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CONNECTIONS IN REGULAR POISSON MANIFOLDS OVER $\mathbb{R}$-LIE FOLIATIONS

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Abstract. The subject of this paper is the notion of the connection in a regular Poisson manifold M, defined as a splitting of the Atiyah sequence of its Lie algebroid. In the case when the characteristic foliation F is an $\mathbb{R}$-Lie foliation, the fibre integral operator along the adjoint bundle is used to define the Euler class of the Poisson manifold M. When M is oriented 3-dimensional, the notion of the index of a local flat connection with singularities along a closed transversal is defined. If, additionally, F has compact leaves (then F is a fibration over S^1), an analogue of the Euler-Poincaré-Hopf index theorem for flat connections with singularities along closed transversals is obtained.

1. Introduction. A Poisson manifold is a couple $(M, \{\cdot,\cdot\})$ consisting of a C^∞ manifold M equipped with an $\mathbb{R}$-Lie algebra structure $\{\cdot,\cdot\}$ in the vector space $C^\infty(M)$ of smooth functions, such that $\{f_1 \cdot f_2, g\} = f_1 \cdot \{f_2, g\} + \{f_1, g\} \cdot f_2,\ f_i, g \in C^\infty(M)$. If $(M, \{\cdot,\cdot\})$ is a Poisson manifold, then, for $f \in C^\infty(M)$, there exists a vector field X_f on M, called a *hamiltonian* of f, such that $X_f(g) = \{f, g\}$, $g \in C^\infty(M)$.

To each Poisson manifold $(M, \{\cdot,\cdot\})$ A. Coste, P. Dazord and A. Weinstein assigned in 1987 [C-D-W] a Lie algebroid with the total space T^*M and the structures:

- the anchor $\gamma : T^*M \to TM$ defined in such a way that

$$\gamma(df) = X_f, \quad \text{i.e.} \quad \gamma(df)(g) = \{f, g\},$$

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- the bracket $[\![\cdot,\cdot]\!]$ in $\operatorname{Sec} T^*M = \Omega^1(M)$ for which

$$[\![df, dg]\!] = d\{f, g\}.$$

In general, the Lie algebroid $(T^*M, \gamma, [\![\cdot,\cdot]\!])$ is not regular, which means that $F := \operatorname{Im}\gamma$ may not be a constant rank distribution (always, the *characteristic foliation* generated by F, i.e. by hamiltonian vector fields, is a foliation with singularities in the sense of P. Stefan, see [K4], [D-S]). The regular case was examined, for example, by P. Dazord, D. Sondaz, G. Hector, F. A. Cuesta and I. Vaisman in [D-S], [He], [C-H], [V1], [V2].

THEOREM 1.1 ([D-S]). *In the regular case, the Lie algebroid T^*M of a Poisson manifold $(M, \{\cdot,\cdot\})$ has the following properties:*

(1) the Atiyah sequence is as follows

$$0 \longrightarrow \nu^*F \hookrightarrow T^*M \xrightarrow{\gamma} F \longrightarrow 0 \tag{1.1}$$

where $F = \operatorname{Im}\gamma$, and $\nu^*F \subset T^*M$ is the transverse bundle of F,

(2) the isotropy Lie algebras $(\nu^*F)_x$ are abelian.

Assume in the sequel that M is a regular Poisson manifold with a characteristic foliation F. A splitting $\lambda : F \to T^*M$ of the vector bundle sequence (1.1) is called a *connection* in the regular Lie algebroid T^*M. The definition of a connection is due to M. Atiyah [A], K. Mackenzie [M], J. Kubarski [K1], [K2]. Connections in transitive Lie algebroids were examined by many authors (see, for example, J. Pradines [P1], [P2], K. Mackenzie [M], [M2], J. Kubarski [K3]) and, in nontransitive regular ones, by J. Kubarski [K1], [K2], [K5]. We add that the definition suggested by K. Mackenzie [M, Def. 5.1 p. 140; 142] fails in nontransitive cases. Each connection λ in T^*M determines two classical objects:

1. the *curvature form* $\Omega \in \Omega^2_F(M; \nu^*F) = \operatorname{Sec}(\bigwedge^2 F^* \otimes \nu^*F)$,

$$\Omega(X, Y) = \lambda \circ [X, Y] - [\![\lambda \circ X, \lambda \circ Y]\!], \qquad X, Y \in \operatorname{Sec}(F)$$

   (which a tangential 2-form on the foliated manifold (M, F)),

2. the *adjoint partial covariant derivative*

$$\nabla_X \nu = [\![\lambda \circ X, \nu]\!], \qquad X \in \operatorname{Sec}(F),\ \nu \in \operatorname{Sec}\nu^*F.$$

Since isotropy Lie algebras are abelian, ∇ is flat: $\nabla^2\nu = -[\Omega, \nu] = 0$, and to all connections λ the same ∇ corresponds.

THEOREM 1.2 ([D-S]). *The adjoint partial covariant derivative ∇ in ν^*F is equal to the Bott connection*

$$\nabla_X \omega = \iota_X(d\omega). \tag{1.2}$$

2. Connections in Poisson manifolds over $\mathbb{R}$-Lie foliations. Assume that the characteristic foliation F of the Poisson manifold $(M, \{\cdot,\cdot\})$ is an $\mathbb{R}$-Lie foliation, i.e. that F is of codimension 1 and $F = \ker\omega$ for a closed non-singular 1-form $\omega \in \Omega^1(M)$. According to (1.2), the form ω is a global ∇-constant cross-section of the adjoint bundle ν^*F. Each F-tangential form Θ with values in ν^*F determines an F-tangential real form $\hat{\Theta}$ (and vice versa)—called a *modified* one—such that

$$\Theta_x(v_1, \ldots, v_k) = \hat{\Theta}_x(v_1, \ldots, v_k) \cdot \omega_x.$$

Let

$$\dim M = m$$

and let $\mathbf{x} = (x_1, \dots, x_m)$ be a distinguished chart of F on $U \subset M$ such that $dx_1 = \omega_{|U}$. The anchor $\gamma_{|U} : T^*M_{|U} \to F_{|U}$ is given by

$$\gamma(dx_1) = 0, \qquad \gamma(dx_i) = \sum_{j \geq 2} \{x_i, x_j\} \frac{\partial}{\partial x_j}, \quad i \geq 2.$$

In particular, for $m = 3$,

$$\gamma(dx_2) = \{x_2, x_3\} \frac{\partial}{\partial x_3}, \qquad \gamma(dx_3) = -\{x_2, x_3\} \frac{\partial}{\partial x_2}.$$

Clearly,

$$W := \det [\{x_i, x_j\}]_{i,j \geq 2} \neq 0 \tag{2.1}$$

(in particular, for $m = 3$, $\{x_2, x_3\} \neq 0$), and the Poisson tensor P on U is given by

$$P_{|U} = \sum_{2 \leq i < j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j};$$

in particular, for $m = 3$, $P_{|U} = \{x_2, x_3\} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$.

LEMMA 2.1. *The general form of a local connection on U, $\lambda : F_{|U} \to T^*M_{|U}$, is*

$$\lambda\left(\frac{\partial}{\partial x_i}\right) = a_i \cdot dx_1 - \sum_{j \geq 2} \lambda_i^j \cdot dx_j, \quad i \geq 2, \tag{2.2}$$

where $a_i \in C^\infty(U)$ are arbitrary and

$$\lambda_i^j = \frac{W_i^j}{W} \tag{2.3}$$

(W_i^j being the algebraic complement of the (i,j)-entry of the matrix $[\{x_k, x_l\}]_{k,l \geq 2}$). In particular, $\lambda_i^j = -\lambda_j^i$, and for $m = 3$,

$$\lambda\left(\frac{\partial}{\partial x_2}\right) = a_2 \cdot dx_1 - \frac{1}{\{x_2, x_3\}} \cdot dx_3, \qquad \lambda\left(\frac{\partial}{\partial x_3}\right) = a_3 \cdot dx_1 + \frac{1}{\{x_2, x_3\}} \cdot dx_2.$$

