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# ISOMORPHISMS OF POISSON AND JACOBI BRACKETS

JANUSZ GRABOWSKI

*Institute of Mathematics, University of Warsaw*
*Banacha 2, 02-097 Warszawa, Poland*
*E-mail: jagrab@mimuw.edu.pl*

**Abstract.** We present a general theorem describing the isomorphisms of the local Lie algebra structures on the spaces of smooth (real-analytic or holomorphic) functions on smooth (resp. real-analytic, Stein) manifolds, as, for example, those given by Poisson or contact structures. We admit degenerate structures as well, which seems to be new in the literature.

**1. Introduction.** We shall admit different classes of smoothness, so by a manifold of class $\mathcal{C}$, where $\mathcal{C} = C^\infty$, $\mathcal{C} = C^\omega$, or $\mathcal{C} = \mathcal{H}$, and by the algebra $\mathcal{C}(M)$ of class $\mathcal{C}$ functions on $M$ we shall mean

a) a real paracompact finite-dimensional smooth manifold and the algebra $C^\infty(M)$ of all real smooth functions on $M$, if $\mathcal{C} = C^\infty$;
b) a real-analytic paracompact finite-dimensional manifold and the algebra $C^\omega(M)$ of all real-analytic functions on $M$, if $\mathcal{C} = C^\omega$;
c) a complex finite-dimensional manifold for which each connected component is Stein and the (complex) algebra of all holomorphic functions on $M$, if $\mathcal{C} = \mathcal{H}$.

A *Jacobi structure* [8] on $M$ is a pair $(\Lambda, D)$ consisting of a vector field $D \in \Gamma(TM)$ and an antisymmetric bi-vector field $\Lambda \in \Gamma(\Lambda^2 TM)$ of class $\mathcal{C}$ satisfying:
*i)* $[D, \Lambda] = 0$, and
*ii)* $[\Lambda, \Lambda] = 2D \wedge \Lambda$,

where $[\cdot, \cdot]$ stands for the Schouten bracket. If $D = 0$, then $[\Lambda, \Lambda] = 0$ and we call $\Lambda$ the *Poisson structure.*

Note that in the case of $\mathcal{C} = \mathcal{H}$, vector fields are holomorphic and of type $(1, 0)$, since we shall understand vector fields as derivations of the algebra $\mathcal{C}(M)$.

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Every Jacobi (in particular, Poisson) structure induces a Lie bracket $\{\cdot,\cdot\}$ on the algebra $\mathcal{C}(M)$, called the *Jacobi* (resp. *Poisson*) *bracket*, by

$$\{f,g\} = \Lambda(f,g) + fD(g) - gD(f). \tag{$*$}$$

This is exactly the local Lie algebra structure investigated by Kirillov [6], Lichnerowicz [7, 8], and Guedira and Lichnerowicz [5] in the special case of a trivial one-dimensional vector bundle over $M$.

Conversely, every Lie bracket on $\mathcal{C}(M)$ given by a bilinear differential operator is of the form $(*)$. The vector $D$ and the bi-vector field $\Lambda$ are uniquely determined by the bracket if we consider $\mathcal{C}(M)$ as an algebra. For example, $D = ad_{\mathbf{1}} = \{\mathbf{1},\cdot\}$. There is, however, an ambiguity when we consider $\mathcal{C}(M)$ as a vector space only (sections of a trivial vector bundle) and we do not know which one is the unit (canonical generator). Consider for instance a linear automorphism $A : \mathcal{C}(M) \to \mathcal{C}(M)$ given by $A(f) = uf$ with $u$ being a nowhere vanishing function from $\mathcal{C}(M)$. We get a new local bracket $\{f,g\}_A = A^{-1}\{A(f),A(g)\}$ corresponding to the Jacobi structure $(u\Lambda, uD + \Lambda(u,\cdot))$.

