q$ be defined by

$$\Phi(\zeta, t) = \big(\zeta, t, -H(\zeta, t)\big).$$

Then the image V of $M = T^*N \times \mathbf{R}$ by Φ is the submanifold of T^*Q defined by $V = K^{-1}(0)$ where

$$K(\zeta, t, p_0) = H(\zeta, t) + p_0.$$

On V, the restriction $X_K|_V$ of the hamiltonian vector field X_K of T^*Q is tangent to V. The projections on the factor $\mathbf{R}^*$ of the trajectories of $X_K|_V$ satisfy the equation

$$p_0 + H = 0$$

which is called the *energy equation* in Mechanics.

The diffeomorphism $\pi|_V : V \to M$ projects $X_K|_V$ onto the vector field $\mathcal{Y}_H = Y_{H_t} + \partial/\partial t$, where Y_{H_t} on T^*N is defined by $i(Y_{H_t})d\theta_N = -dH_t$.

Fore more details see [L.M, chapter V].

Remark. The symplectification $\mathcal{E}_0$ of the contact manifold $M = T^*N \times \mathbf{R}$ is endowed with the form $\lambda(\theta_N - H\,dt)$; as was noticed in 3.5, the foliation defined by the projection $\mathcal{E}_0 \to M$ is not symplectically complete.

4. The vertical Liouville form and the momentum map

4.1. Let $\pi : E \to M$ be a locally trivial fibration, where E is not necessarily symplectic.

As the vertical bundle VE is a completely integrable vector subbundle of TE, it is endowed with a Lie algebroid structure whose bracket is the induced bracket and whose anchor map is the inclusion $\iota : VE \to TE$. According to [D.S], its dual V^*E is endowed with a homogeneous Poisson structure (V^*E, Λ, Z) where Z is the dilation vector field on V^*E; the projection $j : T^*E \to V^*E$ is a homogeneous Poisson morphism. The symplectic leaves of this Poisson structure are the cotangent bundles to the connected components of the fibers of $\pi : E \to M$; these symplectic leaves are also the connected components of the fibers of the fibration $r : V^*E \to M$. With each fiber $F_x = r^{-1}(x)$ we associate the natural Liouville form θ_{F_x}. Thus we obtain a differentiable family of forms θ_{F_x}. Their union Θ_F is a *vertical form* with respect to r, in the sense of 1.2.

The map $j : T^*E \to V^*E$ is a morphism of vector bundles with base E; as $\pi \circ q = r \circ j$, this map is also a morphism of fiber bundles with base M. So according to 1.2, we may define the pull-back $\Sigma = j^*\Theta_F$. This pull-back is a vertical form on T^*E which could be called the natural *vertical Liouville form* with respect to $\alpha = \pi \circ q$.

4.2. Let $\pi : P \to M$ be a principal G-bundle. The action of the Lie group G on P being free and regular, any $y \in P$ determines a diffeomorphism from the fiber P_x (where $x = \pi(y)$) onto G which maps y onto e; hence we get an isomorphism ϖ_y from T_yP_x onto $T_eG = \mathcal{G}$ (where $\mathcal{G}$ is the Lie algebra of G).

We deduce a map $\varpi : VP \to \mathcal{G}$ such that the restriction of ϖ to T_yP_x is ϖ_y. This vertical form ϖ satisfies the relation

$$\varpi_{ys} = \mathrm{Ad}(s^{-1})\varpi_y \quad \text{for any} \quad s \in G.$$

If for every $y \in P$, we consider the contragredient ${}^t\varpi_y^{-1}$ of ϖ_y, we obtain a map $\check{J} : V^*P \to \mathcal{G}^*$.

The natural Liouville form θ_{P_x} on the fiber T^*P_x is the form which, for any $z \in T^*P_x$ and any $v \in T_zT^*P_x$, associates the scalar

$$\big\langle \check{J}(z), \varpi(v)\big\rangle.$$

When x generates M we thus obtain a vertical form Θ_F with respect to $r : V^*P \to M$.

