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CONVERGENCE RATES FOR QUADRATIC FORMS OF RANDOM VARIABLES

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In this paper the rates of convergence in probability to zero of suitable centred quadratic forms of random variables are investigated. The rates are expressed in terms of real coefficients of random quadratic forms.

§ 1. Introduction

For each positive integer $n = 1, 2, \ldots$ let $[a_{kj}(n);\ k, j \geqslant 1]$, $[b_{kj}(n);\ k, j \geqslant 1]$ and $[c_{kj}(n);\ k, j \geqslant 1]$ be matrices of real numbers and let $\{X_i;\ i \geqslant 1\}$ be a sequence of real random variables defined on a probability space $(\Omega, \mathscr{A}, \mathscr{P})$, not necessarily identically distributed. In this paper we shall establish the rates of convergence in probability to zero of quadratic forms

$$S(n) = K(n)+L(n)+M(n) \quad \text{as} \quad n \to \infty, \tag{1}$$

where

$$K(n) = \sum_{k=1}^{\infty} a_{kk}(n)\big(X_k - b_{kk}(n)\big)\big(X_k - c_{kk}(n)\big), \tag{2}$$

$$L(n) = \sum_{j=2}^{\infty}\sum_{k=1}^{j-1} a_{kj}(n)\big(X_k - b_{kj}(n)\big)\big(X_j - c_{kj}(n)\big) \tag{3}$$

and

$$M(n) = \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} a_{kj}(n)\big(X_k - b_{kj}(n)\big)\big(X_j - c_{kj}(n)\big). \tag{4}$$

In formulas (2), (3), (4) summation is taken only over those indices k, j for which $a_{kj}(n) \neq 0$.

To simplify the notation let us denote by F_k and F_k' the distribution functions of X_k and $(X_k - EX_k)$ (if EX_k exists), respectively, and let

$$F(y) = \sup_{i \geqslant 1} P[|X_i| \geqslant y], \tag{5}$$

$$F'(y) = \sup_{i \geqslant 1} P[|X_i - EX_i| \geqslant y]. \tag{6}$$

§ 2. Results

Investigating the asymptotic behaviour of the quadratic forms of random variables $S(n)$ as $n \to \infty$, one can prove that the following theorems are valid:

THEOREM 1. *Let $\{X_i;\ i \geqslant 1\}$ be a sequence of independent random variables.*

(1) *If $0 < t < 1$ and $y^t F(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then*

$$P[|S(n)| > \varepsilon] = O\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} |a_{kj}(n)|^{t/(2-t)}\Big) \tag{7}$$

holds as $n \to \infty$ with the coefficients $b_{kj}(n) = c_{kj}(n) = 0$.

(2) *If $0 < t < 1$, $y^t F(y) \to 0$ as $y \to \infty$, and $\sup_{k,j} |a_{kj}(n)| \to 0$ as $n \to \infty$, then*

$$P[|S(n)| > \varepsilon] = o\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} |a_{kj}(n)|^{t/(2-t)}\Big) \tag{8}$$

holds as $n \to \infty$ with the same coefficients $b_{kj}(n)$ and $c_{kj}(n)$ as in (1).

THEOREM 2. *Let $\{X_i;\ i \geqslant 1\}$ be a sequence of independent random variables.*

(1) *If $t = 1$ and $yF(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then*

$$P[|S(n)| > \varepsilon] = O\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} |a_{kj}(n)|^{2t/(4-t)}\Big) \tag{9}$$

holds as $n \to \infty$ with the coefficients

$$b_{kj}(n) = c_{kj}(n) = (1-\delta_{kj}) \int_{-1/|a_{kj}(n)|^{2/3}}^{1/|a_{kj}(n)|^{2/3}} x\,dF_k(x), \tag{10}$$

where δ_{kj} denotes Kronecker's symbol.

(2) *If $t = 1$, $yF(y) \to 0$ as $y \to \infty$, and $\sup_{k,j} |a_{kj}(n)| \to 0$ as $n \to \infty$, then*

$$P[|S(n)| > \varepsilon] = o\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} |a_{kj}(n)|^{2t/(4-t)}\Big) \tag{11}$$

holds as $n \to \infty$ with the coefficients $b_{kj}(n)$ and $c_{kj}(n)$ given by (10).

