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DEFORMATIONS OF BATALIN–VILKOVISKY ALGEBRAS

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To the memory of Stanisław Zakrzewski

Abstract. We show that a graded commutative algebra A with any square zero odd differential operator is a natural generalization of a Batalin–Vilkovisky algebra. While such an operator of order 2 defines a Gerstenhaber (Lie) algebra structure on A, an operator of an order higher than 2 (Koszul–Akman definition) leads to the structure of a strongly homotopy Lie algebra (L_∞-algebra) on A. This allows us to give a definition of a Batalin–Vilkovisky algebra up to homotopy. We also make a conjecture which is a generalization of the formality theorem of Kontsevich to the Batalin–Vilkovisky algebra level.

1. Introduction. Batalin–Vilkovisky algebras are graded commutative algebras with an extra structure given by a second order differential operator of square 0. The simplest example is the algebra of polyvector fields on a vector space $\mathbb{R}^n$. There is a second order square zero differential operator on this algebra, obtained as an operator dual to the de Rham differential on the algebra of differential forms [W]. Namely, if one chooses a volume form, one can pair differential forms to polyvector fields. This pairing lifts the de Rham differential to polyvector fields and gives a second order square 0 operator.

In this article, we consider the following generalization of the Batalin–Vilkovisky structure: we do not require that the operator be of the second order. The condition that this operator be a differential (of square 0) leads to the structure of L_∞ algebra [HS, GK, LS] (also called a Lie algebra up to homotopy or strong homotopy Lie algebra).

The notion of an algebra up to homotopy is a very useful tool in proving certain deep theorems (like the formality theorem of Kontsevich [K]).

The most important property of algebras up to homotopy is that all the higher homotopies vanish on their cohomology groups. Namely, let A be a $\mathcal{P}$ algebra up to homotopy,

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with a differential d; then the space of its cohomology $H(A,d)$ is a $\mathcal{P}$ algebra, where $\mathcal{P}$ means either Lie, or associative, or commutative, or Poisson, or Gerstenhaber, etc.

We propose a definition of a *commutative strong homotopy Batalin–Vilkovisky algebra.* Its noncommutative version leads to a generalized formality conjecture.

2. Batalin–Vilkovisky algebras (BV-algebras). We work in the category of $\mathbb{Z}$-graded algebras. $A = \oplus A_i$. We denote the degree of a homogeneous element a by $|a|$.

Definition 1. A map $D\colon A \to A$ is of *degree* $|D|$ if $D\colon A_l \to A_{l+|D|}$ for each l. The degree of an element $a_1 \otimes \cdots \otimes a_k \in A^{\otimes k}$ is the sum of degrees $\sum_{j=1}^k |a_j|$.

Let $\mu : A \otimes A \to A$ be a product on A (a priori noncommutative non-associative). Following Akman [A], from any map $D\colon A \to A$ we can inductively define the following linear maps $F_D^k\colon A^{\otimes k} \to A$:

$$\begin{aligned} F_D^1(a) &= Da, \\ F_D^2(a_1,a_2) &= D\mu(a_1,a_2) - \mu(Da_1,a_2) - (-1)^{|a_1||D|}\mu(a_1,Da_2), \\ &\cdots\cdots \\ F_D^{n+1}(a_1,...,a_n,a_{n+1}) &= F_D^n(a_1,...,\mu(a_n,a_{n+1})) \\ &\quad - \mu(F_D^n(a_1,...,a_{n-1},a_n),a_{n+1}) \\ &\quad - (-1)^{|a_n|(|a_1|+\ldots+|a_{n-1}|+|D|)}\mu(a_n,F_D^n(a_1,...,a_{n-1},a_{n+1})). \end{aligned} \tag{1}$$

Definition 2 (Akman). A linear map $D\colon A \to A$ is a *differential operator* of order not higher than k if $F_D^{k+1} \equiv 0$.

Definition 3. A *Batalin–Vilkovisky algebra* (BV-algebra for short) is the following data (A,δ): an associative $\mathbb{Z}$-graded commutative algebra A, and an operator δ of order 2, of degree -1, and of square 0.

