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RECOGNIZING RIGHT-LEFT EQUIVALENCE LOCALLY

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The present paper is a survey of the author's recent results on recognizing C^r right-left equivalence of C^∞ map-germs $(0 \leq r \leq \infty)$.

We say two C^∞ map-germs are C^r *right-left equivalent* if they coincide under germs of appropriate C^r co-ordinate systems of the source space and the target space, where a C^0 co-ordinate system means a co-ordinate system given by a homeomorphism. We often encounter the situations where we would like to decide whether or not given two map-germs are C^r right-left equivalent. In the case that one of them is of full rank (resp. linear), the implicit function theorem (resp. the rank theorem) answers our purpose (possibly except for $r = 0$). However, how can we decide in general case? By using a simple systematic method explained in Section 4, we can obtain many results to the problem. In Section 1, we give a series of criteria for C^r right-left equivalence of C^∞ map-germs $(1 \leq r \leq \infty)$. In Section 2, infinitesimal refinements of criteria for C^∞ right-left equivalence of C^∞ map-germs are given. Next, we consider C^0 right-left equivalence. In Section 3, we give a series of criteria for C^0 right-left equivalence of $\mathcal{K}$-equivalent map-germs. All of the results are derived from one simple idea, which is the key of our systematic method and explained exhaustively in Section 4. In Section 5 we give several applications of our results, which show how useful our method is.

The results for $r = \infty$ are all valid both in the real analytic category and in the complex analytic category as well.

1. Criteria for C^r right-left equivalence $(1 \leq r \leq \infty)$**.** For a given C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, any C^r map-germ $\Phi : \left(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\right) \to (\mathbf{R}^p, 0)$ such that $\Phi(x, 0) = f(x)$ is called a C^r *deformation-germ* of f. A C^r deformation-germ $\Phi : \left(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\right) \to (\mathbf{R}^p, 0)$ of $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be C^r-*trivial* if

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there exist germs of C^r diffeomorphisms $h : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big)$ and $H : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big)$ such that the following diagram $(*)$ commutes, where $\pi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, $\pi' : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, are canonical projections:

$$(*)\qquad \begin{array}{ccccc} \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) & \xrightarrow{(\Phi,\pi)} & \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0)\ . \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & \Vert \\ \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) & \xrightarrow{(f,\pi)} & \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \end{array}$$

For given two C^∞ map-germs $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, we consider the following conditions $(\mathrm{i_r})$, $(\mathrm{ii_r})$, $(\mathrm{iii_r})$ and $(\mathrm{iv_r})$.

$(\mathrm{i_r})$ The map-germ f is C^r right-left equivalent to g.

$(\mathrm{ii_r})$ There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that the following (a) and (b) are satisfied:

(a) $f(x) = M(x)g(s(x))$,

(b) the C^r map-germ $F : \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big) \to (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a C^r-trivial deformation-germ of f.

$(\mathrm{iii_r})$ There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that (a), (b) of condition $(\mathrm{ii_r})$ and the following (c) are satisfied:

(c) The C^r map-germ $G : \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big) \to (\mathbf{R}^p, 0)$ given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a C^r-trivial deformation-germ of g.

$(\mathrm{iv_r})$ There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that (a), (b) of condition $(\mathrm{ii_r})$ and the following (d) are satisfied:

(d) The germ $(H(\{0\} \times \mathbf{R}^p), 0)$ is transverse to the germ $(\{0\} \times \mathbf{R}^p, 0)$, where H is the germ of C^r diffeomorphism of $(\mathbf{R}^p \times \mathbf{R}^p, 0)$ given in the above commutative diagram $(*)$ with k, Φ replaced by p, F.

First, we consider rank zero cases.

Theorem 1.1 ([15]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Then condition* $(\mathrm{ii_r})$ *implies condition* $(\mathrm{i_r})$ *for $1 \le r \le \infty$.*

Next, we consider positive rank cases.