Classical examples of the Jacobi brackets in our sense are: the standard Poisson bracket on a symplectic manifold $M$ of dimension $2n$ and the Lagrange bracket on a contact manifold of dimension $2n+1$. For symplectic Poisson bracket we have the well-known form in canonical coordinates $(q_i,p_i)$:

$$\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

and

$$\{f,g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right).$$

The Lagrange (contact) bracket is given in canonical coordinates $(x_1,\dots,x_{2k+1})$ by

$$\begin{aligned}\{f,g\} = {} & \sum_{i=1}^{k} \left( \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_{i+k}} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_{i+k}} \right) \\ & + \frac{\partial f}{\partial x_{2k+1}} \left( \sum_{i=1}^{2k} x_i \frac{\partial g}{\partial x_i} - 2g \right) - \frac{\partial g}{\partial x_{2k+1}} \left( \sum_{i=1}^{2k} x_i \frac{\partial f}{\partial x_i} - 2f \right).\end{aligned}$$

These brackets are nowhere vanishing, but also Jacobi structures with some degeneracy are of some interest, as for example the Poisson structures on Poisson-Lie groups which always have a singularity at the neutral element (cf. [7] and [11]).

Given a Jacobi manifold, every function $f \in \mathcal{C}(M)$ defines the corresponding (Jacobi-)hamiltonian vector field

$$\hat{f} = \Lambda(f,\cdot) + fD.$$

The mapping

$$\hat{\ }: \mathcal{C}(M) \to \Gamma(TM)$$

is a homomorphism of the Lie algebras with the usual Lie bracket for vector fields. The kernel of this homomorphism is exactly $\mathcal{Z}(M)$, the center of the Lie algebra $\mathcal{C}(M)$. Set

$\mathcal{X}(\Lambda, D)$ for the Lie algebra of hamiltonian vector fields associated with $(\Lambda, D)$. Since

$$(uf)\hat{} = u\Lambda(f,\cdot) + f(uD + \Lambda(u,\cdot)),$$

we have $\mathcal{X}(\Lambda, D) = \mathcal{X}(u\Lambda, uD + \Lambda(u,\cdot))$ for any nowhere vanishing $u \in \mathcal{C}(M)$, which shows that $\Lambda$ and $D$ are not uniquely defined by its Lie algebra of hamiltonian vector fields. We shall see later on that the freedom of choice is exactly the one described above.

Define the support $S$ of a Jacobi bracket $\{\cdot,\cdot\}$ to be the set of those points $p \in M$ for which $\{f,g\}(p) \neq 0$ for some $f, g \in \mathcal{C}(M)$. It is easy to see that $S = supp(\Lambda) \cup supp(D)$ and that it is the support of the Lie algebra of hamiltonian vector fields. It is also clear that the Jacobi bracket $\{f,g\}$ does not recognize the behaviour of $f$ and $g$ outside its support. It makes the description of isomorphisms between general Jacobi brackets a little more complicated than those for nondegenerate brackets.

## 2. Statement of the results

THEOREM 1. *Let $(M_i, (\Lambda_i, D_i))$ be a Jacobi manifold of class $\mathcal{C}$, let $\{\cdot,\cdot\}$ be the corresponding Jacobi bracket with support $S_i$, and let $\mathcal{X}_i$ be the corresponding Lie algebra of hamiltonian vector fields, $i = 1, 2$. If*

$$\phi : (\mathcal{C}(M_1), \{\cdot,\cdot\}_1) \to (\mathcal{C}(M_2), \{\cdot,\cdot\}_2)$$

*is a Lie algebra isomorphism, then there is a diffeomorphism $\psi : S_2 \to S_1$ of class $\mathcal{C}$, a nowhere vanishing function $u \in \mathcal{C}(S_1)$, and a linear map $F : \mathcal{C}(M_1) \to \mathcal{C}(S_2)$ with image in the center of the bracket $\{\cdot,\cdot\}_{2|S_2}$ and vanishing on the derived ideal $\{\mathcal{C}(M_1), \mathcal{C}(M_1)\}$ such that*