Definition 4. A *Gerstenhaber algebra* is a graded space $A = \sum_i A_i$ with

- an associative graded commutative product of degree 1, $\mu : A_i \otimes A_j \to A_{i+j+1}$, $\mu(a \otimes b) = a \cdot b$;
- a graded Lie bracket of degree 0, $l : A_i \wedge A_j \to A_{i+j}$, $l(a \otimes b) = [a,b]$, such that
- the Lie adjoint action is an odd derivation with respect to the product:

$$[a, b \cdot c] = [a,b] \cdot c + (-1)^{|b|\ |c|}[a,c] \cdot b.$$

Lemma 1. *Any BV-algebra (A,δ) is a Gerstenhaber algebra with the Lie bracket given by F_δ^2 up to sign:*

$$[a_1,a_2] = (-1)^{|a_1|}F_\delta^2(a_1,a_2) = (-1)^{|a_1|}\big(\delta\mu(a_1,a_2) - \mu(\delta a_1,a_2) - (-1)^{|a_1|}\mu(a_1,\delta a_2)\big), \tag{2}$$

for $a_1,a_2 \in A$.

A Gerstenhaber algebra which is also a BV-algebra is called “exact” [KS], since the bracket then is given by a δ-coboundary.

Remark 1. In the language of operads one can give another characterization of a Gerstenhaber algebra. A Gerstenhaber algebra is an algebra over the braid operad [G]. Then BV-algebras are algebras over the *cyclic* braid operad [GK]. In other words a

Gerstenhaber algebra structure comes from a BV-operator if the corresponding operad is cyclic.

3. L_∞**-algebras.** The brackets defined by the recursive formulas (1) have interesting relations. We need the notion of an L_∞-algebra to describe them.

We view an L_∞-algebra structure as a codifferential on the exterior coalgebra of a vector space [LM, P]. This is a generalization of the point of view on graded Lie algebras taken in [R].

Let V be a graded vector space. Define the exterior coalgebra structure on ΛV by giving the coproduct on the exterior algebra $\Delta : \Lambda V \to \Lambda V \otimes \Lambda V$:

$$\Delta v = 0 \tag{3}$$

$$\Delta(v_1 \wedge \cdots \wedge v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in Sh(k,n-k)} (-1)^\sigma \epsilon(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} \otimes v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)},$$

where $Sh(k, n-k)$ are the unshuffles of type $(k, n-k)$, that is, those permutations σ of n elements with $\sigma(i) < \sigma(i+1)$ when $i \neq k$. The sign $\epsilon(\sigma)$ is determined by the requirement that

$$v_1 \wedge \cdots \wedge v_n = (-1)^\sigma \epsilon(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)},$$

where $(-1)^\sigma$ is the sign of the permutation σ. Consider the suspension of the space V; $sV = V[1]$.

Definition 5. An L_∞-*algebra structure* on a graded vector space V is a codifferential Q on $\Lambda(sV)$ of degree $+1$, that is, a map $Q : \Lambda(sV) \to \Lambda(sV)[1]$ such that

- Q is a coderivation: $\Delta \circ Q = (Q \otimes 1 + 1 \otimes Q) \circ \Delta$,
- $Q \circ Q = 0$.

A coderivation Q_k is of k-th order if it is defined by a map $Q_k : \Lambda^k(sV) \to sV$. Then the coderivation property provides the extension of the action of Q_k on $\Lambda^n(sV)$ for any n:

$$Q_k : \Lambda^n(sV) \to \Lambda^{n-k+1}(sV) \quad \text{for} \quad n \geq k, \qquad \text{and} \quad Q_k : \Lambda^n(sV) \to 0 \quad \text{otherwise.}$$

This way we can consider sums of coderivations of various orders and define

$$Q(v_1 \wedge \cdots \wedge v_n) = \sum_{k=1}^{n} \sum_{\sigma \in Sh(k,n-k)} (-1)^\sigma \epsilon(\sigma)\ Q_k\big(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}\big) \wedge v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)},$$

where $Q_k : \Lambda^k(sV) \to sV$ and $Q = \sum_{k=1}^\infty Q_k$. Then we can rewrite $Q^2 = 0$ as a sequence of equations for each n:

$$\sum_{k=1}^{n} (-1)^{k(n-k)} \sum_{\sigma \in Sh(k,n-k)} (-1)^\sigma \epsilon(\sigma) Q_{n-k+1}(Q_k(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}) = 0.$$

Remark 2. An L_∞-algebra V has the following geometrical meaning. For each k : $\Lambda^k(sV) = Sym^k V$, the k-th symmetric power of the space V. If V is finite-dimensional, the symmetric powers of the space V are algebraic functions on the dual space V^*, which suggests that Q be a vector field on the dual space. Q_k then are Taylor coefficients of the odd vector field Q. Hence the map Q could be interpreted as an odd vector field of square 0. Such Q is called a homological vector field. The notion of a homological vector field appears in [V], in relation to the Gerstenhaber structure on the exterior algebra of an algebroid. A. S. Schwarz [Schw] calls supermanifolds with a homological vector field Q-manifolds.