Example 1.1. Let $f, g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be given by

$$f(x, y) = (x, y^3 + xy),$$
$$g(x, y) = (x, y^3)$$

and $M : (\mathbf{R}^2, 0) \to (GL(2, \mathbf{R}), E_2)$ be given by

$$M(x, y) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix},$$

where E_2 is the unit 2 by 2 matrix. Then $f(x, y) = M(x, y)g(x, y)$.

It is well known that any C^∞ deformation-germ of the map-germ f is C^∞-trivial. Thus, (ii_∞) is satisfied. However, for any $1 \le r \le \infty$ condition $(\mathrm{i_r})$ does not hold (in fact, f and g are even not topologically right-left equivalent).

This example shows that condition $(\mathrm{ii_r})$ does not necessarily imply condition $(\mathrm{i_r})$ in positive rank cases. Nevertheless, the following holds under no assumptions.

Theorem 1.2 ([15]). *Condition* $(\mathrm{iii_r})$ *implies condition* $(\mathrm{i_r})$ *for* $1 \le r \le \infty$.

Although Theorem 1.2 is interesting in itself, we *prefer* the C^r triviality of the linearly parametrized deformation-germ of *only one* of f or g to those of both of f and g. Thus, we are led to condition $(\mathrm{iv_r})$.

Theorem 1.3 ([15]). *Condition* $(\mathrm{iv_r})$ *implies condition* $(\mathrm{i_r})$ *for* $1 \le r \le \infty$.

In the case $r = \infty$, we have

Theorem 1.4 ([15]). *For any* C^∞ *map germs* $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, *the following hold:*

(1) $(\mathrm{i}_\infty) \Leftrightarrow (\mathrm{iii}_\infty) \Leftrightarrow (\mathrm{iv}_\infty)$.

(2) $(\mathrm{i}_\infty) \Leftrightarrow (\mathrm{ii}_\infty) \Leftrightarrow (\mathrm{iii}_\infty) \Leftrightarrow (\mathrm{iv}_\infty)$ *if the rank of* f *is zero.*

Therefore, we may answer the C^∞ recognition problem completely by using our conditions in principle.

2. Infinitesimal refinements of criteria for C^∞ right-left equivalence. First, we review infinitesimal notations briefly. For details on them, see [9], [14], [15], [21].

Let $\mathcal{E}_n$ (resp. m_n) denote the set of C^∞ function-germs $(\mathbf{R}^n, 0) \to \mathbf{R}$ (resp. $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$). The set $\mathcal{E}_n$ has a natural $\mathbf{R}$-algebra structure and the set m_n is the unique maximal ideal in $\mathcal{E}_n$. For any positive integer ℓ, m_n^ℓ means the ℓ-times product of m_n. For $\ell = 0$, m_n^0 is $\mathcal{E}_n$.

For a C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, let $\theta(f)$ denote the $\mathcal{E}_n$-module consisting of germs of C^∞ vector fields $\zeta : (\mathbf{R}^n, 0) \to T(\mathbf{R}^p)$ such that $\pi_p \circ \zeta = f$, where $\pi_p : T(\mathbf{R}^p) \to \mathbf{R}^p$ denotes the canonical projection. By using the standard identification of $T(\mathbf{R}^p)$ with $\mathbf{R}^p \times \mathbf{R}^p$, $\theta(f)$ may be identified with the free $\mathcal{E}_n$-module with p-generators. When the given f is the identity map-germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, $\theta(f)$ may be denoted by $\theta(n)$.