The *vertical Liouville form* $\Sigma = j^*\Theta_F$ on T^*P may be defined as follows. Let

$$J : T^*P \to \mathcal{G}^*$$

be the composed map $\check{J} \circ j$.

The vertical Liouville form $\Sigma = j^*\Theta_F$ (with respect to $\alpha = \pi \circ q$) is the form which associates with every $u \in \ker T\alpha$ the scalar

$$\big\langle J(z), \varpi\big(Tj(u)\big)\big\rangle,$$

where z is the image of u by the projection $TT^*P \to T^*P$.

The right action of G on P lifts to a symplectic right action on T^*P and the *momentum map is the map J defined above*. Indeed A. Weinstein [W] and M. Kummer [K] have remarked that the momentum map $T^*P \to \mathcal{G}$ associates with any $\varphi \in T_y^*P$ the element $\Phi_y^*\varphi$, where Φ_y is the mapping $\mathcal{G} \to T_yP$ extending the isomorphism $\varpi_y^{-1} : \mathcal{G} \to V_yP$.

The submanifold $J^{-1}(0)$ of T^*P is the kernel of the map $j : T^*P \to V^*P$. According to 1.1, *the submanifold $J^{-1}(0)$ is $\widetilde{P} = P \times_M T^*M$, the space of semi-basic forms on P.*

Let us consider the vector bundle TP/G, with base M, whose sections are the G-equivariant vector fields on P. It is known that TP/G may be identified with the Lie algebroid of the gauge groupoid, quotient of $P \times P$ by the diagonal action of G. The anchor map is the submersion $TP/G \to TM$. Hence the dual T^*P/G is endowed with a Poisson structure such that the injection $T^*M \to T^*P/G$ is a Poisson morphism.

If we limit ourselves to the subbundle $\widetilde{P} = P \times_M T^*M$, the injection $T^*M \to \widetilde{P}/G$ is an isomorphism and $\widetilde{P}/G$ is endowed with a symplectic structure. We have here an example of the reduction procedure. It follows from 1.1 that the 2-form $\Omega_{\widetilde{P}}$ induced by $d\theta_P$ on the submanifold $\widetilde{P} = J^{-1}(0)$ is of constant rank; the characteristic foliation comes from the projection $\mathrm{pr}_2 : \widetilde{P} \to T^*M$ and $\Omega_{\widetilde{P}} = (\mathrm{pr}_2)^*d\theta_M$.

4.3. Let $\omega : TP \to \mathcal{G}$ be a connection form on P inducing a principal connection on $\pi : P \to M$. Let Σ_ω be the differential 1-form on T^*P which associates with each $v \in TT^*P$ the scalar

$$\big\langle J(z), q^*\omega(v)\big\rangle$$

with $z = p(v)$.

This form Σ_ω which is a differential form in the usual sense has been called the "vertical Liouville form" in [A.G.M]. As the restriction of ω to VP is the form ϖ introduced in 4.2, the projection $T^*T^*P \to V^*T^*P$ sends Σ_ω onto Σ.

The principal connection defines a splitting

$$TP = VP \oplus \mathcal{H} \qquad (*)$$

where the horizontal bundle $\mathcal{H} = \ker\omega$ is G-invariant.

By duality we get

(∗∗) $$T^*P = V^*P \oplus \mathcal{H}^*.$$

We may identify $\mathcal{H}^*$ with the annihilator $(VP)^0$ of VP, i.e. with $\widetilde{P} = P \times_M T^*M$; similarly V^*P may be identified with $\mathcal{H}^0$. So we obtain

(∗∗∗) $$T^*P = \widetilde{P} \oplus \mathcal{H}^0.$$

Utilizing the ideas of M. Kummer [K], we shall make precise this decomposition for any $y \in P$. As $\Phi_y : \mathcal{G} \to T_yP$ is the extension of $\varpi_y^{-1} : \mathcal{G} \to V_yP$ and ϖ_y is the restriction of ω_y to V_yP, we have

$$\omega_y \circ \Phi_y = \mathrm{id}_{\mathcal{G}}, \qquad \Phi_y^*\omega_y^* = \mathrm{id}_{\mathcal{G}^*}\,.$$

