RECOGNIZING RIGHT-LEFT EQUIVALENCE LOCALLY

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The present paper is a survey of the author’s recent results on recognizing $C^r$ right-left equivalence of $C^\infty$ map-germs $(0 \leq r \leq \infty)$.

We say two $C^\infty$ map-germs are $C^r$ right-left equivalent if they coincide under germs of appropriate $C^r$ co-ordinate systems of the source space and the target space, where a $C^0$ co-ordinate system means a co-ordinate system given by a homeomorphism. We often encounter the situations where we would like to decide whether or not given two map-germs are $C^r$ right-left equivalent. In the case that one of them is of full rank (resp. linear), the implicit function theorem (resp. the rank theorem) answers our purpose (possibly except for $r = 0$). However, how can we decide in general case? By using a simple systematic method explained in Section 4, we can obtain many results to the problem. In Section 1, we give a series of criteria for $C^r$ right-left equivalence of $C^\infty$ map-germs $(1 \leq r \leq \infty)$. In Section 2, infinitesimal refinements of criteria for $C^\infty$ right-left equivalence of $C^\infty$ map-germs are given. Next, we consider $C^0$ right-left equivalence. In Section 3, we give a series of criteria for $C^0$ right-left equivalence of $K$-equivalent map-germs. All of the results are derived from one simple idea, which is the key of our systematic method and explained exhaustively in Section 4. In Section 5 we give several applications of our results, which show how useful our method is.

The results for $r = \infty$ are all valid both in the real analytic category and in the complex analytic category as well.

1. Criteria for $C^r$ right-left equivalence $(1 \leq r \leq \infty)$. For a given $C^\infty$ map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, any $C^r$ map-germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p, 0)$ such that $\Phi(x, 0) = f(x)$ is called a $C^r$ deformation-germ of $f$. A $C^r$ deformation-germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p, 0)$ of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is said to be $C^r$-trivial if

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there exist germs of \( C^r \) diffeomorphisms \( h : (\mathbb{R}^n \times \mathbb{R}^k, (0,0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^k, (0,0)) \) and \( H : (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) \) such that the following diagram \((*)\) commutes, where \( \pi : (\mathbb{R}^n \times \mathbb{R}^k, (0,0)) \rightarrow (\mathbb{R}^k, 0), \) \( \pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0,0)) \rightarrow (\mathbb{R}^k, 0), \) are canonical projections:

\[
\begin{array}{c}
\begin{array}{c}
(\mathbb{R}^n \times \mathbb{R}^k, (0,0)) \\
\downarrow h
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(\mathbb{R}^p \times \mathbb{R}^k, (0,0)) \\
\downarrow \pi
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(\mathbb{R}^k, 0)
\end{array}
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\begin{array}{c}
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(\mathbb{R}^k, 0)
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\begin{array}{c}
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(\mathbb{R}^p, 0)
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(\mathbb{R}^k, 0)
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(\mathbb{R}^p, 0)
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\end{array}

\]

For given two \( C^\infty \) map-germs \( f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0), \) we consider the following conditions \((i), (ii), (iii), \) and \((iv).\)

\((i)\) The map-germ \( f \) is \( C^r \) right-left equivalent to \( g. \)

\((ii)\) There exist a germ of \( C^r \) diffeomorphism \( s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) and a \( C^r \) map-germ \( M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0)) \) such that the following (a) and (b) are satisfied:

(a) \( f(x) = M(x)g(s(x)), \)
(b) the \( C^r \) map-germ \( F : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \rightarrow (\mathbb{R}^p, 0) \) given by

\[
F(x, \lambda) = f(x) - M(x)\lambda
\]

is a \( C^r \)-trivial deformation-germ of \( f. \)

\((iii)\) There exist a germ of \( C^r \) diffeomorphism \( s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) and a \( C^r \) map-germ \( M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0)) \) such that (a), (b) of condition \((ii)\) and the following (c) are satisfied:

(c) The \( C^r \) map-germ \( G : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \rightarrow (\mathbb{R}^p, 0) \) given by

\[
G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda
\]

is a \( C^r \)-trivial deformation-germ of \( g. \)

\((iv)\) There exist a germ of \( C^r \) diffeomorphism \( s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) and a \( C^r \) map-germ \( M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0)) \) such that (a), (b) of condition \((ii)\) and the following (d) are satisfied:

(d) The germ \( H([0] \times \mathbb{R}^p), 0) \) is transverse to the germ \( ([0] \times \mathbb{R}^p), 0, \) where \( H \) is the germ of \( C^r \) diffeomorphism of \( (\mathbb{R}^p \times \mathbb{R}^p, 0) \) given in the above commutative diagram \((*)\) with \( k, \Phi \) replaced by \( p, F. \)

First, we consider rank zero cases.

**Theorem 1.1** ([15]). Let \( f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0) \) be \( C^\infty \) map-germs with rank zero. Then condition \((ii)\) implies condition \((i)\) for \( 1 \leq r \leq \infty. \)

Next, we consider positive rank cases.

**Example 1.1.** Let \( f, g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \) be given by

\[
f(x, y) = (x, y^3 + xy),
g(x, y) = (x, y^3)
\]
and $M : (\mathbb{R}^2, 0) \to (GL(2, \mathbb{R}), E_2)$ be given by

$$M(x, y) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix},$$

where $E_2$ is the unit 2 by 2 matrix. Then $f(x, y) = M(x, y)g(x, y)$.

It is well known that any $C^\infty$ deformation-germ of the map-germ $f$ is $C^\infty$-trivial. Thus, (ii) is satisfied. However, for any $1 \leq r \leq \infty$ condition (i) does not hold (in fact, $f$ and $g$ are even not topologically right-left equivalent).

This example shows that condition (ii) does not necessarily imply condition (i) in positive rank cases. Nevertheless, the following holds under no assumptions.

**Theorem 1.2** ([15]). Condition (iii) implies condition (i) for $1 \leq r \leq \infty$.

Although Theorem 1.2 is interesting in itself, we prefer the $C^r$ triviality of the linearly parametrized deformation-germ of only one of $f$ or $g$ to those of both of $f$ and $g$. Thus, we are led to condition (iv).

**Theorem 1.3** ([15]). Condition (iv) implies condition (i) for $1 \leq r \leq \infty$.

In the case $r = \infty$, we have

**Theorem 1.4** ([15]). For any $C^\infty$ map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, the following hold:

1. $(i_\infty) \iff (iii_\infty) \iff (iv_\infty)$.
2. $(i_\infty) \iff (ii_\infty) \iff (iii_\infty) \iff (iv_\infty)$ if the rank of $f$ is zero.

Therefore, we may answer the $C^\infty$ recognition problem completely by using our conditions in principle.

### 2. Infinitesimal refinements of criteria for $C^\infty$ right-left equivalence.

First, we review infinitesimal notations briefly. For details on them, see [9], [14], [15], [21].

Let $\mathcal{E}_n$ (resp. $m_n$) denote the set of $C^\infty$ function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ (resp. $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$). The set $\mathcal{E}_n$ has a natural $\mathbb{R}$-algebra structure and the set $m_n$ is the unique maximal ideal in $\mathcal{E}_n$. For any positive integer $\ell$, $m_n^\ell$ means the $\ell$-times product of $m_n$. For $\ell = 0$, $m_n^0$ is $\mathcal{E}_n$.

For a $C^\infty$ map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let $\theta(f)$ denote the $\mathcal{E}_n$-module consisting of germs of $C^\infty$ vector fields $\zeta : (\mathbb{R}^n, 0) \to T(\mathbb{R}^p)$ such that $\pi_p \circ \zeta = f$, where $\pi_p : T(\mathbb{R}^p) \to \mathbb{R}^p$ denotes the canonical projection. By using the standard identification of $T(\mathbb{R}^p)$ with $\mathbb{R}^p \times \mathbb{R}^p$, $\theta(f)$ may be identified with the free $\mathcal{E}_n$-module with $p$-generators. When the given $f$ is the identity map-germ $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\theta(f)$ may be denoted by $\theta(n)$.

