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A NOTE ON A SELECTION PROBLEM

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An optimal rule for a continuous time generalization of the so-called secretary problem is investigated. A lower and upper bounds as well as asymptotic of a sequence which determines the best strategy are obtained.

1. Formulation of the main results

In papers [1], [2] a natural generalization of the "best choice problem" was considered. This generalization called an "apartment problem" was as follows. A man has been allowed a fixed time T in which to find an apartment. Opportunities to inspect apartments occur at the epochs of a stationary Poisson process of intensity λ. He inspects each apartment immediately the opportunity arises, however he must decide at that epoch whether or not to accept the apartment. At any moment, he is able to rank a given apartment amongst all of those inspected to date. The man's objective is to maximize the chance of selecting the best apartment from those (if any) available in the interval $[0, T]$. Let us number the apartments in the order they are inspected, then the following rule is optimal (see [1], [2]):

Select the first apartment that is better than all preceding apartments and which index k and the epoch t it was inspected satisfy

$$T-t \leqslant y_k/\lambda.$$

For every $k = 1, 2, \ldots$ number y_k is the unique solution of the equation:

$$\sum_{n=0}^{+\infty} \frac{y^n}{n!}\frac{1}{n+k} = \sum_{n=1}^{+\infty} \frac{y^n}{n!}\frac{1}{n+k}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{k+n-1}\right), \quad y > 0. \tag{1}$$

Since the optimal policy is directly expressed in terms of the solutions of the transcendental equation (1), therefore it is desirable to tabulate the numbers y_k, as well as to investigate the asymptotic behaviour of the sequence (y_k). A corresponding table was given in [2]. The asymptotic properties of y_k are the scope of this note.

The main results of this note are the following two theorems.

THEOREM 1. *Let y_k, $k = 2, 3, \ldots$ be the unique solution of the equation* (1); *then*

$$(e-1)(k-1) \leqslant y_k \leqslant 4e\big((e-1)k+1\big).$$

THEOREM 2. *Let* (y_k) *be the sequence of the solutions of the equation* (1); *then*

$$\lim_{k\to\infty} \frac{y_k}{k} = e-1.$$

The exact value of the limit in Theorem 2 was suggested to us by numerical results reported in [2].

2. Proof of Theorem 1

The proof will be given in two steps.

1st step. For every $k = 1, 2 \ldots, y_k \geqslant (e-1)(k-1)$. Let us fix $k \geqslant 2$ and let φ be a continuous function defined on $[0, +\infty]$ as follows: φ is a linear function on every interval $[n, n+1]$, $n = 0, 1, \ldots, \varphi(0) = 0$ and $\varphi(n) = \frac{1}{k}+\frac{1}{k+1}+\ldots+$ $+\frac{1}{k+(n-1)}$ for $n = 1, 2, \ldots$ By direct computation we check that φ is an increasing and concave function. Moreover, multiplying equation (1) by y^k and taking into account that $\int_0^y u^{n+k-1}du = \frac{y^{n+k}}{n+k}$ we obtain that (1) is equivalent to every of the following equations:

$$(2)\qquad \int_0^y e^u u^{k-1}\,du = \int_0^y u^{k-1}\left(\sum_{n=0}^{+\infty}\frac{u^n}{n!}\varphi(n)\right)du, \quad y>0,$$

$$(3)\qquad \int_0^y e^u u^{k-1}\,du = \int_0^y e^u u^{k-1} E\big(\varphi(\xi_u)\big)du, \quad y>0.$$

In the latter equation ξ_u denotes a random variable with the Poisson distribution and mean value $u > 0$. Monotonicity of φ implies that the function ψ: $\psi(u) = E(\varphi(\xi_u))$ is increasing. Consequently, if $y = y_k$ is the unique solution of (3), we have

$$\int_0^y e^u u^{k-1}\,du \leqslant \psi(y)\int_0^y e^u u^{k-1}\,du,$$

thus $\psi(y) \geqslant 1$. From Jensen's inequality we obtain

$$\psi(y) = E\big(\varphi(\xi_y)\big) \leqslant \varphi\big(E(\xi_y)\big) = \varphi(y)$$

and we see that also $\varphi(y) \geqslant 1$. Let $z = z_k$ be a natural number such that $\varphi(z)$ $< 1 < \varphi(z+1)$. Since for natural numbers $n, k \geqslant 2$ we have $\ln\frac{n}{k} < \frac{1}{k}+\ldots+\frac{1}{n-1}$ $< \ln\frac{n-1}{k-1}$, therefore

$$(4)\qquad (e-1)(k-1) < z < (e-1)k.$$

This gives the required inequality $y > (e-1)(k-1)$.

2nd step. For every $k = 2, 3, \ldots, y_k \leqslant 4e\big((e-1)k+1\big)$. With the same notation as above equation (1) can be rewritten as

$$(5)\qquad \sum_{n\leqslant z}\frac{1}{k+n}\big(1-\varphi(n)\big)\frac{x^n}{n!} = \sum_{n>z}\frac{1}{k+n}\big(\varphi(n)-1\big)\frac{x^n}{n!}, \quad x>0.$$

If $x < y = y_k$, then the left-hand side of (5) is bigger than the right-hand side of (5). Thus, to finish the proof of Theorem 1 it is sufficient to prove that for $x = \bar{x}$ $= 4e\big((e-1)k+1\big)$ the opposite inequality holds.