*(a) the induced isomorphism $\Phi : \mathcal{X}_1 \simeq \mathcal{C}(M_1)/\mathcal{Z}(M_1) \to \mathcal{X}_2 \simeq \mathcal{C}(M_2)/\mathcal{Z}(M_2)$ is of the form $\psi_*^{-1}$ (outside supports we have simply zero);*

*(b) $\phi(f)_{|S_2} = (u^{-1}f) \circ \psi + F(f)$ for any $f \in \mathcal{C}(M_1)$;*

*(c) $\psi_*(\Lambda_{2|S_2}) = u\Lambda_{1|S_1}$ and $\psi_*(D_{2|S_2}) = \Lambda_{1|S_1}(u,\cdot) + uD_{1|S_1}$.*

If the Jacobi brackets are nondegenerate then $\psi$ and $u$ are defined everywhere and describe the isomorphism $\phi$ completely up to $F$. We get in particular:

COROLLARY 1. *Every automorphism $\phi$ of the Poisson bracket on a connected Poisson manifold $(M, \Lambda)$ of class $\mathcal{C}$ with non-singular (i.e. nowhere-vanishing) Poisson bracket is of the form*

$$\phi(f) = (u^{-1} \cdot f) \circ \psi + F(f), \tag{2}$$

*where*

*(a) $u$ is a non-vanishing Casimir function;*

*(b) $\psi$ is a diffeomorphism of $M$ of class $\mathcal{C}$ such that $\psi_*(\Lambda) = u\Lambda$;*

*(c) $F$ is a linear map $F : \mathcal{C}(M) \to \mathcal{Z}(M)$ vanishing on $\{\mathcal{C}(M), \mathcal{C}(M)\}$.*

*In particular, for a symplectic Poisson bracket, $F = 0$ if $M$ is non-compact and $F(f) = a \cdot \int_M f\eta$ for a constant $a$ and $\eta$ being the Liouville volume form, if $M$ is compact.*

*Each of $\psi$, $u$, and $F$ are uniquely determined by $\phi$. Conversely, any $u, \psi, F$ as in (a),(b),(d) define $\phi$ by (2) which is an automorphism of Poisson brackets if only it is bijective.*

Note that the above result for symplectic manifolds was obtained earlier by Atkin and Grabowski [1].

Corollary 2. *Every automorphism $\phi$ of the Lagrange (contact) bracket on a contact manifold $(M,\beta)$ of class $\mathcal{C}$ is of the form $\phi(f) = v\cdot(f\circ\psi)$ for a nowhere vanishing $v\in\mathcal{C}(M)$ and a diffeomorphism $\psi$ of $M$ of class $\mathcal{C}$ such that $v\cdot\psi^*(\beta)=\beta$.*

The $C^\infty$ case of the above result is due to Omori [9].

To get the flavour of how the presented general theorem works in degenerate cases, consider the following examples.

Example 1. Let $\{\cdot,\cdot\}$ be the Jacobi bracket on $C^\infty(\mathbf{R})$ defined by

$$\{f,g\}(x) = x(f(x)g'(x) - f'(x)g(x))$$

(where prime stands for the derivative), i.e. $\Lambda = 0$ and $D = x\frac{\partial}{\partial x}$. The support $S$ of the bracket is clearly $\mathbf{R}^* = \mathbf{R}\setminus\{0\}$ and the center is trivial. If $\phi$ is an automorphism of the defined bracket, then by our theorem $\phi(f) = v^{-1}\cdot(f\circ\psi)$ with $v$ being a nowhere vanishing function of $C^\infty(\mathbf{R}^*)$ and $\psi$ being a diffeomorphism of $\mathbf{R}^*$ such that

$$\psi_*\left(x\frac{\partial}{\partial x}\right) = v(\psi^{-1}(x))\cdot x\frac{\partial}{\partial x}.$$

Since $\phi(\mathbf{1})_{|\mathbf{R}^*} = v^{-1}$, we can consider $v$ as a smooth non-vanishing function on $\mathbf{R}$. It is not hard to verify now that $\psi$ extends to a diffeomorphism of $\mathbf{R}$ with $0\in\mathbf{R}$ as a fixed point. One can easily compute that $v(x) = \frac{x\psi\prime(x)}{\psi(x)}$. Observe that the quotient makes sense at 0 if we pass to the limit. This proves the following.