For a $C^\infty$ map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let $Tf : (T\mathbb{R}^n, \pi_n^{-1}(0)) \to (T\mathbb{R}^p, \pi_p^{-1}(0))$ denote the tangent map-germ of $f$, where $\pi_n : T\mathbb{R}^n \to \mathbb{R}^n$, $\pi_p : T\mathbb{R}^p \to \mathbb{R}^p$ are canonical projections. We define

$$tf : \theta(n) \to \theta(f), \quad wf : \theta(p) \to \theta(f)$$
by \( tf(ξ) = Tf ∘ ξ, wf(η) = η ∘ f \). By using \( tf \) and \( wf \), we define
\[
T_A(f) = tf(m_nθ(n)) + wf(m_pθ(p)) \quad \text{and} \quad T_εA(f) = tf(θ(n)) + wf(θ(p)).
\]

**Theorem 2.1.** Let \( f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be \( C^∞ \) map-germs with rank zero. Suppose that there exist a germ of \( C^∞ \) diffeomorphism \( s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and a \( C^∞ \) map-germ \( M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0)) \) such that \( f(x) = M(x)g(s(x)) \). Suppose furthermore that there exists an integer \( k \) \((k ≥ 0)\) such that
(a) each entry of \( M - M(0) \) belongs to \( m_n^{k+1} \) and
(b) \( m_n^kθ(f) \subset T_eA(f) \).

Then \( f \) and \( g \) are \( C^∞ \) right-left equivalent.

Although there are no proofs of Theorem 2.1 in a series of author’s papers [13]–[18], by the proof of Theorem 2.2 below it is clear that conditions (a) and (b) of Theorem 2.1 imply the \( C^∞ \)-triviality of \( f(x) - M(x)λ \). Thus, Theorem 2.1 follows from Theorem 1.1.

**Theorem 2.2 ([14]).** Let \( f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be \( C^∞ \) map-germs. Suppose that there exist a germ of \( C^∞ \) diffeomorphism \( s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and a \( C^∞ \) map-germ \( M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0)) \) such that \( f(x) = M(x)g(s(x)) \). Suppose furthermore that there exists an integer \( k \) \((k ≥ 0)\) such that
(a) each entry of \( M - M(0) \) belongs to \( m_n^{k+1} \),
(b) \( m_n^kθ(f) \subset T_eA(f) \) and
(c) \( m_n^kθ(g) \subset T_eA(g) \).

Then \( f \) and \( g \) are \( C^∞ \) right-left equivalent.

Theorem 2.2 was stated (but not proved) first by A. A. du Plessis ([19], page 128). Conditions (a) and (b) (resp. (a) and (c)) of Theorem 2.2 imply the \( C^∞ \)-triviality of \( f(x) - M(x)λ \) (resp. \( g(x) - M(s^{-1}(x))^{-1}λ \)). Thus, Theorem 2.2 follows from Theorem 1.2 (for details, see [14]).

**Theorem 2.3 ([14]).** Let \( f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be \( C^∞ \) map-germs. Suppose that there exist a germ of \( C^∞ \) diffeomorphism \( s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and a \( C^∞ \) map-germ \( M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0)) \) such that \( f(x) = M(x)g(s(x)) \). Suppose furthermore that there exists a positive integer \( k \) such that
(a) each entry of \( M - M(0) \) belongs to \( m_n^k \) and
(b) \( m_n^kθ(f) \subset T_A(f) \).

Then \( f \) and \( g \) are \( C^∞ \) right-left equivalent.

Conditions (a) and (b) imply condition (iv). Thus Theorem 2.3 follows from Theorem 1.3 (for details, see [14]). Although we can deduce infinitesimal results from Theorem 1.3 in a different way, Theorem 2.3 is the most standard infinitesimal refinement of Theorem 1.3 and quite useful as shown in the following examples and §5.3.
Example 2.1 (taken from [12]). This example is almost the same as Example 1.1 in [14]. Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be given by
\[
f(x, y) = (x, xy + y^3, xy^2 + y^4).
\]
Thanks to D. Mond, the following have been known as information on \( f \) ([14], Theorem 4.2.2.7).