For a C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, let $Tf : (T\mathbf{R}^n, \pi_n^{-1}(0)) \to (T\mathbf{R}^p, \pi_p^{-1}(0))$ denote the tangent map-germ of f, where $\pi_n : T\mathbf{R}^n \to \mathbf{R}^n$, $\pi_p : T\mathbf{R}^p \to \mathbf{R}^p$ are canonical projections. We define

$$tf : \theta(n) \to \theta(f), \qquad wf : \theta(p) \to \theta(f)$$

by $tf(\xi) = Tf \circ \xi$, $wf(\eta) = \eta \circ f$. By using tf and wf, we define

$$T\mathcal{A}(f) = tf(m_n\theta(n)) + wf(m_p\theta(p)) \quad \text{and}$$
$$T_e\mathcal{A}(f) = tf(\theta(n)) + wf(\theta(p)).$$

Theorem 2.1. *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists an integer $k \quad (k \geq 0)$ such that*

(a) *each entry of $M - M(0)$ belongs to m_n^{k+1} and*

(b) $m_n^k\theta(f) \subset T_e\mathcal{A}(f)$.

Then f and g are C^∞ right-left equivalent.

Although there are no proofs of Theorem 2.1 in a series of author's papers [13]–[18], by the proof of Theorem 2.2 below it is clear that conditions (a) and (b) of Theorem 2.1 imply the C^∞-triviality of $f(x) - M(x)\lambda$. Thus, Theorem 2.1 follows from Theorem 1.1.

Theorem 2.2 ([14]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists an integer $k \quad (k \geq 0)$ such that*

(a) *each entry of $M - M(0)$ belongs to m_n^{k+1},*

(b) $m_n^k\theta(f) \subset T_e\mathcal{A}(f)$ *and*

(c) $m_n^k\theta(g) \subset T_e\mathcal{A}(g)$.

Then f and g are C^∞ right-left equivalent.

Theorem 2.2 was stated (but not proved) first by A. A. du Plessis ([19], page 128). Conditions (a) and (b) (resp. (a) and (c)) of Theorem 2.2 imply the C^∞-triviality of $f(x) - M(x)\lambda$ (resp. $g(x) - M(s^{-1}(x))^{-1}\lambda$). Thus, Theorem 2.2 follows from Theorem 1.2 (for details, see [14]).

Theorem 2.3 ([14]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to \big(GL(p, \mathbf{R}), M(0)\big)$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists a positive integer k such that*

(a) *each entry of $M - M(0)$ belongs to m_n^k and*

(b) $m_n^k\theta(f) \subset T\mathcal{A}(f)$.

Then f and g are C^∞ right-left equivalent.

Conditions (a) and (b) imply condition (iv$_\infty$). Thus Theorem 2.3 follows from Theorem 1.3 (for details, see [14]). Although we can deduce infinitesimal results from Theorem 1.3 in a different way, Theorem 2.3 is the most standard infinitesimal refinement of Theorem 1.3 and quite useful as shown in the following examples and §5.3.

EXAMPLE 2.1 (taken from [12]). This example is almost the same as Example 1.1 in [14]. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be given by

$$f(x, y) = (x, xy + y^3, xy^2 + y^{10}).$$

Thanks to D. Mond, the following have been known as information on f ([12], Theorem 4.2.2:7).

(2.1.1) The map-germ f is 10-determined with respect to C^∞ right-left equivalence.
(2.1.2) The map-germ f is not 9-determined with respect to C^∞ right-left equivalence.
(2.1.3) $m_2^8\theta(f)$ is contained in $T\mathcal{A}(f)$.

Let $N(x, y)$ be a 3 by 3 matrix with entries belonging to m_2^8. Then, by Theorem 2.3, $g = f + Nf$ is C^∞ right-left equivalent to f. In fact, $f = (E_3 + N)^{-1}g$ and $M = (E_3 + N)^{-1}$ satisfies $M(0) = E_3$ and each entry of $M - M(0)$ belongs to m_2^8.