PROOF. Since λ is a connection if and only if $\gamma \circ \lambda = \mathrm{id}$, we obtain that (2.2) is a connection if and only if, for each $i \geq 2$, the coefficients λ_i^j satisfy the following system of algebraic equations

$$\lambda_i^2 \cdot \{x_2, x_k\} + \lambda_i^3 \cdot \{x_3, x_k\} + \ldots + \lambda_i^m \cdot \{x_m, x_k\} = \delta_{ik}, \quad k = 2, 3, \ldots, m,$$

equivalent to

$$\lambda_i^2 \cdot \{x_k, x_2\} + \lambda_i^3 \cdot \{x_k, x_3\} + \ldots + \lambda_i^m \cdot \{x_k, x_m\} = -\delta_{ik}, \quad k = 2, 3, \ldots, m.$$

According to (2.1), this system is a Cramer system and (2.3) is its solution. The rest is easy. ■

If $\mathbf{y} = (y_1, \dots, y_m)$ is a second distinguished chart of F on $U \subset M$ such that $dy_1 = \omega_{|U} = dx_1$ and $\frac{\partial}{\partial y_i} = \sum_{j=1}^m A_i^j \frac{\partial}{\partial x_j}$ $(A_i^1 = \delta_i^1)$ and $\lambda(\frac{\partial}{\partial y_i}) = \tilde{a}_i \cdot dy_1 - \sum_{j \geq 2} \tilde{\lambda}_i^j \cdot dy_j$, $i \geq 2$, then

$$a_i = \tilde{a}_i + \sum_{j\geq 2} \tilde{\lambda}_i^j \cdot \left(A^{-1}\right)_1^j, \quad \lambda_i^k = \sum_{j\geq 2} \tilde{\lambda}_i^j \cdot \left(A^{-1}\right)_k^j.$$

Now, we calculate the curvature form Ω of λ. After simple algebraic calculations we obtain, for $i, j \geq 2$,

$$\begin{aligned}
&\Omega\left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\right) \\
&\quad = \lambda\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] - [\![\lambda\frac{\partial}{\partial x_i}, \lambda\frac{\partial}{\partial x_j}]\!] \\
&\quad = -[\![a_i \cdot dx_1 - \sum_{k\geq 2} \lambda_i^k \cdot dx_k, a_j \cdot dx_1 - \sum_{r\geq 2} \lambda_j^r \cdot dx_r]\!] \\
&\quad = \left(\sum_{k,r\geq 2} \{x_k, x_r\} \cdot \left(\lambda_i^k \cdot \frac{\partial a_j}{\partial x_r} - \lambda_j^k \cdot \frac{\partial a_i}{\partial x_r}\right) - \sum_{k,r\geq 2} \lambda_i^k \cdot \lambda_j^r \cdot \frac{\partial\{x_k, x_r\}}{\partial x_1}\right) dx_1 \\
&\quad = \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} - \sum_{k,r\geq 2} \frac{W_i^k \cdot W_j^r}{W^2} \cdot \frac{\partial\{x_k, x_r\}}{\partial x_1}\right) dx_1,
\end{aligned}$$

i.e.

$$\hat{\Omega} = \sum_{2\leq i<j} \left(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} - \sum_{k,r\geq 2} \frac{W_i^k \cdot W_j^r}{W^2} \cdot \frac{\partial\{x_k, x_r\}}{\partial x_1}\right) d_F x_i \wedge d_F x_j.$$

In particular, for $m = 3$,

$$\hat{\Omega} = \left(\frac{\partial a_2}{\partial x_3} - \frac{\partial a_3}{\partial x_2} + \frac{\partial}{\partial x_1}\left(\frac{1}{\{x_2, x_3\}}\right)\right) d_F x_2 \wedge d_F x_3.$$

Let M be oriented and odd dimensional. The question:

- does there exist, for any symplectic $\mathbb{R}$-Lie foliation $F = \ker\omega$ and F-tangential closed 2-form Ω, a Poisson structure on M with the characteristic foliation F, for which Ω is the curvature form of some connection λ?

is open, see [K8].