4. Deformations of Batalin–Vilkovisky algebras. The brackets (1) are skew-symmetric when the product μ is graded commutative. Hence they can be restricted to the exterior powers of A :

$$F_D^k : \Lambda^k A \to A.$$

We now extend each linear map F_D^k to a coderivation of ΛA. We are going to show that the sum of all these coderivations is of square zero.

We need just another notion related to the degree:

Definition 6. A linear map $D : A \to A$, where $A = \sum_i A_i$ is a $\mathbb{Z}$-graded vector space, is called *odd* if $D : A_i \to \sum_k A_{i+2k+1},\ k \in \mathbb{Z}$ for each i.

Proposition 2. [1] *Consider an odd operator D on a graded commutative algebra (A,μ). Then $D^2 = 0$ if and only if the sum of brackets $Q_D = \sum F_D^n$ is a codifferential on ΛA defining an L_∞-structure, in other words $\sum_{k+l=n+1} F_D^k \circ F_D^l = 0$ for each $n \geq 1$.*

Proof. The "if" direction is obvious — it is given by the first equation in the series of equations above: $n = k = l = 1$. The proof of the "only if" part is a tedious calculation.

For a graded commutative algebra, Akman's definition of the brackets (1) coincides with the definition of Koszul [Ko], which we reformulate in the following terms. Define a product on the exterior algebra $M : A \wedge A \to A$ by $M(a_1 \wedge a_2) = a_1 \cdot a_2$. We can extend it to any exterior power $M(a_1 \wedge \ldots \wedge a_n) = a_1 \cdot \ldots \cdot a_n$. Then we can define an M-coproduct as a map $\Lambda A \to A \otimes A :\ \Delta_M = (M \otimes M)\Delta$:

$$\Delta_M(a_1 \wedge \ldots \wedge a_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in Sh(k,n-k)} (-1)^\sigma \epsilon(\sigma) a_{\sigma(1)} \cdot \ldots \cdot a_{\sigma(k)} \otimes a_{\sigma(k+1)} \cdot \ldots \cdot a_{\sigma(n)}.$$

Koszul's definition of multi-brackets is the following:

$$F_D^n(a_1 \wedge \ldots \wedge a_n) = M(D \otimes 1)(a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_n \otimes 1 - 1 \otimes a_n).$$

It can be reformulated as

$$F_D^n(a_1 \wedge \ldots \wedge a_n) = M(D \otimes 1)\Delta_M(a_1 \wedge \ldots \wedge a_n). \tag{4}$$

Then the lemma states that

$$(M(D \otimes 1)\Delta_M)(M(D \otimes 1)\Delta_M \otimes 1)\Delta = 0$$

[1] While finishing this article, I learned about the paper [BDA] which contains a result similar to this proposition. However, the aim and the language of [BDA] are somewhat different.

iff $D^2 = 0$. We see that in the left hand side of this equation there are either summands containing D^2 or summands which are present twice with opposite signs, due to the fact that the operator D is odd. ■

Notice that the brackets F_D^n form an L_∞ structure with homotopies with respect to the operator D, since the bracket F_D^2 gives a Lie algebra structure on $H(A, D)$, the cohomology of A with respect to the operator D.

Remark 3 (Order and degree). There is a filtration on the algebra of differential operators defined by their order. For the operator D however we would like to obtain an unambiguous splitting $D = \Sigma_{n\geq 1} D_n$, where D_n are homogeneous operators of n-th order. All we know is that for the first D_1, $F_{D_1}^n \equiv 0$ for $n > 1$. Then $F_{D_2}^n \equiv 0, n > 2$, but $F_{D_2}^2 \neq 0$, but there is already an ambiguity when defining D_2.

To obtain the splitting into homogeneous operators we use degree.