Corollary 3. *The mapping $\phi : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ is an automorphism of the bracket $\{f,g\}(x) = x(f(x)g'(x) - f'(x)g(x)$ if and only if*

$$\phi(f)(x) = \frac{\psi(x)}{x\psi\prime(x)}f(\psi(x))$$

*for a diffeomorphism $\psi : \mathbf{R}\to\mathbf{R}$, $\psi(0)=0$.*

Example 2. The Kirillov-Kostant-Souriau bracket on $\mathrm{su}(2)^*$, the dual space of the Lie algebra su(2), is the linear Poisson bracket defined in linear coordinates $(x_1,x_2,x_3)$ by

$$[x_1,x_2] = x_3,\quad [x_2,x_3] = x_1,\quad [x_3,x_1] = x_2.$$

The corresponding Poisson tensor $\Lambda$ is singular at 0 and the symplectic leaves consist of spheres

$$S_r = \{(x_1,x_2,x_3)\in\mathrm{su}(2)^* : x_1^2+x_2^2+x_3^2 = r^2,\quad r\geq 0\}.$$

Let us take a diffeomorphism $\psi$ of $\mathrm{su}(2)^*$ given by $\psi(x) = e^t x$, where $t\in\mathbf{R}$. This diffeomorphism preserves the support $\mathrm{su}(2)^*\setminus\{0\}$ of $\Lambda$ and $\psi_*(\Lambda) = e^t\Lambda$.

Take now $F : C^\infty(\mathrm{su}(2)^*) \to C^\infty(\mathrm{su}(2)^*)$ given by

$$F(f)(x) = \int_{S_{\|e^t x\|}} f\,\mathrm{d}\mu_{\|e^t x\|},$$

where $\mu_r$ is the canonical SU(2)-invariant measure on the sphere $S_r$, $\mu_r(S_r) = 4\pi r^2$. It is easy to see that the values of $F$ consist of functions constant on spheres (i.e. Casimir elements of the Poisson bracket) and that $F$ vanishes on the derived algebra, since on the symplectic leaves the elements of the derived algebra have mean value zero (since the leaves are compact symplectic manifolds, see [7]). According to Theorem 1, we look for automorphisms $\psi$ of the Poisson bracket in the form

$$\phi(f)(x) = e^{-t} f(e^t x) + a \int_{S_{\|e^t x\|}} f \, \mathrm{d}\mu_{\|e^t x\|},$$

for some $a \in \mathbf{R}$. Integrating the function $\phi(f)$ on the sphere $S_r$, we get, after changing coordinates,

$$\begin{aligned}\int_{S_r} \phi(f)(x) \mathrm{d}\mu_r &= e^{-t} \int_{S_r} f(e^t x) \mathrm{d}\mu_r + 4a\pi e^{2t} r^2 \int_{S_{(e^t r)}} f(x) \mathrm{d}\mu_{e^t r} \\ &= (e^{-3t} + 4a\pi e^{2t} r^2) \int_{S_{(e^t r)}} f(x) \mathrm{d}\mu_{e^t r}.\end{aligned}$$

Hence, if $a < 0$, then for the value

$$r = \sqrt{\frac{-e^{-5t}}{4\pi a}}$$

we have

$$\int_{S_r} \phi(f)(x) \mathrm{d}\mu_r = 0$$

for any function $f$, so that $\phi$ cannot be a bijection. If $a \geq 0$, then $\phi$ is invertible, since one calculates explicitly

$$f(x) = e^t \phi(f)(e^{-t} x) - \frac{a e^{3t}}{e^{-t} + 4a\pi \|x\|^2} \int_{S_{\|e^{-t} x\|}} \phi(f) \mathrm{d}\mu_{\|e^{-t} x\|}.$$

Thus, our $\phi$ is an automorphism of the linear Poisson bracket. This automorphism is identical on linear functions: $\phi(x_i) = x_i$, but not identical in the whole, since $\phi(1) = e^{-t} + 4a\pi e^{2t} \|x\|^2$ and $\phi(\|x\|^2) = e^t \|x\|^2 + 4a\pi e^{4t} \|x\|^4$.