THEOREM 3. *Let $\{X_i;\ i \geqslant 1\}$ be a sequence of independent random variables.*

(1) *If $1 < t < 2$ and $y^t F(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then* (9) *holds as $n \to \infty$ with the coefficients $b_{kj}(n) = EX_k$ and $c_{kj}(n) = EX_j$.*

(2) *If $1 < t < 2$, $y^t F(y) \to 0$ as $y \to \infty$ and $\sup_{k,j} |a_{kj}(n)| \to 0$ as $n \to \infty$, then* (11) *holds as $n \to \infty$ with the same coefficients $b_{kj}(n)$ and $c_{kj}(n)$ as in* (1).

THEOREM 4. *Let $\{X_i;\ i \geqslant 1\}$ be a sequence of independent random variables. If $t = 2$ and $y^t F(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then*

$$P[|S(n)| > \varepsilon] = O\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} a_{kj}^2(n)\big(a_{kj}^2(n) - \log|a_{kj}(n)|\big)^2\Big) \tag{12}$$

holds as $n \to \infty$ with the coefficients

$$b_{kk}(n) = -c_{kk}(n) = \Big(\int_{-1/|a_{kk}(n)|^{1/2}}^{1/|a_{kk}(n)|^{1/2}} x^2\,dF_k(x)\Big)^{1/2}, \tag{13}$$
$$b_{kj}(n) = EX_k,\ c_{kj}(n) = EX_j \quad \text{for} \quad k \neq j.$$

If $2 < t < 4$ and $y^t F(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then

$$P[|S(n)| > \varepsilon] = O\Big(\sum_k |a_{kk}(n)|^{t/2} + \sum_{k\neq j} a_{kj}^2(n)\Big) \tag{14}$$

holds as $n \to \infty$ with the coefficients

$$b_{kk}(n) = -c_{kk}(n) = (EX_k^2)^{1/2}, \tag{15}$$
$$b_{kj}(n) = EX_k,\ c_{kj}(n) = EX_j \quad \text{for} \quad k \neq j.$$

If $t = 4$ and $y^t F(y) \leqslant M < \infty$ for $0 \leqslant y < \infty$, then

$$P[|S(n)| > \varepsilon] = O\Big(\sum_k a_{kk}^2(n)\big(a_{kk}^2(n) - \log|a_{kk}(n)|\big) + \sum_{k\neq j} a_{kj}^2(n)\Big) \tag{16}$$

holds as $n \to \infty$ with the coefficients $b_{kj}(n)$ and $c_{kj}(n)$ given by (15).

§ 3. Proofs

The proofs of all the theorems are based on the same method. Therefore only the exact proof of the first theorem will be presented here. However, at the end of this section short comments connected with the proofs of Theorems 2, 3, and 4 will be given.

Proof of Theorem 1. (1) To prove the theorem it suffices to investigate the asymptotic properties of $K(n)$ and $L(n)$ as $n \to \infty$, because $M(n)$ has similar asymptotic properties to those of $L(n)$. By using random variables $U_k(n)$, where

$$U_k(n) = \begin{cases} 0 & \text{if} \quad |a_{kk}(n)|^{1/2}|X_k| \geqslant 1, \\ X_k^2 & \text{if} \quad |a_{kk}(n)|^{1/2}|X_k| < 1, \end{cases} \tag{17}$$

the following estimate can be obtained:

$$P[|K(n)| > \varepsilon] \leqslant \sum_k P[|a_{kk}(n)|^{1/2}|X_k| \geqslant 1] + P\Big[\Big|\sum_k a_{kk}(n)\,U_k(n)\Big| > \varepsilon\Big]. \tag{18}$$

