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A CLASSIFICATION OF POISSON HOMOGENEOUS SPACES OF COMPLEX REDUCTIVE POISSON-LIE GROUPS

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Abstract. Let G be a complex reductive connected algebraic group equipped with the Sklyanin bracket. A classification of Poisson homogeneous G-spaces with connected isotropy subgroups is given. This result is based on Drinfeld's correspondence between Poisson homogeneous G-spaces and Lagrangian subalgebras in the double $D(\mathfrak{g})$ (here $\mathfrak{g} = \operatorname{Lie} G$). A geometric interpretation of some Poisson homogeneous G-spaces is also proposed.

1. Introduction. Let G be a Poisson-Lie group, $\mathfrak{g} = \operatorname{Lie} G$, let $D(\mathfrak{g})$ be the double corresponding to the Lie bialgebra $\mathfrak{g}$. We say that a subalgebra $\mathfrak{l} \subset D(\mathfrak{g})$ is *Lagrangian* if it is a maximal isotropic subspace with respect to the natural scalar product in $D(\mathfrak{g})$. It follows from [3] that there is a one-to-one correspondence between Poisson homogeneous G-spaces (up to isomorphism) with connected stabilizers and Lagrangian subalgebras $\mathfrak{l} \subset D(\mathfrak{g})$ such that $\mathfrak{l} \cap \mathfrak{g}$ is a Lie algebra of a certain closed subgroup in G (up to G-conjugacy).

Now let G be a connected complex reductive algebraic group equipped with the Sklyanin bracket. By $\langle \cdot\,, \cdot \rangle$ denote any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that its restriction to a compact real form of $[\mathfrak{g}, \mathfrak{g}]$ is positive definite. Then $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$, and the natural scalar product in $D(\mathfrak{g})$ is given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle, \tag{1}$$

where $x_1, x_2, y_1, y_2 \in \mathfrak{g}$ (see Section 2).

In this paper we obtain a description of orbits of the diagonal G-action on the set of all Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ (see Theorem 3.1) and specify the orbits of Lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{g}$ such that the subalgebra $\mathfrak{l} \cap \mathfrak{g}_{diag} \subset \mathfrak{g}_{diag} \simeq \mathfrak{g}$ corresponds to a certain closed subgroup in G (see Theorem 3.2; here by $\mathfrak{g}_{diag}$ we denote the diagonal

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image of $\mathfrak{g}$ in $\mathfrak{g}\times\mathfrak{g}$). Thus we get a classification of all Poisson homogeneous G-spaces with connected stabilizers.

Note that the description of G-orbits on the set of Lagrangian subalgebras $\mathfrak{l}\subset\mathfrak{g}\times\mathfrak{g}$ such that $\mathfrak{l}\cap\mathfrak{g}_{diag}=0$ was obtained in [1]; this result is related to a classification of the solutions of the classical Yang-Baxter equation. A classification of structures of a Poisson homogeneous space on G/H, where H is a Cartan subgroup, was independently obtained by Jiang-Hua Lu; these structures are closely related to the solutions of the classical dynamical Yang-Baxter equation (see [8]).

This paper is organized as follows. In Section 2 we recall the definition of the Sklyanin bracket on G. In Section 3 we formulate the classification theorems. Section 4 presents methods of the proof of Theorem 3.1. In Section 5 we propose a geometric interpretation of some Poisson homogeneous G-spaces, i.e., we construct a Poisson manifold X with a Poisson G-action such that G-orbits on X are Poisson homogeneous G-spaces, and different orbits are not isomorphic (note that in the case when the Poisson bracket on G is zero, an analogue of X is $\mathfrak{g}^*$ with the Kirillov bracket and the coadjoint action of G).

Note that in this paper we only formulate the main results and give a brief description of methods of proofs. The complete proofs will be presented elsewhere.

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2. Poisson structure on G. Let us recall the definition of the Poisson structure on G. Fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Let $\mathbf{R}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\mathbf{R}_+$ the set of positive roots with respect to a certain system of simple roots $\Gamma\subset\mathbf{R}$. Set

$$\mathfrak{n}_+=\bigoplus_{\alpha\in\mathbf{R}_+}\mathfrak{g}_\alpha,\ \mathfrak{n}_-=\bigoplus_{\alpha\in\mathbf{R}_+}\mathfrak{g}_{-\alpha},$$

$\mathfrak{b}_+=\mathfrak{h}\oplus\mathfrak{n}_+$, $\mathfrak{b}_-=\mathfrak{h}\oplus\mathfrak{n}_-$.