(2.1.1) The map-germ \( f \) is 10-determined with respect to \( C^\infty \) right-left equivalence.

(2.1.2) The map-germ \( f \) is not 9-determined with respect to \( C^\infty \) right-left equivalence.

(2.1.3) \( m_2^3 \theta(f) \) is contained in \( T \mathcal{A}(f) \).

Let \( N(x, y) \) be a 3 by 3 matrix with entries belonging to \( m_3^2 \). Then, by Theorem 2.3, \( g = f + Nf \) is \( C^\infty \) right-left equivalent to \( f \). In fact, \( f = (E_3 + N)^{-1} g \) and \( M = (E_3 + N)^{-1} \) satisfies \( M(0) = E_3 \) and each entry of \( M - M(0) \) belongs to \( m_2^3 \).

Example 2.2 (taken from [6]). Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be given by
\[
f(x, y) = (x, y^3 + x^2 y, y^4 + xy^2).
\]
Thanks to T. Gaffney and A. A. du Plessis, the following has been known as one of information on \( f \) ([6], Example (3.6)):

(2.2.1) \( m_2^3 \theta(f) \subset T \mathcal{A}(f) \).

Let \( N(x, y) \) be a 3 by 3 matrix with entries belonging to \( m_2^3 \). Then, by Theorem 2.3, \( g = f + Nf \) is \( C^\infty \) right-left equivalent to \( f \). In fact, \( f = (E_3 + N)^{-1} g \) and \( M = (E_3 + N)^{-1} \) satisfies \( M(0) = E_3 \) and each entry of \( M - M(0) \) belongs to \( m_2^3 \). Combining this result with direct co-ordinate manipulations yields the same \( M \)-determinacy result as in Example (3.6) of [6]. Note that the only information which we require is (2.2.1).

3. Criteria for \( C^0 \) right-left equivalence. A \( C^\infty \) deformation-germ \( \Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p, 0) \) of \( f \) is said to be Thom trivial (resp. transversely Thom trivial) if there exist C-regular stratifications in the sense of Bekka ([2]), \( S \) of \( \mathbb{R}^n \times \mathbb{R}^k \), \( T \) of \( \mathbb{R}^p \times \mathbb{R}^k \) and \( \{ \mathbb{R}^k \} \) of \( \mathbb{R}^k \) such that the following (T1) and (T2) (resp. (T1), (T2) and (T3)) hold:

(T1) The map-germ
\[
(\Phi, \pi) : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p \times \mathbb{R}^k, (0, 0))
\]
is a Thom map-germ with respect to \( S \) and \( T \).

(T2) The map-germ
\[
\pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^k, 0)
\]
is a stratified map-germ (or equivalently in this case, a Thom map-germ) with respect to \( T \) and \( \{ \mathbb{R}^k \} \).

(T3) The stratum \( T_0 \) of \( T \), which contains the origin \( (0, 0) \) of \( \mathbb{R}^p \times \mathbb{R}^k \), is transverse to \( \{ 0 \} \times \mathbb{R}^k (\subset \mathbb{R}^p \times \mathbb{R}^k) \).

Here \( \pi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^k, 0) \), \( \pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^k, 0) \) are canonical projections. For the definition of a C-regular stratification, see [2]. We remark that the notion of a C-regular stratification is an extended one of a Whitney stratification and it
is known that every C-regular stratification admits a controlled tube system ([2]). For the definitions of a stratified map-germ and a Thom map-germ, see [2], [7], [11].