D acts on a graded algebra, so D is a sum of operators of different degrees. It turns out that degree and order are in correspondence. It is natural to ask that the classical BV structure is a particular case of the generalized structure. Hence, we may start with the requirement that D_1 is of order 1 and of degree $+1$, and D_2 is of order 2 and of degree -1. This defines the grading: the operator D is unambiguously represented as a sum of homogeneous operators.

Lemma 3. *Consider an operator $D : A \to A$ such that $D^2 = 0$ and assume that D is the sum of an operator of order 1 and of degree $+1$, $D_1 : A_\bullet \to A_{\bullet+1}$ and higher order operators. Then D can be represented as a sum*

$$D = \sum_{n\geq 1} D_n$$

where each D_n is an operator of order n and of degree $3 - 2n$ (in other words: $F_{D_n}^{n+1} \equiv 0$ and $D_n : A_\bullet \to A_{\bullet+3-2n}$).

This lemma is an easy consequence of the condition $D^2 = 0$. Of course we can also weigh each operator of a certain degree by some corresponding power of a formal parameter.

Remark 4 (Differential BV-algebra). If the operator D is of order n we see that the highest homotopy is given by the n-th bracket.

In particular, the second bracket

$$F_D^2(a, b) = D(ab) - Da\ b - (-1)^{|a|} a\ Db$$

gives a classical BV-bracket for the case when $D_n = 0$ for $n \geq 3$. Then the operator D is of order 2, that is, $D = D_1 + D_2$. Such a D describes the case of a differential BV-algebra which is the starting point of [BK], see also [M].

On the other hand, given a differential algebra (A, μ, d) with additional second order differential operator δ one can define a generalized BV-algebra by adding operators of higher order to $d + \delta$, requiring that their sum

$$D = D_1 + D_2 + D_3 + \dots$$

be of square 0 (here $D_1 = d, D_2 = \delta$). Comparing with the differential BV-algebra case we see that there are still two differentials on the generalized algebra, D and D_1 (the fact that D_1 is a differential follows from $D^2 = 0$). The following lemma is easy to prove.

LEMMA 4. *An operator on the algebra* (A, μ), $D = \sum D_n$, *such that* $D^2 = 0$ *is a derivation of the bracket* $[a, b] = (-1)^{|a|} F_D^2(a, b)$, *but not of the product* μ, *while* D_1 *is a derivation of the product but not of the bracket.*

REMARK 5 (Generalization to Leibniz algebras). If we start with a non-commutative associative algebra structure, the brackets F_D^n (1) still make sense for a differential operator D, (Definition 2). However since there is no antisymmetry condition anymore, the homotopy structure we get from $D^2 = 0$ is not L_∞. Instead, one gets a Leib_∞-algebra ([Li]), a homotopy version of a Leibniz algebra ([L]).

5. Commutative $\mathbf{BV}_\infty$-algebra. We now propose a definition of a strong homotopy Batalin–Vilkovisky algebra (BV_∞-algebra). Here we will restrict ourselves to the case of commutative algebras.

DEFINITION 7. A triple (A, d, D) is a *commutative BV_∞-algebra* when

- A is a graded commutative algebra,
- $d : A \to A$ is a degree 1 differential of the algebra A,
- $D : A \to A$ is an odd square zero differential operator such that the degree of $D - d$ is negative.

There are various ways to define a BV_∞-algebra. In our definition the commutative structure is preserved. One can imagine deforming the commutative structure as well. In Remark 5 we mentioned one of the generalizations, the one leading to the Leib_∞-algebras. However, all definitions should lead to the following property: a BV_∞-algebra should have a BV-algebra structure on its cohomology. Indeed in our case:

THEOREM 5. *The cohomology* $H(A, d)$ *of a commutative BV_∞-algebra* (A, d, D) *is a BV-algebra.*

PROOF. Consider the condition $D^2 = 0$. Since the degree of $D - d$ is negative, it means that D is the sum of d, a derivation of degree $+1$, and of negative degree operators: $D_2 + D_3 + \cdots$. From $D^2 = 0$ it follows that $dD_2 + D_2 d = 0$, that is, D_2 acts on the cohomology $H(A, d)$. Moreover, $D_2^2 = dD_3 + D_3 d$, which means that on $H(A, d)$, $D_2^2 = 0$. Since D_2 is a second order operator, it defines the structure of a BV-algebra on $H(A, d)$. ∎

6. Possible applications. It would be interesting if we could extend the formality theorem of Kontsevich [K] to the quasi-isomorphism of BV_∞-algebras.