Consider $r=\frac{1}{2}t_0+t_1$ (here the tensor $t=t_0+t_1+t_2\in\mathfrak{g}\otimes\mathfrak{g}$ corresponds to the bilinear form $\langle\cdot,\cdot\rangle$, $t_0\in\mathfrak{h}\otimes\mathfrak{h}$, $t_1\in\mathfrak{n}_+\otimes\mathfrak{n}_-$, $t_2\in\mathfrak{n}_-\otimes\mathfrak{n}_+$. We have $r=r_{sym}+r_{alt}$, where r_{sym} is symmetric and r_{alt} is skew-symmetric. Let $r^{\mu\nu}$ be the components of the tensor r in some basis $\{e_\mu\}\subset\mathfrak{g}$. Denote by ∂_μ (respectively by ∂'_μ) the right-invariant (respectively left-invariant) vector field corresponding to e_μ. Since r satisfies the classical Yang-Baxter equation and r_{sym} is $\mathfrak{g}$-invariant (see [4, §4]), we see that Sklyanin's formula

$$\{\phi,\psi\}=r^{\mu\nu}(\partial'_\mu\phi\cdot\partial'_\nu\psi-\partial_\mu\phi\cdot\partial_\nu\psi)$$

(here ϕ,ψ are regular functions on G) defines the structure of Poisson-Lie group on G. The structure of a Lie bialgebra on $\mathfrak{g}=\operatorname{Lie}G$ is defined by the Manin triple $(\mathfrak{g}\times\mathfrak{g},\mathfrak{g}_{diag},\mathfrak{m})$, where $\mathfrak{g}\times\mathfrak{g}$ equipped with the scalar product (1),

$$\mathfrak{m}=\{(x,y)\in\mathfrak{b}_-\times\mathfrak{b}_+\mid x_\mathfrak{h}+y_\mathfrak{h}=0\},$$

$x_\mathfrak{h}$ (respectively $y_\mathfrak{h}$) is the image of x (respectively of y) in $\mathfrak{h}$ (see [4, §3, Example 3.2]). In particular, the double $D(\mathfrak{g})$ is equal to $\mathfrak{g}\times\mathfrak{g}$ equipped with the scalar product (1).

3. Classification theorems. Fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Let $\mathbf{R}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Let $\mathbf{P}, \mathbf{P}' \subset \mathbf{R}$ be parabolic subsets (see [2, Ch.6, §1.7]). Set

$$\mathfrak{p} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \mathbf{P}} \mathfrak{g}_\alpha \Big), \ \mathfrak{p}' = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \mathbf{P}'} \mathfrak{g}_\alpha \Big).$$

Then $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras in $\mathfrak{g}$. Set $\mathbf{A} = \mathbf{P} \cap (-\mathbf{P})$, $\mathbf{A}' = \mathbf{P}' \cap (-\mathbf{P}')$. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be the semisimple subalgebras in $\mathfrak{g}$ generated by $\mathbf{A}$ and $\mathbf{A}'$ respectively. Let $\tilde{\mathfrak{h}} = \mathfrak{a} \cap \mathfrak{h}$, $\tilde{\mathfrak{h}}' = \mathfrak{a}' \cap \mathfrak{h}$. Note that $\tilde{\mathfrak{h}}$ (respectively $\tilde{\mathfrak{h}}'$) is the linear span of the coroots $\alpha^\vee \in \mathfrak{h}$ such that $\alpha \in \mathbf{A}$ (respectively $\alpha \in \mathbf{A}'$).