EXAMPLE 2.2 (taken from [6]). Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be given by

$$f(x, y) = (x, y^3 + x^2y, y^4 + xy^2).$$

Thanks to T. Gaffney and A. A. du Plessis, the following has been known as one of information on f ([6], Example (3.6)):

$$m_2^3\theta(f) \subset T\mathcal{A}(f). \tag{2.2.1}$$

Let $N(x, y)$ be a 3 by 3 matrix with entries belonging to m_2^3. Then, by Theorem 2.3, $g = f + Nf$ is C^∞ right-left equivalent to f. In fact, $f = (E_3 + N)^{-1}g$ and $M = (E_3 + N)^{-1}$ satisfies $M(0) = E_3$ and each entry of $M - M(0)$ belongs to m_2^3. Combining this result with direct co-ordinate manipulations yields the same M-determinacy result as in Example (3.6) of [6]. Note that the only information which we require is (2.2.1).

3. Criteria for C^0 right-left equivalence. A C^∞ deformation-germ $\Phi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^p, 0)$ of f is said to be *Thom trivial* (resp. *transversely Thom trivial*) if there exist C-regular stratifications in the sense of Bekka ([2]), $\mathcal{S}$ of $\mathbf{R}^n \times \mathbf{R}^k$, $\mathcal{T}$ of $\mathbf{R}^p \times \mathbf{R}^k$ and $\{\mathbf{R}^k\}$ of $\mathbf{R}^k$ such that the following (T1) and (T2) (resp. (T1), (T2) and (T3)) hold:

(T1) The map-germ

$$(\Phi, \pi) : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big)$$

is a Thom map-germ with respect to $\mathcal{S}$ and $\mathcal{T}$.

(T2) The map-germ

$$\pi' : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$$

is a stratified map-germ (or equivalently in this case, a Thom map-germ) with respect to $\mathcal{T}$ and $\{\mathbf{R}^k\}$.

(T3) The stratum T_0 of $\mathcal{T}$, which contains the origin $(0,0)$ of $\mathbf{R}^p \times \mathbf{R}^k$, is transverse to $\{0\} \times \mathbf{R}^k (\subset \mathbf{R}^p \times \mathbf{R}^k)$.

Here $\pi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, $\pi' : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$ are canonical projections. For the definition of a C-regular stratification, see [2]. We remark that the notion of a C-regular stratification is an extended one of a Whitney stratification and it is known that every C-regular stratification admits a controlled tube system ([2]). For the definitions of a stratified map-germ and a Thom map-germ, see [2], [7], [11].

By Thom's second isotopy lemma ([2], [7], [11]), we see that for any Thom trivial deformation-germ $\Phi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^p, 0)$ of f, there exist germs of homeomorphisms $h : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big)$ and $H : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^p \times \mathbf{R}^k, (0,0))$ such that the diagram $(*)$ in Section 1 commutes, where $\pi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, $\pi' : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$ are canonical projections.

For given two C^∞ map-germs $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, we consider the following four conditions.

(i) The map-germ f is topologically equivalent to g.

(ii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that the following (a) and (b) are satisfied:

(a) $f(x) = M(x)g(s(x))$,

(b) the C^∞ map-germ $F : \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big) \to (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a Thom trivial deformation-germ of f.

(iii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that (a), (b) of condition (ii) and the following (c) are satisfied:

(c) The C^∞ map-germ $G : \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big) \to (\mathbf{R}^p, 0)$ given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a Thom trivial deformation-germ of g.

(iv) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that (a) of condition (ii) and the following (d) are satisfied:

(d) The C^∞ map-germ $F : \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big) \to (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a transversely Thom trivial deformation-germ of f.

Theorem 3.1 ([16]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Then condition* (ii) *implies condition* (i).

Theorem 3.2 ([16]). *Condition* (iii) *implies condition* (i).

Theorem 3.3 ([18]). *Condition* (iv) *implies condition* (i).

Note that Example 1.1

$$f(x, y) = (x, y^3 + xy)$$
$$g(x, y) = (x, y^3)$$

again shows that condition (ii) does not necessarily imply condition (i) in positive rank case.