Fix a connection $\lambda : F \to T^*M$ and let $\hat{\Omega}$ be a modified curvature form of λ. Another connection λ_1 differs from λ by a tensor $t : F \to \nu^*F$, $\lambda_1 - \lambda = t$. The connection λ_1 is flat if and only if $d_F\left(\hat{t}\right) = \hat{\Omega}$. Indeed, $\lambda_1 = \lambda + t$ is flat if and only if $\lambda_1[X, Y] - [\![\lambda_1 X, \lambda_1 Y]\!] = 0$, but

$$\begin{aligned}
&\lambda_1[X, Y] - [\![\lambda_1 X, \lambda_1 Y]\!] \\
&\quad = (\lambda + t)[X, Y] - [\![(\lambda + t) X, (\lambda + t) Y]\!] \\
&\quad = \lambda[X, Y] + t[X, Y] - [\![\lambda X, \lambda Y]\!] - [\![tX, \lambda Y]\!] - [\![\lambda X, tY]\!] - [\![tX, tY]\!] \\
&\quad = \Omega(X, Y) + t[X, Y] - [\![\hat{t}X \cdot \omega, \lambda Y]\!] - [\![\lambda X, \hat{t}Y \cdot \omega]\!] \\
&\quad = \hat{\Omega}(X, Y) \cdot \omega + \hat{t}[X, Y] \cdot \omega + Y\left(\hat{t}X\right) \cdot \omega - X\left(\hat{t}Y\right) \cdot \omega \\
&\quad = (\hat{\Omega}(X, Y) - d_F\left(\hat{t}\right)(X, Y)) \cdot \omega.
\end{aligned}$$

We also observe that the cohomology class $[\hat{\Omega}]$ is independent of the choice of a connection and T^*M admits a flat connection if and only if $[\hat{\Omega}] = 0$. The class $[\hat{\Omega}]$ is the Pontryagin class of the regular Lie algebroid T^*M, corresponding to the Ad-invariant cross-section $\varepsilon^* \in \mathrm{Sec}\,(\nu^* F)^*$ for which $\langle \varepsilon^*, \omega \rangle = 1$. Indeed, let $h : I \to H_F(M)$ be the Chern-Weil homomorphism of the regular Lie algebroid T^*M; then $h(\varepsilon^*) = [\langle \varepsilon^*, \Omega \rangle] = [\hat{\Omega}]$ (for the construction of h, see [K5]).

As an example, consider a 3-manifold M and assume that the foliation F is a fibration with non-compact leaves; then $H_F^2(M) = 0$, which means that $[\hat{\Omega}] = 0$, therefore T^*M is a flat algebroid.

Using the 1-form ω, we can define the integration operator [K6]

$$\fint_{T^*M} : \Omega^*_{T^*M}(M) \longrightarrow \Omega_F^{*-1}(M),$$

$$\left(\fint_{T^*M} \Phi^k \right)(x; v_1 \wedge \ldots \wedge v_{k-1}) = (-1)^k \Phi^k(x; \omega_x \wedge \bar{v}_1 \wedge \ldots \wedge \bar{v}_{k-1}),$$

where $\bar{v}_i \in T_x^* M$, $\gamma(\bar{v}_i) = v_i$. The operator $\fint_{T^*M}$ is an epimorphism and commutes with exterior derivatives, giving a homomorphism on cohomology