Let $\sigma : \mathbf{A} \to \mathbf{A}'$ be an isomorphism of the root systems such that σ preserves the scalar product. Set

$$\mathbf{U} = \{\alpha \in \mathbf{A}' \mid \sigma^{-k}(\alpha) \in \mathbf{A}' \ \ \forall k \in \mathbb{N}\}.$$

Since the sets $\mathbf{A}, \mathbf{A}' \subset \mathbf{R}$ are finite, and $\sigma : \mathbf{A} \to \mathbf{A}'$ is a bijection, we have

$$\mathbf{U} = \{\alpha \in \mathbf{A} \mid \sigma^{k}(\alpha) \in \mathbf{A} \ \ \forall k \in \mathbb{N}\} =$$

$$= \{\alpha \in \mathbf{A} \cap \mathbf{A}' \mid \sigma^{l}(\alpha) \in \mathbf{A} \cap \mathbf{A}' \ \ \forall l \in \mathbb{Z}\}.$$

It is easy to prove that

$$\mathfrak{u} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \mathbf{U}} \mathfrak{g}_\alpha \Big)$$

is a Levi subalgebra in $\mathfrak{g}$ (i.e., a reductive Levi subalgebra of a certain parabolic subalgebra in $\mathfrak{g}$), and $\mathbf{U}$ is the root system of $\mathfrak{u}$. We consider only the case when σ preserves a certain system of simple roots in $\mathbf{U}$.

Let $\xi : \mathfrak{a} \to \mathfrak{a}'$ be an isomorphism such that $\xi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$ for all $\alpha \in \mathbf{A}$; then $\xi(\tilde{\mathfrak{h}}) = \tilde{\mathfrak{h}}'$, and ξ preserves $\langle \cdot\,, \cdot \rangle$. Let the linear map $\sigma^\vee : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}'$ be given by $\sigma^\vee(\alpha^\vee) = \sigma(\alpha)^\vee$, where $\alpha \in \mathbf{A}$; then $\xi(x) = \sigma^\vee(x)$ for all $x \in \tilde{\mathfrak{h}}$. Note that

$$[\mathfrak{u}, \mathfrak{u}]^\xi = \{x \in [\mathfrak{u}, \mathfrak{u}] \mid \xi(x) = x\}$$

is a reductive Lie algebra, and $\mathfrak{h}^\xi = [\mathfrak{u}, \mathfrak{u}]^\xi \cap \mathfrak{h}$ is a Cartan subalgebra in $[\mathfrak{u}, \mathfrak{u}]^\xi$ (see [9, Ch.4, §4.2]).

Consider a nilpotent element $x \in [\mathfrak{u}, \mathfrak{u}]^\xi$ (we say that an element $x \in \mathfrak{g}$ is nilpotent if $x \in [\mathfrak{g}, \mathfrak{g}]$ and $\mathrm{ad}x$ is nilpotent). Let $h \in \mathfrak{h}^\xi$ be the characteristic of the nilpotent element x (see [5, Ch.6, §2.1]; recall that one can reconstruct x from h uniquely up to conjugation). Let the isomorphism $\theta : \mathfrak{a} \to \mathfrak{a}'$ be given by $\theta = \xi \cdot \exp(\mathrm{ad}x)$.

Let $\mathfrak{z}$ (respectively $\mathfrak{z}'$) be the orthogonal complement to $\tilde{\mathfrak{h}}$ (respectively $\tilde{\mathfrak{h}}'$) in $\mathfrak{h}$. Note that the natural maps $\mathfrak{z} \to \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$ and $\mathfrak{z}' \to \mathfrak{p}'/[\mathfrak{p}', \mathfrak{p}']$ are isomorphisms. Consider $\mathfrak{z} \times \mathfrak{z}'$ equipped with the scalar product (1). Let $\mathfrak{l}_0 \subset \mathfrak{z} \times \mathfrak{z}'$ be a Lagrangian subspace. Consider

$$\mathfrak{l} = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid x \in \mathfrak{p}, \ y \in \mathfrak{p}', \ \theta(x_\mathfrak{a}) = y_{\mathfrak{a}'}, \ (x_\mathfrak{z}, y_{\mathfrak{z}'}) \in \mathfrak{l}_0\};$$

here $x_\mathfrak{a}$ is the image of x in $\mathfrak{a}$, $x_\mathfrak{z}$ is the image of x in $\mathfrak{z} = \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$, $y_{\mathfrak{a}'}$ is the image of y in $\mathfrak{a}'$, $y_{\mathfrak{z}'}$ is the image of y in $\mathfrak{z}' = \mathfrak{p}'/[\mathfrak{p}', \mathfrak{p}']$. Then $\mathfrak{l}$ is a Lagrangian subalgebra in $\mathfrak{g} \times \mathfrak{g}$. By $L(\mathbf{P}, \mathbf{P}', \sigma, \xi, h, \mathfrak{l}_0)$ denote the class of G-conjugacy of $\mathfrak{l}$.