4. The simple systematic method. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ and $\Phi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^p, 0)$ be a C^r deformation-germ of f, where $0 \le r \le \infty$. Suppose that there exists a C^r $\mathcal{A}$-morphism from Φ to f. By the definition of a C^r $\mathcal{A}$-morphism, there exist C^r map-germs $h : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big)$, $H : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big)$ and $\varphi : (\mathbf{R}^k, 0) \to (\mathbf{R}^k, 0)$ such that the following (4.1) and (4.2) hold (for the definition of C^r $\mathcal{A}$-morphism, see [17]).

(4.1) For any representatives $\widetilde{h}$ of h and $\widetilde{H}$ of H, there exist neighborhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^k$ and W of the origin in $\mathbf{R}^p$ such that the restrictions $\widetilde{h}|_{U\times\{\lambda\}}$ and $\widetilde{H}|_{W\times\{\lambda\}}$ are C^r diffeomorphisms for any $\lambda \in V$.

(4.2) The following diagram commutes, where $\pi : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, $\pi' : \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \to (\mathbf{R}^k, 0)$, are canonical projections:

$$\begin{tikzcd}
\big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \arrow[r, "(\Phi,\pi)"] \arrow[d, "h"'] & \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \arrow[r, "\pi'"] \arrow[d, "H"'] & (\mathbf{R}^k, 0) \arrow[d, "\varphi"'] \\
\big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \arrow[r, "(f,\pi)"] & \big(\mathbf{R}^p \times \mathbf{R}^k, (0,0)\big) \arrow[r, "\pi'"] & (\mathbf{R}^k, 0).
\end{tikzcd}$$

By (4.2), we may write

$$h(x,\lambda) = (h_1(x,\lambda), \varphi(\lambda)) \qquad \text{and} \qquad H(y,\lambda) = (H_1(y,\lambda), \varphi(\lambda)).$$

Let $\varphi'_H : (\mathbf{R}^k, 0) \to (\mathbf{R}^p, 0)$ be the C^r map-germ given by

$$\varphi'_H(\lambda) = H_1(0,\lambda). \tag{4.3}$$

The map-germ (4.3) is *the key idea* in our study.

We put also $h' : \big(\mathbf{R}^n \times \mathbf{R}^k, (0,0)\big) \to \big(\mathbf{R}^n \times \mathbf{R}^p, (0,0)\big)$ as

$$h'(x,\lambda) = (h_1(x,\lambda), \varphi'_H(\lambda))$$

and $H' : \big(\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^p, (0,0,0)\big) \to \big(\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^p, (0,0,0)\big)$ as

$$H'(x,\lambda,y) = (h'(x,\lambda), H_1(y,\lambda) - \varphi'_H(\lambda)).$$

Then we can show that $\{h', H', \varphi'_H\}$ is a C^r $\mathcal{K}$-morphism from Φ to F, where F is the graph deformation-germ of f given by $F(x,y) = f(x) - y$. (For details, see [17]. In [17] the argument is pursued only for C^∞ deformation-germs. However, C^r deformation-germs can be treated by the exactly parallel argument.)

Next, returning to the situations in Sections 1–3, we let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs. We suppose that there exist a C^r map-germ $M : (\mathbf{R}^n, 0) \to (GL(p,\mathbf{R}), M(0))$ and a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $f(x) = M(x)g(s(x))$. We concentrate on considering the following C^r deformation-germ of f:

$$f(x) - M(x)\lambda. \tag{4.4}$$

This deformation-germ is linear with respect to parameter variables. Remark that the parameter space of (4.4) is p-dimensional. Thus, if there exists a C^r $\mathcal{A}$-morphism $\{h, H, \varphi\}$ from Φ to f, then the map-germ (4.3) is a map-germ between the same dimensional spaces. We suppose furthermore that the deformation-germ (4.4) is C^r-trivial. Then, there exists a C^r $\mathcal{A}$-morphism $\{h, H, \varphi\}$ from Φ to f. Thus, from the above argument we see that there exists a C^r $\mathcal{K}$-morphism $\{h', H', \varphi'_H\}$ from (4.4) to the graph deformation-germ F of f. In particular, we have the following equality (for details, see [15], [16]):

$$f\bigl(h_1\bigl(x, g(s(x))\bigr)\bigr) = H_1\bigl(0, g(s(x))\bigr).$$

Finally, we can show the following.