It is easy to observe that

$$\sum_k P[|a_{kk}(n)|^{1/2}|X_k| \geqslant 1] \leqslant \sum_k F(|a_{kk}(n)|^{-1/2}) \leqslant M \sum_k |a_{kk}(n)|^{t/2}, \tag{19}$$

which proves that the first term on the right-hand side of (18) fulfil assertion (7). To determine the upper boundary of the second term, let us compute the expected value of the random variable $|U_k(n)|$. Integrating by parts, one can state that

$$(20)\qquad E|U_k(n)| = \int_{-|a_{kk}(n)|^{-1/2}}^{|a_{kk}(n)|^{-1/2}} x^2 dF_k(x)$$

$$= -(a_{kk}^2(n))^{-1/2}F_k(-|a_{kk}(n)|^{-1/2}) + \int_0^{|a_{kk}(n)|^{-1/2}} F_k(-x)\,dx^2$$

$$+(a_{kk}^2(n))^{-1/2}F_k(|a_{kk}(n)|^{-1/2}) + \int_0^{|a_{kk}(n)|^{-1/2}} -F_k(x)\,dx^2$$

$$\leqslant \int_0^{|a_{kk}(n)|^{-1/2}} (1-F_k(x)+F_k(-x))\,dx^2$$

$$\leqslant \int_0^{|a_{kk}(n)|^{-1/2}} 2xF(x)\,dx \leqslant \frac{2M}{2-t}|a_{kk}(n)|^{(t-2)/2}.$$

Thus, applying Markov's inequality, we get

$$(21)\qquad P\Big[\Big|\sum_k a_{kk}(n)U_k(n)\Big| > \varepsilon\Big] \leqslant \frac{1}{\varepsilon}\sum_k |a_{kk}(n)|E|U_k(n)|$$

$$\leqslant \frac{2M}{\varepsilon(2-t)}\sum_k |a_{kk}(n)|^{t/2}.$$

Let us now establish the asymptotic behaviour of $L(n)$ as $n \to \infty$. The following inequality holds:

$$(22)\qquad P[|L(n)| > \varepsilon] \leqslant \sum_{k<j}\big(P[|a_{kj}(n)|^{1/(2-t)}|X_k| \geqslant 1] + P[|a_{kj}(n)|^{1/(2-t)}|X_j| \geqslant 1]\big) +$$

$$+P\Big[\Big|\sum_{k<j} a_{kj}(n)Y_{kj}(n)Z_{kj}(n)\Big| > \varepsilon\Big],$$

where

$$(23)\qquad Y_{kj}(n) = \begin{cases} 0 & \text{if } |a_{kj}(n)|^{1/(2-t)}|X_k| \geqslant 1,\\ X_k & \text{if } |a_{kj}(n)|^{1/(2-t)}|X_k| < 1,\end{cases}$$
$$Z_{kj}(n) = \begin{cases} 0 & \text{if } |a_{kj}(n)|^{1/(2-t)}|X_j| \geqslant 1,\\ X_j & \text{if } |a_{kj}(n)|^{1/(2-t)}|X_j| < 1,\ k<j.\end{cases}$$

By analogy to (19), it is easy to observe that

$$(24)\qquad \sum_{k<j}\big(P[|a_{kj}(n)|^{1/(2-t)}|X_k| \geqslant 1] + P[|a_{kj}(n)|^{1/(2-t)}|X_j| \geqslant 1]\big)$$

$$\leqslant 2\sum_{k<j} F\big(|a_{kj}(n)|^{-1/(2-t)}\big) \leqslant 2M\sum_{k<j}|a_{kj}(n)|^{t/(2-t)}.$$

Since

$$(25)\qquad E|Y_{kj}(n)| \leqslant \int_0^{|a_{kj}(n)|^{-1/(2-t)}} F(x)\,dx \leqslant \frac{M}{1-t}|a_{kj}(n)|^{(t-1)/(2-t)}$$

and since $E|Z_{kj}(n)|$ has the same property, we have

$$(26)\qquad P\Big[\Big|\sum_{k<j} a_{kj}(n)Y_{kj}(n)Z_{kj}(n)\Big| > \varepsilon\Big] \leqslant \frac{1}{\varepsilon}\sum_{k<j}|a_{kj}(n)|E|Y_{kj}(n)|E|Z_{kj}(n)|$$

$$\leqslant \frac{M^2}{\varepsilon(1-t)^2}\sum_{k<j}|a_{kj}(n)|^{2(t-1)/(2-t)+1},$$

therefore (7) is proved.