Theorem 3.1. 1) *Any G-orbit on the set of all Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ is of the form $L(\mathbf{P}, \mathbf{P}', \sigma, \xi, h, \mathfrak{l}_0)$.*

2) $L(\mathbf{P}, \mathbf{P}', \sigma, \xi, h, \mathfrak{l}_0) = L(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}', \tilde{\sigma}, \tilde{\xi}, \tilde{h}, \tilde{\mathfrak{l}}_0)$ *iff* $(\mathbf{P}, \mathbf{P}', \xi, h, \mathfrak{l}_0)$ *and* $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}', \tilde{\xi}, \tilde{h}, \tilde{\mathfrak{l}}_0)$ *are $N(\mathfrak{h})$-conjugate (here by $N(\mathfrak{h})$ we denote the normalizer of $\mathfrak{h}$ in G).*

Remark 1. Let W be the Weyl group of the root system $\mathbf{R}$. If $(\mathbf{P},\mathbf{P}',\xi,h,\mathfrak{l}_0)$ and $(\tilde{\mathbf{P}},\tilde{\mathbf{P}}',\tilde{\xi},\tilde{h},\tilde{\mathfrak{l}}_0)$ are $N(\mathfrak{h})$-conjugate, then $(\mathbf{P},\mathbf{P}',\sigma,h)$ and $(\tilde{\mathbf{P}},\tilde{\mathbf{P}}',\tilde{\sigma},\tilde{h})$ are W-conjugate.

Remark 2. Every class of G-conjugacy of Lagrangian subalgebras in $\mathfrak{g}\times\mathfrak{g}$ depends on the discrete parameters $(\mathbf{P},\mathbf{P}',\sigma,h)$ and the continuous parameters $(\xi,\mathfrak{l}_0)$. Fix $(\mathbf{P},\mathbf{P}',\sigma)$ and denote by Ξ (respectively by Λ) the space of parameters ξ (respectively $\mathfrak{l}_0$) such that ξ (respectively $\mathfrak{l}_0$) corresponds to $(\mathbf{P},\mathbf{P}',\sigma)$. Let $\langle\mathbf{A}\rangle$ be the linear span of $\mathbf{A}$, $n=\dim\mathfrak{z}$. It can be proved that

$$\dim\Xi=\dim\,\{\alpha\in\langle\mathbf{A}\rangle\mid\sigma(\alpha)=\alpha\},$$

$\dim\Lambda=\frac{n(n-1)}{2}$ (note that Λ is the Lagrangian Grassmann manifold for $\mathfrak{z}\times\mathfrak{z}'$).

We shall say that a class of G-conjugacy $L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$ is *integrable* (respectively *algebraic integrable*) if the subalgebra $\mathfrak{l}\cap\mathfrak{g}_{diag}\subset\mathfrak{g}_{diag}\simeq\mathfrak{g}$ corresponds to a closed (respectively Zariski closed) subgroup in G for a certain (and then for every) Lagrangian subalgebra $\mathfrak{l}\in L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$. Theorem 3.2 gives a test of the integrability and the algebraic integrability of $L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$.

Let $H\subset G$ be the connected subgroup such that $\operatorname{Lie}H=\mathfrak{h}\subset\mathfrak{g}$.

Theorem 3.2. *A class of G-conjugacy $L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$ is integrable (respectively algebraic integrable) iff the subspace*

$$V=\{x\in\mathfrak{h}\mid(x_{\mathfrak{z}},x_{\mathfrak{z}'})\in\mathfrak{l}_0,\ \sigma^\vee(x_{\tilde{\mathfrak{h}}})=x_{\tilde{\mathfrak{h}}'}\}\subset\mathfrak{h}$$

(here $x_{\mathfrak{z}}$ is the image of x in $\mathfrak{z}$, ets.) is the Lie algebra of a closed (respectively Zariski closed) subgroup in H.

Remark 3. It follows from the Theorem 3.2 that the (algebraic) integrability of a G-conjugacy class $L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$ depends only on σ and $\mathfrak{l}_0$ (and is independent of ξ and h).