$$\fint^{\#}_{T^*M} : H^*_{T^*M}(M) \longrightarrow H_F^{*-1}(M).$$

We can consider $\ker \fint_{T^*M}$ with the differential $d_{T^*M} | \ker \fint_{T^*M}$ and obtain the cohomology space $H\left(\ker \fint_{T^*M}\right)$. Clearly,

$$\gamma^{\#} : H_F(M) \xrightarrow{\approx} H\left(\ker \fint_{T^*M}\right)$$

is an isomorphism, which is crucial to form the Gysin sequence [K8], [K7]

$$\cdots \longrightarrow H_F^k(M) \xrightarrow{D^k} H_F^{k+2}(M) \xrightarrow{\gamma^{\#}} H_A^{k+2}(M) \xrightarrow{\fint^{\#}_A} H_F^{k+1}(M) \longrightarrow \cdots$$

where $D\alpha = (-1)^{\deg \alpha + 1} \left(\gamma^{\#}\right)^{-1}(\partial \alpha)$, $\partial : H_F^*(M) \to H^{*+2}\left(\ker \fint_{T^*M}\right)$ being the connecting homomorphism for the long cohomology sequence corresponding to the short sequence of graded differential spaces

$$0 \longrightarrow \ker \fint^*_{T^*M} \longrightarrow \Omega^*_{T^*M}(M) \xrightarrow{\fint_{T^*M}} \Omega_F^{*-1}(M) \longrightarrow 0.$$

We notice that $\partial\left[\varphi^k\right] = (-1)^k [\gamma^*(\hat{\Omega} \wedge \varphi^k)]$. Indeed, $\varphi^k = \fint_{T^*M} \Phi^{k+1}$ for $\Phi^{k+1} = (-1)^k \hat{\Lambda} \wedge \gamma^* \varphi^k$ where $\hat{\Lambda} \in \Omega^1_{T^*M}(M)$ is given by $\hat{\Lambda}(x; \omega_x) = 1$ and $\hat{\Lambda} | \operatorname{Im} \lambda_x = 0$ (i.e. $\Lambda(x; u) = \hat{\Lambda}(x; u) \cdot \omega_x$ is the connection form of λ); it remains to show that $d_{T^*M}((-1)^k \Lambda \wedge \gamma^* \varphi^k) = (-1)^k \gamma^*(\Omega \wedge \varphi^k)$, which follows directly from the closedness of φ^k and the equality $d_{T^*M}(\Lambda) = \gamma^* \hat{\Omega}$ shown below:

$$\begin{aligned}
& d_{T^*M}(\Lambda)(f \cdot \omega + \lambda X, g \cdot \omega + \lambda Y) \\
& = X(\Lambda(g \cdot \omega + \lambda Y)) - Y(\Lambda(f \cdot \omega + \lambda X)) - \Lambda([\![f \cdot \omega + \lambda X, g \cdot \omega + \lambda Y]\!]) \\
& = Xg - Yf - \Lambda(\lambda[X, Y] - \Omega(X, Y) + X(g) \cdot \omega - Y(f) \cdot \omega) \\
& = \hat{\Omega}(X, Y) \\
& = \gamma^* \hat{\Omega}(f \cdot \omega + \lambda X, g \cdot \omega + \lambda Y).
\end{aligned}$$

According to this,

$$D\alpha = -[\hat{\Omega}] \wedge \alpha$$

and (conventionally), the class $\chi := D(1) = -[\hat{\Omega}]$ is called the *Euler class* of the Poisson manifold $(M, \{\cdot,\cdot\})$ (or of the Lie algebroid T^*M of this Poisson manifold).

Fix two flat connections $\sigma_1, \sigma_2 : F \to T^*M$ and take the tensor $t : \sigma_2 - \sigma_1 : F \to \nu^*F$. The 1-form $\hat{t} \in \Omega^1_F(M)$ is closed. Indeed, $d\hat{t}$ is equal to the modified curvature tensor $\hat{\Omega} = 0$ of the connection σ_1. The cohomology class $[\sigma_1, \sigma_2] := [(\sigma_2 - \sigma_1)\hat{}\,]$ is called the *difference class* for flat connections σ_1 and σ_2. The fundamental property of the Euler class is given below.