(2) At the very beginning an inequality will be derived on which the next steps of the proof are based. Let $\tau > 0$ be an arbitrary real number. Under the assumptions of the theorem, $m_h = \sup_{y \geqslant h} y^t F(y) \to 0$ as $h \to \infty$. Thus, choosing an H such that for each $h \geqslant H m_h(1-t)^{-1} < \tau/2$, and then a $T \geqslant H$ such that $HT^{t-1} < \tau/2$, one can obtain the inequality

$$(27)\qquad T^{t-2}\int_0^T yF(y)\,dy \leqslant T^{t-1}\int_T^0 F(y)\,dy$$

$$\leqslant HT^{t-1} + T^{t-1}\int_H^T m_H y^{-t}\,dy < \tau.$$

If $n \geqslant n_1$, then $\big(\sup_k |a_{kk}(n)|\big)^{-1/2} \geqslant T$. Therefore, taking into account (18), (19), (20), and (21) one can obtain the inequality

$$(28)\qquad P[|K(n)| > \varepsilon] \leqslant \sum_k F(|a_{kk}(n)|^{-1/2}) + \frac{1}{\varepsilon}\sum_k |a_{kk}(n)|\int_0^{|a_{kk}(n)|^{-1/2}} 2yF(y)\,dy$$

$$\leqslant \sum_k \frac{\tau}{2}|a_{kk}(n)|^{t/2} + \frac{2\tau}{\varepsilon}\sum_k |a_{kk}(n)|^{1+(t-2)/2}.$$

Similarly, $n \geqslant n_2$ implies that $\big(\sup_{k<j}|a_{kj}(n)|\big)^{-1/(2-t)} \geqslant T$, and on the strength of (22), (24), (25), and (26) the following inequality holds:

$$(29)\qquad P[|L(n)| > \varepsilon] \leqslant \sum_{k<j} 2F\big(|a_{kj}(n)|^{-1/(2-t)}\big) +$$

$$+\frac{1}{\varepsilon}\sum_{k<j}|a_{kj}(n)|\Big(\int_0^{|a_{kj}(n)|^{-1/(2-t)}} F(x)\,dx\Big)^2$$

$$\leqslant \tau\sum_{k<j}|a_{kj}(n)|^{t/(2-t)} + \frac{\tau^2}{\varepsilon}\sum_{k<j}|a_{kj}(n)|^{1+2(t-1)/(2-t)}.$$

Since τ is arbitrary, the left-hand sides of (28) and (29) fulfil assertion (8), so the proof of Theorem 1 is completed.

Proving Theorem 2 let us note that almost all the considerations connected with the expression $K(n)$ from the proof of Theorem 1 remain valid. To estimate the probability $P[|L(n)| > \varepsilon]$ as $n \to \infty$, one must cut random variables X_k and X_j in such a way as to make the expected values of new random variables equal to $b_{kj}(n)$ or $c_{kj}(n)$, respectively. Having such random variables, it is very easy to obtain assertion (9) and (11). Only small changes are needed in the proof of Theorem 1, so the details will be omitted.

In the proof of Theorem 3 the following fact is very useful: for $t > 1$ under the assumptions of the theorem expected values of all the random variables X_k exist and are bounded by the same constant, and, moreover, function F' has the same asymptotical properties as F.

The assertion of the theorem can now be obtained without much trouble: in all the inequalities it is only necessary to put function F' instead of F, and expression $P[|L(n)| > \varepsilon]$ has to be divided into three parts, of which one will contain the sum of the expected values of cut random variables multiplied by the coefficients $a_{kj}(n)$.

To prove the last theorem one may apply the same method as that used in the proof of Theorem 1. The only important change is the estimate of the modulus of expectation of cut random variables in the consideration of $P[|L(n)| > \varepsilon]$. All the random variables are now centred at the expectation, and thus the modulus of expectation of cut below $-\alpha$ and above α such random variable (for example $X_k - EX_k$) is bounded by the sum of integrals

$$\Big|\int_{-\infty}^{-\alpha} x\,dF_k'(x) + \int_{\alpha}^{\infty} x\,dF_k'(x)\Big|.$$

It is also necessary to emphasize that for $t > 2$ the expectations of squared random variables are bounded by a certain constant. By using the above facts the proof can easily be completed.