By Thom’s second isotopy lemma ([2], [7], [11]), we see that for any Thom trivial deformation-germ \( \Phi : (R^n \times R^k, (0, 0)) \to (R^p, 0) \) of \( f \), there exist germs of homeomorphisms \( h : (R^n \times R^k, (0, 0)) \to (R^n \times R^k, (0, 0)) \) and \( H : (R^p \times R^k, (0, 0)) \to (R^p \times R^k, (0, 0)) \) such that the diagram (\( * \)) in Section 1 commutes, where \( \pi : (R^n \times R^k, (0, 0)) \to (R^k, 0) \), \( \pi' : (R^p \times R^k, (0, 0)) \to (R^k, 0) \) are canonical projections.

For given two \( C^\infty \) map-germs \( f, g : (R^n, 0) \to (R^p, 0) \), we consider the following four conditions.

(i) The map-germ \( f \) is topologically equivalent to \( g \).

(ii) There exist a germ of \( C^\infty \) diffeomorphism \( s : (R^n, 0) \to (R^n, 0) \) and a \( C^\infty \) map-germ \( M : (R^n, 0) \to (GL(p, R), M(0)) \) such that the following (a) and (b) are satisfied:

(a) \( f(x) = M(x)g(s(x)) \),

(b) The \( C^\infty \) map-germ \( F : (R^n \times R^p, (0, 0)) \to (R^p, 0) \) given by

\[
F(x, \lambda) = f(x) - M(x)\lambda
\]

is a Thom trivial deformation-germ of \( f \).

(iii) There exist a germ of \( C^\infty \) diffeomorphism \( s : (R^n, 0) \to (R^n, 0) \) and a \( C^\infty \) map-germ \( M : (R^n, 0) \to (GL(p, R), M(0)) \) such that (a), (b) of condition (ii) and the following (c) are satisfied:

(c) The \( C^\infty \) map-germ \( G : (R^n \times R^p, (0, 0)) \to (R^p, 0) \) given by

\[
G(x, \lambda) = g(x) - M(x^{-1}(x))^{-1}\lambda
\]

is a Thom trivial deformation-germ of \( g \).

(iv) There exist a germ of \( C^\infty \) diffeomorphism \( s : (R^n, 0) \to (R^n, 0) \) and a \( C^\infty \) map-germ \( M : (R^n, 0) \to (GL(p, R), M(0)) \) such that (a) of condition (ii) and the following (d) are satisfied:

(d) The \( C^\infty \) map-germ \( F : (R^n \times R^p, (0, 0)) \to (R^p, 0) \) given by

\[
F(x, \lambda) = f(x) - M(x)\lambda
\]

is a transversely Thom trivial deformation-germ of \( f \).

**Theorem 3.1** ([16]). Let \( f, g : (R^n, 0) \to (R^p, 0) \) be \( C^\infty \) map-germs with rank zero. Then condition (ii) implies condition (i).

**Theorem 3.2** ([16]). Condition (iii) implies condition (i).

**Theorem 3.3** ([18]). Condition (iv) implies condition (i).

Note that Example 1.1

\[
f(x, y) = (x, y^3 + xy)
g(x, y) = (x, y^3)
\]

again shows that condition (ii) does not necessarily imply condition (i) in positive rank case.
4. The simple systematic method. Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be a \( C^\infty \) map-germ and \( \Phi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p, 0) \) be a \( C^r \) deformation-germ of \( f \), where \( 0 \leq r \leq \infty \.

Suppose that there exists a \( C^r \) \( \mathcal{A} \)-morphism from \( \Phi \) to \( f \). By the definition of a \( C^r \) \( \mathcal{A} \)-morphism, there exist \( C^r \) map-germs \( h : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)), H : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \) and \( \varphi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0) \) such that the following (4.1) and (4.2) hold (for the definition of (4.1) and (4.2) hold (for the definition of

\[ \begin{align*}
\Phi &= (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \\
\varphi &= (\mathbb{R}^k, 0) \\
\end{align*} \]

By the definition of a \( C^r \) \( \mathcal{A} \)-morphism, see [17]).