The formality theorem of Kontsevich states that two differential graded Lie algebras defined on any manifold M, the algebra of local Hochschild cochains and the algebra of polyvector fields, are quasi-isomorphic as L_∞-algebras.

Let A denote the algebra of smooth functions on M, $A = C^\infty(M)$, with the pointwise commutative product. Let D be the algebra of polydifferential operators on M : $D = \oplus D^k$, $D^k = Hom_{loc}(A^{\otimes k+1}, A)$, and let T be the algebra of polyvector fields on

$M\colon T = \oplus T^k$, $T^k = \Gamma(\Lambda^{k+1}TM)$, both with the degree shifted by 1. Then there are the following corresponding structures on these two algebras:

Graded space	**Polyvector fields** $T = \oplus T^\bullet = \oplus\Gamma(\Lambda^{\bullet+1}TM)$	**Polydifferential Operators** $D = \oplus D^\bullet = \oplus Hom_{loc}(A^{\bullet+1}, A)$
Differential	$d = 0$	Hochschild $b : D^\bullet \to D^{\bullet+1}$
Lie bracket	Schouten–Nijenhuis	Gerstenhaber
Product	$\wedge$ — exterior product	$\cup$ — cup product
BV-operator	δ	??

One can check that T is in fact a Gerstenhaber algebra while D is a Gerstenhaber algebra up to homotopy, since the $\cup$-product on D is commutative only up to homotopy. However the Lie adjoint action on D is still an odd derivation with respect to the product.

Recently Dima Tamarkin [T] proved a generalization of Kontsevich's formality theorem, he showed the existence of a morphism of Gerstenhaber algebras up to homotopy between T and D. In other words, the algebra of polydifferential operators is G-formal: the algebra of polydifferential operators and the algebra of polyvector fields are quasi-isomorphic as G_∞-algebras (Gerstenhaber algebras up to homotopy).

We would like to see if one could prove the formality not only as G_∞-algebras but as BV_∞-algebras.

If the first Chern class of a manifold M is 0, then the algebra of polyvector fields on M is a BV-algebra. There is a one-to-one correspondence between BV-structures on a manifold M and flat connections on the determinant bundle (bundle of polyvector fields in the top degree: $\Lambda^{top}TM$). Such a structure on real manifolds was studied in many papers [Ko, Xu, H, W], on Calabi–Yau manifolds one can refer to [Sch, BK]. We conjecture that in these cases there should be some BV_∞-structure leading to the Gerstenhaber bracket on polydifferential operators.

Conjecture 1. *There is a structure of a BV_∞-algebra on the space of polydifferential operators on a manifold with a zero first Chern class.*

Conjecture 2. *The BV_∞-algebra of polydifferential operators on a manifold is formal: it is quasi-isomorphic as a BV_∞-algebra to its cohomology, the BV-algebra of polyvector fields.*

For these conjectures we will need a more general definition than definition 7, since the cup product on the algebra of polydifferential operators is commutative only up to homotopy. This generalization should not pose a problem, it will be done in a subsequent article.

From the conjecture, it would follow the Maurer–Cartan equation (MC-equation) for the BV operator on the algebra of polydifferential operators (probably tensored with some graded commutative algebra). Moreover, a quasi-isomorphism of BV_∞-algebras would map solutions of the MC-equation on one algebra to solutions of the MC-equation on the other algebra.

We know from [BK] that the formal moduli space of solutions to the MC-equation, modulo gauge invariance on polyvector fields tensored with the algebra of anti-holomorphic forms on a Calabi–Yau manifold carries a natural structure of Frobenius manifold. If a quasi-isomorphism $T \to D$ of BV-structures up to homotopy exists it would define a Frobenius manifold structure on the solutions of the MC-equation modulo gauge invariance on polydifferential operators tensored with the algebra of anti-holomorphic forms.

Another instance where we could expect to find generalized BV-structures is in the theory of vertex operator algebras. There is a structure of a Batalin–Vilkovisky algebra on the cohomology of a vertex operator algebra (see [LZ], [PS]). It is natural to ask what structure exists on the vertex operator algebra itself. This shows the need for a suitable definition of a BV_∞-structure. Besides it should fit into the general picture outlined by Stasheff [S].

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