Now we recall a well-known method to verify that a subspace $V\subset\mathfrak{h}$ is the Lie algebra of a closed (respectively Zariski closed) subgroup in H.

Consider the lattice

$$\mathcal{H}=\operatorname{Ker}(\exp:\mathfrak{h}\to H)\subset\mathfrak{h}.$$

Proposition 3.3 (see [9, Ch.3, §2, Theorem 5]). *A subspace $V\subset\mathfrak{h}$ corresponds to a Zariski closed subgroup in H iff V is defined over $\mathbb{Q}$ with respect to the lattice $\mathcal{H}$, i.e., $V=\mathcal{V}\otimes\mathbb{C}$ for a certain sublattice $\mathcal{V}\subset\mathcal{H}$.*

Let $\mathfrak{t}=\mathcal{H}\otimes\mathbb{R}\subset\mathfrak{h}$.

Proposition 3.4. *A subspace $V\subset\mathfrak{h}$ corresponds to a closed subgroup in H iff $V\cap\mathfrak{t}$ is defined over $\mathbb{Q}$ with respect to the lattice $\mathcal{H}$, i.e., $V\cap\mathfrak{t}=\mathcal{V}\otimes\mathbb{R}$ for a certain sublattice $\mathcal{V}\subset\mathcal{H}$.*

4. Methods of proof of Theorem 3.1. Now we present a way to prove Theorem 3.1.

Let $\mathfrak{p},\mathfrak{p}'\subset\mathfrak{g}$ be parabolic subalgebras. We have $\mathfrak{p}/\mathfrak{p}^\perp=\mathfrak{a}\oplus\mathfrak{z}$, where $\mathfrak{a}$ is semisimple, and $\mathfrak{z}$ is abelian; the same holds for $\mathfrak{p}'$. Let $\theta:\mathfrak{a}\to\mathfrak{a}'$ be an isomorphism such that θ

preserves $\langle\cdot\,,\cdot\rangle$. We shall say that a triple $(\mathfrak{p},\mathfrak{p}',\theta)$ is *admissible*. By $T(\mathfrak{g})$ denote the set of all admissible triples.

Consider $(\mathfrak{p},\mathfrak{p}',\theta)\in T(\mathfrak{g})$. Let $\mathfrak{l}_0\subset\mathfrak{z}\times\mathfrak{z}'$ be a Lagrangian subspace with respect to the bilinear form (1). The quadruple $(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$ is called *admissible*. Suppose $(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$ is an admissible quadruple; then set

$$\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0):=\{(x,y)\in\mathfrak{p}\times\mathfrak{p}'\mid\theta(x_{\mathfrak{a}})=y_{\mathfrak{a}'},\ (x_{\mathfrak{z}},y_{\mathfrak{z}'})\in\mathfrak{l}_0\}\subset\mathfrak{g}\times\mathfrak{g},$$

where $x_{\mathfrak{a}}$ is the image of x in $\mathfrak{a}$, $x_{\mathfrak{z}}$ is the image of x in $\mathfrak{z}$, $y_{\mathfrak{a}'}$ is the image of y in $\mathfrak{a}'$, $y_{\mathfrak{z}'}$ is the image of y in $\mathfrak{z}'$. It is not hard to prove the following proposition.

PROPOSITION 4.1. 1) *$\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$ is a Lagrangian subalgebra.*

2) *The correspondence $(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)\mapsto\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$ is a G-equivariant bijection between the set of all Lagrangian subalgebras in $\mathfrak{g}\times\mathfrak{g}$ and the set of all admissible quadruples $(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$.*

3) *Lagrangian subalgebras $\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\theta,\mathfrak{l}_0)$ and $\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\theta,\tilde{\mathfrak{l}}_0)$ are G-conjugate iff $\mathfrak{l}_0=\tilde{\mathfrak{l}}_0$.*

Thus a classification of Lagrangian subalgebras is reduced to a classification of admissible triples up to G-conjugacy. It can be shown that the theory of admissible triples is quite similar to the theory of automorphisms of complex semisimple Lie algebras. In fact, there exists a natural notion of a semisimple admissible triple; we can define a notion of an invariant subalgebra for an admissible triple; for any semisimple admissible triple there exists an invariant Cartan subalgebra; it is possible, using invariant Cartan subalgebras, to give a complete description of semisimple admissible triples up to G-conjugacy; for any admissible triple there exists an analogue of the Jordan decomposition, etc. The realization of this program leads us to Theorem 3.1.