THEOREM 2.2. *Suppose that there are an open covering $\{U, V\}$ of M and flat connections $\sigma_1 : F_{|U} \to T^*M_{|U}$, $\sigma_2 : F_{|V} \to T^*M_{|V}$. The difference class*

$$\left[\sigma_{1|U\cap V}, \sigma_{2|U\cap V}\right] \in H^1_F(U \cap V)$$

of the restrictions of σ_1 and σ_2 to $F_{|U\cap V}$ is defined. Let $\tilde{\partial} : H_F(U \cap V) \to H_F(M)$ denote the connecting homomorphism for the Mayer-Vietoris sequence of the triple (M, U, V) for the F-tangential cohomology [M-S]. Then the Euler class of T^*M is given by

$$\chi = \tilde{\partial}\left[\sigma_{1|U\cap V}, \sigma_{2|U\cap V}\right].$$

PROOF. Fix an arbitrary global connection $\lambda : F \to T^*M$ with the curvature form Ω. The form $-\hat{\Omega}$ represents the Euler class. Consider the inclusions $j_1 : F_{|U\cap V} \to F_{|U}$ and $j_2 : F_{|U\cap V} \to F_{|V}$. Take the tensors $t_1 = \sigma_1 - \lambda_{|U} : F_{|U} \to \nu^*F_{|U}$, $t_2 = \sigma_2 - \lambda_{|V} : F_{|V} \to \nu^*F_{|V}$. Since σ_1, σ_2 are flat,

$$d_{F_{|U}}\hat{t}_1 = \hat{\Omega}_{|U}, \qquad d_{F_{|V}}\hat{t}_2 = \hat{\Omega}_{|V}. \tag{2.4}$$

The form

$$\begin{aligned}(\sigma_2 - \sigma_1)\hat{} &= \left(\sigma_{2|U\cap V} - \lambda_{|U\cap V} - \sigma_{1|U\cap V} + \lambda_{|U\cap V}\right)\hat{} \\ &= \hat{t}_{2|U\cap V} - \hat{t}_{1|U\cap V} \\ &= j_1^*\left(-\hat{t}_1\right) - j_2^*\left(-\hat{t}_2\right)\end{aligned}$$

represents the difference class $[\sigma_1, \sigma_2]$. Since $d\left(-\hat{t}_1\right) = -\hat{\Omega}_{|U}$ and $d\left(-\hat{t}_2\right) = -\hat{\Omega}_{|V}$, we obtain $\tilde{\partial}\left[\sigma_{1|U\cap V}, \sigma_{2|U\cap V}\right] = [-\hat{\Omega}]$. ∎

3. Flat connections with singularities along closed transversals. Since the foliation F given by a closed 1-form ω, $F = \ker\omega$, is an $\mathbb{R}$-Lie foliation, we have that, for a compact manifold M [H-H], F admits a closed transversal, and that any closed transversal is total (we also have that the Euler characteristic of the clean manifold M is zero).

Assume that M is a compact oriented m-manifold. Given a closed transversal $S^1 \cong N \hookrightarrow M$ of F, we can choose a tubular neighbourhood W of N in M [H-H] such that the components of W in the leaf topology are open disks. The projection $p : W \to N$ along these disks is trivial, $W \cong S^1 \times D$ (D the standard open disk in $\mathbb{R}^{m-1}$), since F is oriented. The neighbourhood W (also the fibration (W, p, N)) is called *simple.* Orienting each fibre $W_x = p^{-1}(x)$ by inducing orientation from the leaf L_x of L, we obtain an oriented bundle.