We show even more: for $m = 0, 1, 2, \ldots, z$,

$$(6)\qquad \frac{1}{k+z-m}\big(1-\varphi(z-m)\big)\frac{\bar{x}^{z-m}}{(z-m)!} < \frac{1}{k+m+z+2}\big(\varphi(z+m+2)-1\big)\frac{\bar{x}^{z+m+2}}{(z+m+2)!}.$$

Taking into account inequality (4), as well as the definition of the number z, we obtain that $k+z+m+2/k+z-m < 2e-1+2/k$,

$$\big(1-\varphi(z-m)\big)/\big(\varphi(z+m+2)-1\big) < (k+z+m+1)/k+z+m$$
$$< 2e-1+1/k,$$
$$(z+m+2)!/(z-m)! < (2z+2)^{2m+2}.$$

Thus (6) holds if for $m = 0, 1, \ldots, z$

$$(7)\qquad (2e-1+1/k)(2e-1+2/k)(2z+2)^{2m+2} < \bar{x}^{2m+2}$$

and therefore certainly holds for $\bar{x} = 4e\big((e-1)k+1\big)$. The proof of the theorem is completed.

3. Proof of Theorem 2

The inequality $\varliminf_{k\to+\infty}\frac{y_k}{k} \geqslant e-1$ follows from Theorem 1. To show that $\varlimsup_{k\to+\infty}\frac{y_k}{k}$ $\leqslant e-1$ we shall need the following lemma:

LEMMA. *There exists a number $C > 0$ such that for every $x \in \mathbf{R}$ and $\lambda > 0$,*

$$(8)\qquad \left|e^{-\lambda}\Big(\sum_{k\leqslant\lambda+x\sqrt{\lambda}}\frac{\lambda^k}{k!}\Big)-\Phi(x)\right| \leqslant \frac{C}{\sqrt{\lambda}},$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du.$$

Proof of the Lemma. Let n and m be numbers such that $\lambda = nm$ and n is natural. Let

$$\Phi_n(x) = P\left(\frac{\xi_1+\ldots+\xi_n-nm}{\sqrt{nm}} \leqslant x\right), \quad x \in \boldsymbol{R},$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables with mean m and the Poisson law. From Berry–Essen's theorem [3], p. 480,

$$|\Phi_n(x)-\Phi(x)| < \frac{33}{4}\cdot\frac{E|\xi_1-m|^3}{\sqrt{n}\,(E(\xi_1-m)^2)^{3/2}}.$$

Since $E\xi = m$, $E\xi^2 = m^2+m$, $E\xi^3 = m^3+3m^2+m$, therefore $E|\xi_1-m|^3 \leqslant E(\xi_1+m)^3 \leqslant \overline{C}m(1+m^2)$ for some positive $\overline{C}$ and $E(\xi_1-m)^2 = m$. Thus

$$|\Phi_n(x)-\Phi(x)| \leqslant \frac{33}{4}\,\frac{\overline{C}m(1+m^2)}{\sqrt{n}\sqrt{m}\,m} \leqslant \frac{33}{4}\,\frac{\overline{C}(1+(\lambda/n)^2)}{\sqrt{\lambda}}.$$

Taking $n\to+\infty$ we obtain (8) with $C = \frac{33}{4}\cdot\overline{C}$. This finishes the proof of the Lemma.

If $\overline{\lim\limits_{k\to+\infty}}\,\frac{y_k}{k} > e-1$, then for some $a > e-1$, the set $K = \{k;\ y_k \geqslant ak\}$ would be infinite. Since

$$\frac{1}{k}+\ldots+\frac{1}{k+n-1} > \begin{cases} 1 & \text{for} \quad n \geqslant z+1, \\ \ln(1+a) = \alpha > 1 & \text{for} \quad n \geqslant ak, \end{cases}$$

therefore for $y = y_k$ and $k\in K$ we would have

$$\sum_{n\geqslant 0}\frac{y^n}{n!}\,\frac{1}{n+k} \geqslant \sum_{z<n\leqslant y}\frac{y^n}{n!}\,\frac{1}{n+k}+\alpha\sum_{n>y}\frac{y^n}{n!}\,\frac{1}{n+k},$$

$$\sum_{n\leqslant z}\frac{y^n}{n!}\,\frac{1}{n+k} \geqslant (\alpha-1)\cdot\sum_{n>y}\frac{y^n}{n!}\,\frac{1}{n+k}. \tag{9}$$

Let b be a number such that $y_k \leqslant bk$ for all k (by Theorem 1 such a number exists). Then from inequality (9) we obtain for $y = y_k$

$$\frac{1}{k}\sum_{n\leqslant z}\frac{y^n}{n!} \geqslant (\alpha-1)\,\frac{1}{2bk+k}\sum_{y<n\leqslant 2bk}\frac{y^n}{n!}.$$

Thus, there exist numbers $\beta < 1$ and $\gamma > 0$ such that for $y = y_k$, $k\in K$

$$e^{-y}\sum_{n\leqslant\beta y}\frac{y^n}{n!} \geqslant \gamma e^{-y}\sum_{y<n\leqslant 2y}\frac{y^n}{n!}. \tag{10}$$

Since $y = y_k \to +\infty$ as $k\to+\infty$, $k\in K$, applying the Lemma we obtain

that

$$e^{-y}\sum_{n\leqslant\beta y}\frac{y^n}{n!}\to 0,$$

$$\gamma e^{-y}\sum_{y<n\leqslant 2y}\frac{y^n}{n!}\to\frac{\gamma}{2}.$$

This contradicts inequality (10). The proof of Theorem 1 is completed.

Remark 1. Some weaker estimations than those formulated in Theorem 1 were announced in [1].

Remark 2. The proven theorems suggest that the following strategy:

Accept the apartment inspected at epoch t and with an index k if

(1) the apartment is better than all preceding,

(2) the residual time $T-t$ is less than $\frac{(e-1)k}{\lambda}$,

is almost as good as the optimal strategy.

References

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