5. A geometric interpretation. In this section we give a geometric interpretation of some Poisson homogeneous G-spaces.

By $\bar{G}$ denote the group of all automorphisms $g:\mathfrak{g}\to\mathfrak{g}$ such that the following conditions hold: (1) g preserves the scalar product $\langle\cdot\,,\cdot\rangle$; (2) g is equal to the identity mapping on the center of $\mathfrak{g}$. Suppose $g\in\bar{G}$ and set

$$\mathfrak{l}_g=\{(x,y)\mid x=g(y)\}\subset\mathfrak{g}\times\mathfrak{g}.$$

Then $\mathfrak{l}_g$ is a Lagrangian subalgebra. Note that the Lagrangian subalgebras $\mathfrak{l}_g$ form the G-conjugacy classes $L(\mathbf{P},\mathbf{P}',\sigma,\xi,h,\mathfrak{l}_0)$ such that $\mathbf{P}=\mathbf{P}'=\mathbf{R}$ and $\mathfrak{l}_0$ is the image of the center of $\mathfrak{g}$ under the diagonal mapping $\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$.

Let us give a geometric interpretation of Poisson homogeneous G-spaces corresponding to Lagrangian subalgebras of the form $\mathfrak{l}_g$. Note that the connected component of the center of G acts trivially on the subalgebras $\mathfrak{l}_g$; therefore it is enough to consider the case when G is semisimple, i.e., $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. In the following part of this section we consider the case $G=\operatorname{Int}\mathfrak{g}$.

Let ϕ,ψ be regular functions on $\bar{G}$. Consider

$$\begin{aligned}\{\phi,\psi\}=&-r^{\mu\nu}_{alt}\cdot(\partial'_\mu\phi-\partial_\mu\phi)\cdot(\partial'_\nu\psi-\partial_\nu\psi)\\&+r^{\mu\nu}_{sym}\cdot(\partial'_\mu\phi-\partial_\mu\phi)\cdot(\partial'_\nu\psi+\partial_\nu\psi),\end{aligned}\tag{2}$$

where r, ∂_μ and ∂'_μ are defined in Section 2. By X we denote the manifold $\bar{G}$ equipped with the bracket (2).

THEOREM 5.1. *The bracket* (2) *is a Poisson bracket, the action of* G *on* X *by conjugations is Poisson, and the orbits of this action are Poisson homogeneous* G*-spaces such that the Lagrangian subalgebra* $\mathfrak{l}_g$ *corresponds to a point* $g \in X$*.*

REMARK 4. The bracket (2) is a special case of the bracket from [10, Theorem 3.1], when $J_1 = -J_2$ (using the notation from [10]). See also [7].

Theorem 5.1 can be proved by using the following general result (see Theorem 5.2). Suppose G is an arbitrary Poisson-Lie group. A *double* of G is a Lie group D such that the following conditions hold: (1) $\mathrm{Lie}\, D = D(\mathfrak{g})$; (2) The natural scalar product in $D(\mathfrak{g})$ is invariant with respect to the adjoint action of D (then D becomes a Poisson-Lie group by means of the canonical element $r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset D(\mathfrak{g}) \otimes D(\mathfrak{g})$, see [4, §13]); (3) G is a closed Poisson-Lie subgroup in D.

THEOREM 5.2 ([6]). *Let* G *be a Poisson-Lie group,* $\mathfrak{g} = \mathrm{Lie}\, G$*. Let* D *be a double of* G*. Consider the action of* G *on the Poisson manifold* D/G *by left translations. Suppose* $w \in D$ *and denote by* x *the image of* w *in* D/G*; then* $X = G \cdot x$ *is a Poisson homogeneous* G*-space, and the Lagrangian subalgebra* $\mathfrak{l}_x := w \cdot \mathfrak{g} \cdot w^{-1} \subset D(\mathfrak{g})$ *corresponds to the pair* (X, x)*.*

In our case take $D = \bar{G} \times \bar{G}$; then Theorem 5.1 follows from Theorem 5.2.

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