PROPOSITION 3.1. *If* $\dim M = 3$, *then the restricted Lie algebroid* $T^*M_{|W}$ *is flat.*

Proof. Indeed, since the fibres W_x are contractible, the vector bundle of tangential vertical 2-cohomologies is a zero bundle; therefore $H_F^2(W) = 0$ [M-S] which implies that $\left[\Omega_{|W}\right] = 0$. This, in turn, is equivalent to the flatness of $T^*M_{|W}$. ∎

Definition 3.2. By a *local connection* with singularities along a closed transversal $S^1 \cong N \hookrightarrow M$ we mean a connection σ in $T^*M_{|\dot{U}}$ where $N \subset U$ (U open) and $\dot{U} = U\backslash N$.

For an arbitrary flat local connection σ in $T^*M_{|\dot{U}}$ and a flat connection λ in $T^*M_{|U'}$, $N \subset U' \subset U$, we define the difference class $[\dot{\lambda}, \sigma_{|\dot{U}'}] \in H_F^1(\dot{U}')$ where $\dot{\lambda} = \lambda_{|\dot{U}'}$.

Let (W, p, N) be a simple fibration for a closed transversal $N \hookrightarrow M$. Choose neighbourhoods V and K of N in W such that $N \subset V \subset K \subset W$ and the components of V and K in the leaf topology are open and closed disks, respectively. Also take a function $g \in C^\infty(W)$ such that $g \geq 0$, $g|V \equiv 0$, $g|W\backslash K \equiv 1$ and consider the tangential 1-form $d_F(g|\dot{W}) \in \Omega_F^1(\dot{W})$, $\dot{W} = W\backslash N$. Its support in each fibre W_x is compact. By the *canonical mapping* for W we mean [K8]

$$\alpha_W : H_F^{m-2}(\dot{W}) \longrightarrow C^\infty(N)\,,\ [\varphi] \longmapsto \int_{\dot{W}} d_F(g|\dot{W}) \wedge \varphi$$

where $\left(\int_{\dot{W}} \Psi\right)(x) := \int_{\dot{W}_x} (i_x^*\Psi)\,,\ i_x : \dot{W}_x \hookrightarrow \dot{W},\ x \in N.$

If $\dim F = 2$ (thus $\dim M = 3$), then $\alpha_W : H_F^1(\dot{W}) \to C^\infty(N)$.

Definition 3.3. If M is a compact oriented 3-manifold and σ a local flat connection with singularities along a closed transversal N and W a simple neighbourhood of N contained in the domain of σ, then the smooth function

$$j_N(\sigma) := \alpha_W[\dot{\lambda}, \sigma_{|\dot{W}}],$$

where λ is an arbitrary flat connection in $T^*M_{|W}$, is called the *local index* of σ along N.

The function $j_N(\sigma)$ is independent of the auxiliary flat connection λ and the choice of the simple neighbourhood $W \supset N$.

The group of periods of the foliation F (F is given by a closed 1-form on a compact manifold) may be cyclic or dense [H-H]. The first case holds if and only if F is given by a fibration $M \to S^1$ (in the second, all leaves of F are dense in M). Assuming the first case, for an arbitrary closed transversal N and any leaf L of F, the set $N \cap L$ is finite. For a mapping $f : N \to \mathbb{R}$, we define $\bar{f} : M \to \mathbb{R}$ by the formula

$$\bar{f}(x) = \sum_{y \in N \cap L_x} f(y)\,,$$

where L_x is the leaf of F through x. The function $\bar{f}$ is constant along leaves of F.

If, additionally, $\dim M = 3$, the function $\overline{\alpha_W(\beta)}$ (for $\beta \in H_F^1(\dot{W})$) is a smooth basic function. This follows from the commutativity of the following diagram

$$\begin{array}{ccc} H_F^1(\dot{W}) & \xrightarrow{\alpha_W} & C^\infty(N) \\ \Big\downarrow \partial & & \Big\downarrow {\scriptstyle f \mapsto \bar{f}} \\ H_F^1(M) & \xrightarrow{\int_M^\#} & \Omega_b(M, F) \end{array}$$

where ∂ is the connecting homomorphism of the triple (M, W, V) for the F-tangential differential forms ($V = M \backslash N$) and $\left(\int_M^{\#} [\varphi]\right)(x) = \int_{L_x} i_x^* \varphi$, $i_x : L_x \hookrightarrow M$.