Lemma 4.1 ([15], [16]). *If the map-germ* (4.3) *is a germ of C^r diffeomorphism, then the endomorphism-germ of* $(\mathbf{R}^n, 0)$ *given by*

$$x \mapsto h_1\bigl(x, g(s(x))\bigr)$$

is also a germ of C^r diffeomorphism.

Thus, we see that for $0 \le r \le \infty$ in order to obtain C^r right-left equivalence of f and g, it suffices to find that the map-germ (4.3) is a germ of C^r diffeomorphism.

5. Several applications

5.1. C^r right-left equivalence of C^r-stable map-germs $(1 \le r \le \infty)$

Definition 5.1. A C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be *C^r-stable* if every C^∞ deformation-germ of f is C^r-trivial $(0 \le r \le \infty)$.

There are several apparently different definitions of C^r-stability. For the relation between them, see [20]. Our definition of C^r-stability is called P-C^r-stability in [20].

From the argument in Section 4, we have the following (for the definition of C^r $\mathcal{K}$-versality, see [17]).

Theorem 5.1 ([17]). *For any C^r-stable map-germ f, its graph deformation-germ is C^r $\mathcal{K}$-versal for $0 \le r \le \infty$.*

For $1 \le r < \infty$, the uniqueness of C^r $\mathcal{K}$-versal deformation-germ of a given map-germ may be proved easily by a slight modification of Martinet's proof of the uniqueness of C^∞ $\mathcal{K}$-versality (pp. 155–156 of [1], pp. 21–22 of [8]), because in order to prove the uniqueness we need only one implication, *the C^r $\mathcal{K}$-versality implies the infinitesimal C^{r-1} $\mathcal{K}$-versality*, which is clear. Thus, by using Martinet's argument (p. 158 of [1], p. 28 of [8]), we see that Theorem 5.1 yields a C^r generalization of Mather's classification theorem $(1 \le r \le \infty)$ without any difficulty. Note that Theorem 1.2 also yields the same generalization of Mather's classification theorem as a trivial corollary.

Theorem 5.2 ([15], [17]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^r-stable map-germs $(1 \le r \le \infty)$. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then f and g are C^r right-left equivalent.*

5.2. C^0 right-left equivalence of C^0-stable map-germs

Definition 5.2. A C^∞ map germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be *MT-stable* if the jet extension of it is multi-transverse to the Thom-Mather canonical stratification of the jet space.

Concerning C^0 right-left equivalence of MT-stable map-germs, there is a well-known theorem due to M. Fukuda and T. Fukuda.

Theorem 5.3 ([3]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be MT-stable map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, they are C^0 right-left equivalent.*

Definition 5.3. A C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be *Thom stable* if every C^∞ deformation-germ of f is Thom trivial.

As a consequence of the definition of MT-stability, every C^∞ deformation-germ of an MT-stable map-germ is Thom trivial (see [7], [11]]). Thus, every MT-stable map-germ is Thom stable. By Thom's second isotopy lemma ([2], [7], [11]), every Thom stable map-germ is C^0-stable in the sense of Definition 5.1. As a trivial corollary of Theorem 3.2, we obtain a generalization of Theorem 5.3.

Theorem 5.4 ([16]). *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be Thom stable map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that*

$$f(x) = M(x)g(s(x)).$$

Then, they are C^0 right-left equivalent.