Let $\varphi \in T_y^*P$; as $J(\varphi) = \Phi_y^*\varphi$, we get

$$J\omega_y^*J(\varphi) = \Phi_y^*\omega_y^*J(\varphi) = J(\varphi)$$

and

$$J\big(\varphi - \omega_y^*J(\varphi)\big) = 0.$$

So the form $\varphi - \omega_y^*J(\varphi)$ belongs to $\widetilde{P}$ while the form $\omega_y^*J(\varphi)$ which vanishes on $\ker\omega$ belongs to $\mathcal{H}^0$. The vector subspace $\mathcal{H}_y^0$ of T_y^*P is the vector space $\omega_y^*\mathcal{G}$.

M. Kummer [K] has studied the submanifold $J^{-1}(\mu)$ where $\mu \in \mathcal{G}^*$ is G-invariant. For $y \in P$, the subspace $J^{-1}(\mu) \cap T_y^*P$ is an affine subspace of T_y^*P deduced from the vector subspace $J^{-1}(0) \cap T_y^*P = \{y\} \times T_y^*M$ by the translation $\varphi \to \varphi + \omega_y^*\mu$. So the connection ω induces a diffeomorphism $\mathcal{D}_\omega$ from $J^{-1}(\mu)/G$ onto T^*M. In general $\mathcal{D}_\omega$ is not a symplectomorphism for the symplectic structure $(T^*M, d\theta_M)$. The symplectic structure of the reduced manifold $J^{-1}(\mu)/G$ is symplectomorphic to $(T^*M, d\theta_M + q^*\Omega_\mu)$, where Ω_μ is the μ-component of the curvature of ω, considered as a form on M. Moreover Kummer has proved that the additional term can be transformed away if and only if the bundle $\pi : P \to M$ admits a connection such that Ω_μ is exact.

Before Kummer's investigations, A. Weinstein [W], by using the reduction procedure, had built a "universal" symplectic manifold in the following situation: he considered a principal G-bundle and a manifold Q on which G acts by a Hamiltonian action; then $T^*P \times Q$ admits a Hamiltonian action, hence a momentum map J. The reduced symplectic manifold associated with the zero value of J is the "universal" symplectic manifold $(T^*P \times G)_0$. The choice of a principal connection on P defines a diffeomorphism from $(T^*P \times G)_0$ to the associated bundle $\widetilde{P} \times_G Q \to T^*M$, where $\widetilde{P}$ is $T \times_M T^*M$ as was seen before. The symplectic structure on the bundle $\widetilde{P} \times_G Q \to T^*M$ associated with a connection on P was introduced by S. Sternberg dealing with a classical particle in Yang-Mills field.

Remarks. 1) If P is the trivial bundle $M \times G$, the curvature of the natural connection vanishes. The condition stated by Kummer for $J^{-1}(\mu)/G$ to be symplectically diffeomorphic to $(T^*M, d\theta_M)$ is fulfilled.

2) A principal connection on $\pi : P \to M$ (where the bundle is no more trivial) may be defined as a map $c : TM \to TP/G$ such that $\rho \circ c = \mathrm{id}_{TM}$ where ρ is the anchor map $TP/G \to TM$. By duality the connection may be defined by a map $f : T^*P/G \to T^*M$ which is a morphism of vector bundles with base M. Then the manifold T^*P/G which

is already endowed with a natural Poisson structure (as seen in 4.2) has a presymplectic structure given by the form $f^*d\theta_M$ which is of rank $2n$. The form $\eta = f^*\theta_M$ is semi-basic. Conversely a 1-form η on T^*P/G which is semi-basic with respect to the projection $T^*P/G \to M$ and such that $d\eta$ is of rank $2n$ determines a connection on P.

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