(4.1) For any representatives \( \tilde{h} \) of \( h \) and \( \tilde{H} \) of \( H \), there exist neighborhoods \( U \) of the origin in \( \mathbb{R}^n \), \( V \) of the origin in \( \mathbb{R}^k \) and \( W \) of the origin in \( \mathbb{R}^p \) such that the restrictions \( \tilde{h}|_{U \times \{0\}} \) and \( \tilde{H}|_{W \times \{0\}} \) are \( C^r \) diffeomorphisms for any \( \lambda \in \mathbb{V} \).

(4.2) The following diagram commutes, where \( \pi : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^k, 0), \pi' : (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^k, 0), \) are canonical projections:

\[
\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) & \xrightarrow{(\Phi, \pi)} & (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \\
\downarrow h & & \downarrow H \\
(\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) & \xrightarrow{(f, \pi')} & (\mathbb{R}^p \times \mathbb{R}^k, (0, 0)) \\
\end{array}
\]

By (4.2), we may write

\[
h(x, \lambda) = (h_1(x, \lambda), \varphi(\lambda)) \quad \text{and} \quad H(y, \lambda) = (H_1(y, \lambda), \varphi(\lambda)).
\]

Let \( \varphi'_H : (\mathbb{R}^k, 0) \to (\mathbb{R}^p, 0) \) be the \( C^r \) map-germ given by

\[\varphi'_H(\lambda) = H_1(0, \lambda).\]

The map-germ (4.3) is the key idea in our study.

We put also \( h' : (\mathbb{R}^n \times \mathbb{R}^k, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \) as

\[h'(x, \lambda) = (h_1(x, \lambda), \varphi'_H(\lambda))\]

and \( H' : (\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p, (0, 0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0)) \) as

\[H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - \varphi'_H(\lambda)).\]

Then we can show that \( \{h', H', \varphi'_H\} \) is a \( C^r \) \( \mathcal{K} \)-morphism from \( \Phi \) to \( F \), where \( F \) is the graph deformation-germ of \( f \) given by \( F(x, y) = f(x) - y \). (For details, see [17]. In [17] the argument is pursued only for \( C^\infty \) deformation-germs. However, \( C^r \) deformation-germs can be treated by the exactly parallel argument.)

Next, returning to the situations in Sections 1–3, we let \( f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be \( C^\infty \) map-germs. We suppose that there exist a \( C^r \) map-germ \( M : (\mathbb{R}^n, 0) \to (\text{GL}(p, \mathbb{R}), M(0)) \) and a germ of \( C^r \) diffeomorphism \( s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) such that \( f(x) = M(x)g(s(x)) \).

We concentrate on considering the following \( C^r \) deformation-germ of \( f \):

\[f(x) - M(x)\lambda.\]

This deformation-germ is linear with respect to parameter variables. Remark that the parameter space of (4.4) is \( p \)-dimensional. Thus, if there exists a \( C^r \) \( \mathcal{A} \)-morphism \( \{h, H, \varphi\} \) from \( \Phi \) to \( f \), then the map-germ (4.3) is a map-germ between the same dimensional spaces.

We suppose furthermore that the deformation-germ (4.4) is \( C^r \)-trivial. Then, there exists
a $C^r$ $A$-morphism $\{h, H, \varphi\}$ from $\Phi$ to $f$. Thus, from the above argument we see that there exists a $C^r$ $K$-morphism $\{h', H', \varphi_H\}$ from (4.4) to the graph deformation-germ $F$ of $f$. In particular, we have the following equality (for details, see [15], [16]):

$$f(h_1(x, g(s(x)))) = H_1(0, g(s(x))).$$

Finally, we can show the following.

**Lemma 4.1** ([15], [16]). If the map-germ (4.3) is a germ of $C^r$ diffeomorphism, then the endomorphism-germ of $(R^n, 0)$ given by

$$x \mapsto h_1(x, g(s(x)))$$

is also a germ of $C^r$ diffeomorphism.

Thus, we see that for $0 \leq r \leq \infty$ in order to obtain $C^r$ right-left equivalence of $f$ and $g$, it suffices to find that the map-germ (4.3) is a germ of $C^r$ diffeomorphism.