**3. The idea of the proof.** We start with a purely algebraic setting for the Jacobi structures as presented in [4]: a *Jacobi structure* on an associative commutative algebra $A$ is a pair $(\Lambda, D)$ consisting of a derivation $D$ and an antisymmetric bilinear derivation $\Lambda$ of $A$ satsfying $[D, \Lambda] = 0$ and $[\Lambda, \Lambda] = 2D \wedge \Lambda$, where $[\cdot, \cdot]$ stands for the Richardson-Nijenhuis bracket.

We obtain the corresponding Jacobi bracket $\{\cdot, \cdot\}$ on $A$ by the formula $(*)$ which gives for $A = \mathcal{C}(M)$ the model considered previously. We define the *associative spectrum* $\mathcal{M}(A)$ to be the family of all maximal finite-codimensional associative ideals of $A$ and the *Lie spectrum* $\Sigma(A)$ of $A$ to be the family of all maximal finite-codimensional Lie subalgebras of $A$ containing no finite-codimensional Lie ideals of $A$.

Note that for the asssociative algebra $\mathcal{C}(M)$ the ideals of the spectrum $\mathcal{M}(\mathcal{C}(M))$ consist of functions vanishing at a given point of $M$ and for the Lie algebra of all class $\mathcal{C}$ vector fields on $M$ the Lie subalgebras from its Lie spectrum consist of vector fields vanishing at a given point (cf. [3]).

For an ideal $I \in \mathcal{M}(A)$ put $N(I) := \{f \in A : \{f, I\} \subset I\}$ and set

$$\mathcal{M}^*(A) = \{I \in \mathcal{M}(A) : 0 < codim_A N(I) < \infty\}.$$

We have:

PROPOSITION 1. *If $A^2 = A$ (e.g. $A$ is unital) then the mapping*

$$\mathcal{M}^*(A) \ni I \mapsto N(I)$$

*establishes a one-one correspondence between $\mathcal{M}^*(A)$ and $\Sigma(A)$.*

One can easily prove that in the case of $A = \mathcal{C}(M)$ the elements of $\mathcal{M}^*(A)$ correspond to the points of the support $S$ of the Jacobi bracket and, hence, that

$$S \ni p \mapsto \{f \in \mathcal{C}(M) : \hat{f}(p) = 0\}$$

is a one-one correspondence between the support and the Lie spectrum $\Sigma(\mathcal{C}(M))$. It implies the following.

PROPOSITION 2. *The mapping*

$$S \ni p \mapsto L(p) := \{X \in \mathcal{X}(\Lambda, D) : X(p) = 0\}$$

*is a one-one correspondence between $S$ and the Lie spectrum $\Sigma(\mathcal{X}(\Lambda, D))$ of the Lie algebra of hamiltonian vector fields.*

Having now a Lie algebra isomorphism $\phi : \mathcal{C}(M_1) \to \mathcal{C}(M_2)$ and the induced isomorphism $\Phi : \mathcal{X}(M_1) \to \mathcal{X}(M_2)$ of the corresponding Lie algebras of hamiltonian vector fields and putting $\psi(p) = L^{-1}(\Phi^{-1}(L(p)))$, we get a bijection $\psi : S_2 \to S_1$ on the level of supports such that $\Phi(X)(p) = 0$ if and only if $X(\psi(p)) = 0$ for all $X \in \mathcal{X}(M_1)$ and all $p \in S_2$.

One can prove, with some effort, that $\psi$ is in fact a diffeomorphism of class $\mathcal{C}$ and that $\Phi(X) = \psi_*^{-1}(X)$. The last (and non-trivial) part of the proof is to show that this form of $\Phi$ implies the form of $\phi$ described in the theorem. This can be done with the help of certain ideas due to Skriabin [10].

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