5.3. An estimate of the order of C^∞ determinacy. As an application of Theorem 2.3, we show the following.

Theorem 5.5. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ. Suppose that there exist positive integers k, ℓ such that*

(a) $m_n^k \theta(f) \subset T\mathcal{A}(f)$ *and*

(b) $m_n^\ell \theta(f) \subset T\mathcal{K}(f)$.

Then, f is $(k + \ell - 1)$-determined with respect to C^∞ right-left equivalence.

The set $T\mathcal{K}(f)$ is defined by

$$T\mathcal{K}(f) = tf(m_n \theta(n)) + f^* m_p \theta(f),$$

where $f^*(u) = u \circ f$. This set is the tangent space of the orbit through f by the action of the group $\mathcal{K}$ which was introduced by Mather in [9]. For details on the group $\mathcal{K}$, $T\mathcal{K}(f)$ and the definition of determinacy, see [9], [21]. Theorem 5.5 is a similar estimate to the well-known estimate due to Gaffney ([4]). Theorem 5.5 has been stated already in [13] without the proof. Several applications of Theorem 5.5 to divergent diagrams have been obtained in [13].

Proof. Let ${}_{k-1}\mathcal{K}$ be the set of all pairs of (s, M), where s is a germ of C^∞ diffeomorphism $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), E_p)$ is a C^∞ map-germ with each entry of $M - M(0)$ belonging to m_n^k. Here, E_p is the p by p unit matrix. The set ${}_{k-1}\mathcal{K}$ is a group by the operation $(s_1, M_1) * (s_2, M_2) = (s_1 \circ s_2, M_1 M_2)$, where $s_1 \circ s_2$ is the composition of s_1 and s_2 and $M_1 M_2$ is the product of matrices of M_1 and M_2. The group ${}_{k-1}\mathcal{K}$ is a subgroup of the group $\mathcal{K}$ and the tangent space of the orbit through f by the action of the group ${}_{k-1}\mathcal{K}$ is

$$T_{k-1}\mathcal{K}(f) = tf(m_n\theta(n)) + f^* m_p m_n^k \theta(f). \tag{5.3.1}$$

Condition (b) of Theorem 5.5 implies

$$m_n^{k+\ell}\theta(f) \subset tf(m_n^{k+1}\theta(n)) + f^* m_p m_n^k \theta(f) \subset T_{k-1}\mathcal{K}(f). \tag{5.3.2}$$

By (5.3.2), we see that f is $(k+\ell)$-determined with respect to the group ${}_{k-1}\mathcal{K}$.

Let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ with $j^{k+\ell-1}f(0) = j^{k+\ell-1}g(0)$. Then, since $k > 0$, (5.3.2) implies

$$m_n^{k+\ell}\theta(g) \subset tg(m_n^{k+1}\theta(n)) + g^* m_p m_n^k \theta(g) + m_n^{k+\ell+1}\theta(g). \tag{5.3.3}$$

By Mather's lemma (Lemma 3.1 of [10]), (5.3.3) implies that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), E_p)$ such that

(5.3.4) $f(x) = M(x)g(s(x))$ and

(5.3.5) each entry of $M - M(0)$ belongs to m_n^k.

By Theorem 2.3, (5.3.4), (5.3.5) and condition (a) of Theorem 5.5 imply that f and g are C^∞ right-left equivalent. ∎

Remark 5.1. Note that in the proof of Theorem 5.5 we use Mather's lemma only for orbits by ${}_{k-1}\mathcal{K}$ group action, whose tangent spaces are much simpler than $T\mathcal{A}(g)$.

Since $T\mathcal{A}(f) \subset T\mathcal{K}(f)$, Theorem 5.5 yields the following well-known estimate due to du Plessis and Wall as a trivial corollary.

THEOREM 5.6 ([19], [21]). *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ. Suppose that there exists a positive integer k such that*

$$m_n^k\theta(f) \subset T\mathcal{A}(f).$$

Then, f is $(2k-1)$-determined with respect to C^∞ right-left equivalence.

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