### 5. Several applications

#### 5.1. $C^r$ right-left equivalence of $C^r$-stable map-germs ($1 \leq r \leq \infty$)

**Definition 5.1.** A $C^\infty$ map-germ $f : (R^n, 0) \to (R^p, 0)$ is said to be $C^r$-stable if every $C^\infty$ deformation-germ of $f$ is $C^r$-trivial ($0 \leq r \leq \infty$).

There are several apparently different definitions of $C^r$-stability. For the relation between them, see [20]. Our definition of $C^r$-stability is called $P$-$C^r$-stability in [20].

From the argument in Section 4, we have the following (for the definition of $C^r$ $K$-versality, see [17]).

**Theorem 5.1** ([17]). For any $C^r$-stable germ $f$, its graph deformation-germ is $C^r$ $K$-versal for $0 \leq r \leq \infty$.

For $1 \leq r < \infty$, the uniqueness of $C^r$ $K$-versal deformation-germ of a given map-germ may be proved easily by a slight modification of Martinet’s proof of the uniqueness of $C^\infty$ $K$-versality (pp. 155–156 of [1], pp. 21–22 of [8]), because in order to prove the uniqueness we need only one implication, the $C^r$ $K$-versality implies the infinitesimal $C^{r-1}$ $K$-versality, which is clear. Thus, by using Martinet’s argument (p. 158 of [1], p. 28 of [8]), we see that Theorem 5.1 yields a $C^r$ generalization of Mather’s classification theorem ($1 \leq r \leq \infty$) without any difficulty. Note that Theorem 1.2 also yields the same generalization of Mather’s classification theorem as a trivial corollary.

**Theorem 5.2** ([15], [17]). Let $f, g : (R^n, 0) \to (R^p, 0)$ be $C^r$-stable map-germs ($1 \leq r \leq \infty$). Suppose that there exist a germ of $C^\infty$ diffeomorphism $s : (R^n, 0) \to (R^n, 0)$ and a $C^\infty$ map-germ $M : (R^n, 0) \to (GL(p, R), M(0))$ such that $f(x) = M(x)g(s(x))$. Then $f$ and $g$ are $C^r$ right-left equivalent.
5.2. $C^0$ right-left equivalence of $C^0$-stable map-germs

**Definition 5.2.** A $C^\infty$ map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is said to be $MT$-stable if the jet extension of it is multi-transverse to the Thom-Mather canonical stratification of the jet space.

Concerning $C^0$ right-left equivalence of $MT$-stable map-germs, there is a well-known theorem due to M. Fukuda and T. Fukuda.

**Theorem 5.3** ([3]). Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be $MT$-stable map-germs. Suppose that there exist a germ of $C^\infty$ diffeomorphism $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a $C^\infty$ map-germ $M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, they are $C^0$ right-left equivalent.

**Definition 5.3.** A $C^\infty$ map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is said to be Thom stable if every $C^\infty$ deformation-germ of $f$ is Thom trivial.

As a consequence of the definition of $MT$-stability, every $C^\infty$ deformation-germ of an $MT$-stable map-germ is Thom trivial (see [7], [11]). Thus, every $MT$-stable map-germ is Thom stable. By Thom’s second isotopy lemma ([2], [7], [11]), every Thom stable map-germ is $C^0$-stable in the sense of Definition 5.1. As a trivial corollary of Theorem 3.2, we obtain a generalization of Theorem 5.3.

**Theorem 5.4** ([16]). Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be Thom stable map-germs. Suppose that there exist a germ of $C^\infty$ diffeomorphism $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a $C^\infty$ map-germ $M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, they are $C^0$ right-left equivalent.

5.3. An estimate of the order of $C^\infty$ determinacy.

As an application of Theorem 2.3, we show the following.

**Theorem 5.5.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map-germ. Suppose that there exist positive integers $k, \ell$ such that

(a) $m^k_\ell \theta(f) \subset TA(f)$ and
(b) $m^k_\ell \theta(f) \subset TK(f)$.

Then, $f$ is $(k + \ell - 1)$-determined with respect to $C^\infty$ right-left equivalence.