§ 4. Concluding remarks

Rohatgi ([4]) was interested in finding sufficient conditions under which a martingale quadratic form converges to some proper random variable; he did not describe the rate of convergence. Hanson and Wright ([3]) investigated the exponential rates of convergence in probability to zero of random quadratic forms. Griffiths, Platt and Wright ([1]) established the algebraic rates of convergence in probability to zero of random quadratic forms. However, they introduced in their paper ([1]), besides F, the function $G(y) = \sup_{k,j} P[|X_k X_j| \geqslant y]$, and made some additional assumptions about it. In the above theorems there is no assumption made about G, and, moreover, if $a_{kj}(n) = 0$ for $k \neq j$, then the results presented here reduce to those in [2] for weighted sums of positive random variables.

Let us now consider the following simple example:

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables such that $P[X_1 = 1] = P[X_1 = -1] = 1/2$ and let $X_2, X_3, \ldots$ have the same normal distribution $\mathcal{N}(0, 1)$. Moreover, let

$$a_{1j}(n) = 1/n \quad \text{if} \quad 2 \leqslant j \leqslant n^2+1,$$

and

$$a_{kj}(n) = 0 \quad \text{for} \quad \text{other indices } k, j.$$

Then $S(n)$ has a normal distribution $\mathcal{N}(0, 1)$ and $P[|S(n)| > \varepsilon] \nrightarrow 0$ in spite of $\sup_{k,j} |a_{kj}(n)| \to 0$ as $n \to \infty$ and $y^t F(y) \to 0$ as $y \to \infty$ for each fixed $t > 2$. This is the reason why in Theorem 4 only the estimate $O(.)$ can be obtained.

A similar example can be constructed for $t > 4$. Namely, let us consider a sequence $X_1, X_2, \ldots$ of independent, identically distributed random variables such that $y^t F(y) \to 0$ as $y \to \infty$ for some $t > 4$ and $EX_1^4 - (EX_1^2)^2 = \sigma^2 X_1^2 = 1$. Furthermore, let

$$a_{kk}(n) = 1/n \quad \text{if} \quad 1 \leqslant k \leqslant n^2,$$

and

$$a_{kj}(n) = 0 \text{ otherwise.}$$

Then $S(n)$ satisfies the central limit theorem, and thus $P[|S(n)| > \varepsilon] \nrightarrow 0$ as $n \to \infty$. Consequently, for $t > 4$ the best possible rate of convergence of $P[|S(n)| > \varepsilon]$ to zero as $n \to \infty$ is always

$$O\Big(\sum_k a_{kk}^2(n) + \sum_{k \neq j} a_{kj}^2(n)\Big).$$

At the end of these considerations it is worth to mentioning that in the case $0 < t < 2$ the estimate of the rate of convergence in probability to zero of random quadratic forms $S(n)$ can also be obtained for dependent random variables; indeed under similar assumptions to those in Theorem 1 this rate is equal to

$$O\Big(\sum_{k,j} |a_{kj}(n)|^{t/2}\Big) \quad \text{or} \quad o\Big(\sum_{k,j} |a_{kj}(n)|^{t/2}\Big),$$

respectively.

References

[1] G. N. Griffiths, R. D. Platt, F. T. Wright, *Convergence to zero of quadratic forms in independent random variables*, Ann. Prob. 1 5 (1973), pp. 838–848.

[2] D. L. Hanson, F. T. Wright, *Some convergence results for weighted sums of independent random variables*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 19 (1971), pp. 81–89.

[3] —, —, *A bound on tail probabilities for quadratic forms in independent random variables*, Ann. Math. Statist. 42 3 (1971), pp. 1079–1083.

[4] V. K. Rohatgi, *Almost sure convergence of quadratic forms in random variables*, J. Austral. Math. Soc. 15 3 (1973), pp. 257–259.

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