Definition 3.4. *If M is a compact oriented 3-manifold, then the smooth basic function $\alpha_W[\dot\lambda, \sigma_{|\dot W}] \in \Omega_b(M; F)$ is called the* global index *of a local connection σ.*

The following theorem is an analogue of the classical Euler-Poincaré-Hopf theorem (from the theory of sphere bundles) in the geometry of Poisson manifolds.

Theorem 3.5. *Let M be a 3-dimensional compact oriented Poisson manifold with the characteristic $\mathbb{R}$-Lie foliation F having compact leaves. Let $N^1, \ldots, N^k$ be disjoint closed transversals of F and let $\sigma : F_{|V} \to T^* M_{|V}$, $V = M \backslash \bigcup_{i=1}^k N^i$, be a flat connection (such a connection always exists). If $\chi \in H_F^2(M)$ is the Euler class of the Lie algebroid T^*M, then*

$$\int_M^{\#} \chi = \sum_{i=1}^{k} \overline{j_{N^i}(\sigma)},$$

equivalently,

$$\chi = \sum_{i=1}^{k} \overline{j_{N^i}(\sigma)} \cdot \omega_F$$

where $\omega_F \in H_F^2(M)$ is the tangential orientation class, i.e. the one for which $\int_M^{\#} \omega_F \equiv 1$.

Proof. For $i = 1, \ldots, k$, choose a simple neighbourhood $W^i \supset N^i$ such that $W^1, \ldots, W^k$ are pairwise disjoint. Put $W = \bigcup_{i=1}^k W^i$, $V = M \backslash \bigcup_{i=1}^k N^i$. Then $M = W \cup V$ and $W \cap V = \bigcup_{i=1}^k \dot W^i$. Take arbitrary flat connections $\tilde\lambda^i : F_{|W^i} \to T^* M_{|W^i}$. The family $\{\tilde\lambda^i\}$ determines a flat connection $\tilde\lambda : F_{|W} \to T^*M_{|W}$. Define $\check\lambda = \tilde\lambda_{|W\cap V}$ and $\check\sigma = \sigma_{|W \cap V}$. According to Theorem 2.2, $\chi = \partial[\check\lambda, \check\sigma]$. Further, put $\lambda^i = \tilde\lambda^i_{|\dot W^i}$ and let $\sigma^i = \sigma_{|\dot W^i}$. Clearly, $[\check\lambda, \check\sigma] = \oplus^i \left[\lambda^i, \sigma^i\right]$. According to the commutativity of the diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^k H_F^1(\dot W^i) & \xrightarrow{\oplus \alpha_{W^i}} & \bigoplus_{i=1}^k C^\infty\left(N^i\right) \\
\Big\downarrow \partial & & \Big\downarrow \begin{smallmatrix} (f_1, \ldots, f_k) \\ \downarrow \\ \overline{f_1} + \cdots + \overline{f_k} \end{smallmatrix} \\
H_F^1(M) & \xrightarrow{\int_M^{\#}} & \Omega_b(M, F)
\end{array}$$

we finally obtain

$$\int_M^{\#} \chi = \int_M^{\#} \partial\left[\check\lambda, \check\sigma\right] = \int_M^{\#} \partial\left(\oplus^i \left[\lambda^i, \sigma^i\right]\right) = \overline{\oplus^i \alpha_{W^i}\left(\left[\lambda^i, \sigma^i\right]\right)} = \sum_{i=1}^{k} \overline{j_{N^i}(\sigma)}. \ \blacksquare$$

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