The set $TK(f)$ is defined by

$TK(f) = tf(m_\ell \theta(n)) + f^* m_\ell \theta(f),$

where $f^*(u) = u \circ f$. This set is the tangent space of the orbit through $f$ by the action of the group $K$ which was introduced by Mather in [9]. For details on the group $K$, $TK(f)$ and the definition of determinacy, see [9], [21]. Theorem 5.5 is a similar estimate to the well-known estimate due to Gaffney ([4]). Theorem 5.5 has been stated already in [13] without the proof. Several applications of Theorem 5.5 to divergent diagrams have been obtained in [13].
Proof. Let \( k_{-1}K \) be the set of all pairs of \((s,M)\), where \( s \) is a germ of \( C^\infty \) diffeomorphism \((\mathbb{R}^n,0) \to (\mathbb{R}^n,0)\) and \( M : (\mathbb{R}^n,0) \to (GL(p,\mathbb{R}),E_p) \) is a \( C^\infty \) map-germ with each entry of \( M - M(0) \) belonging to \( m_k^k \). Here, \( E_p \) is the \( p \) by \( p \) unit matrix. The set \( k_{-1}K \) is a group by the operation \((s_1,M_1) \ast (s_2,M_2) = (s_1 \circ s_2,M_1M_2)\), where \( s_1 \circ s_2 \) is the composition of \( s_1 \) and \( s_2 \) and \( M_1M_2 \) is the product of matrices of \( M_1 \) and \( M_2 \). The group \( k_{-1}K \) is a subgroup of the group \( K \) and the tangent space of the orbit through \( f \) by the action of the group \( k_{-1}K \) is

\[
(5.3.1) \quad T_{k_{-1}K}(f) = tf(m_n \theta(n)) + f^* m_p m_k^k \theta(f).
\]

Condition (b) of Theorem 5.5 implies

\[
(5.3.2) \quad m_n^{k+1} \theta(f) \subset tf(m_n^{k+1} \theta(n)) + f^* m_p m_k^k \theta(f) \subset T_{k_{-1}K}(f).
\]

By (5.3.2), we see that \( f \) is \((k+\ell)\)-determined with respect to the group \( k_{-1}K \).

Let \( g : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0) \) be a \( C^\infty \) map-germ with \( j^{k+\ell-1}f(0) = j^{k+\ell-1}g(0) \). Then, since \( k > 0 \), (5.3.2) implies

\[
(5.3.3) \quad m_n^{k+\ell} \theta(g) \subset tg(m_n^{k+1} \theta(n)) + g^* m_p m_k^k \theta(g) + m_n^{k+\ell+1} \theta(g).
\]

By Mather’s lemma (Lemma 3.1 of [10]), (5.3.3) implies that there exist a germ of \( C^\infty \) diffeomorphism \( s : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0) \) and a \( C^\infty \) map-germ \( M : (\mathbb{R}^n,0) \to (GL(p,\mathbb{R}),E_p) \) such that

\[
(5.3.4) \quad f(x) = M(x)g(s(x)) \quad \text{and} \quad (5.3.5) \quad \text{each entry of } M - M(0) \text{ belongs to } m_k^k.
\]

By Theorem 2.3, (5.3.4), (5.3.5) and condition (a) of Theorem 5.5 imply that \( f \) and \( g \) are \( C^\infty \) right-left equivalent.

Remark 5.1. Note that in the proof of Theorem 5.5 we use Mather’s lemma only for orbits by \( k_{-1}K \) group action, whose tangent spaces are much simpler than \( T_kA(g) \).

Since \( T_kA(f) \subset TK(f) \), Theorem 5.5 yields the following well-known estimate due to du Plessis and Wall as a trivial corollary.

Theorem 5.6 ([19], [21]). Let \( f : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0) \) be a \( C^\infty \) map-germ. Suppose that there exists a positive integer \( k \) such that

\[
m_k^k \theta(f) \subset T_kA(f).
\]

Then, \( f \) is \((2k-1)\)-determined with respect to \( C^\infty \) right-left equivalence.

References


