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ON THE STABILITY OF INFINITE-DIMENSIONAL LINEAR STOCHASTIC SYSTEMS

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The aim of the note is to obtain a necessary and sufficient condition for the stability of infinite-dimensional linear stochastic systems and to show how this result can be applied to the investigation of the stability of stochastic linear systems with delay.

1. Preliminaries

Let H be a Hilbert space and $A, B_1, \ldots, B_p$ linear operators on H such that:

(1) A is a generator of a semigroup $(T_t)_{t\geqslant 0}$ on H,

(2) $B_1, \ldots, B_p$ are bounded operators on H.

Let $(w_t^1)_{t\geqslant 0}, \ldots, (w_t^p)_{t\geqslant 0}$ be independent real-valued Wiener processes. By a linear stochastic equation we mean, in this paper, an equation of the form:

$$dx = Ax\,dt + \sum_{s=1}^{p} B_j x\,dw_t^j, \qquad t \geqslant 0,\ x_0 \in H. \tag{1}$$

A *mild solution* of equation (1) is a stochastic process $(x_t)_{t\geqslant 0}$ with values in H, continuous in the mean-square sense, such that x_t is $\sigma(w_s^1, \ldots, w_s^p, s \leqslant t\}$-measurable, and that the following integral equation is satisfied:

$$x_t = T_t x + \sum_{j=1}^{p} \int_0^t T_{t-s} B_j x_s\,dw_s^j \tag{2}$$

for all $t \geqslant 0$. A mild solution of equation (1) always exists and is unique [1].

If the operators B_j, $j = 1, \ldots, p$ are of a special type

$$B_j(x) = b_j(c_j, x), \qquad x \in H,\ j = 1, \ldots, p,$$

where b_j, c_j are given elements in H, then system (1) is said to be of the *Lurie type*. In this case we denote operators B_j shortly by $b_j|c_j$.

If for every initial state $x_0 \in H$ the corresponding mild solution $(x_t)_{t\geq 0}$ of (1) satisfies

$$E\Big(\int_0^{+\infty} |x_t|^2 dt\Big) < +\infty,$$

then we say that *system* (1) *is stable.*

In this note we give a necessary and sufficient condition for the stability of system (1) in terms of Liapunov's equation (3). More concrete sufficient conditions will also be formulated. From the necessary and sufficient condition, in the case of Lurie systems, we deduce an effective criterion for stability, generalizing that given by Jakubovich and Levit for finite-dimensional systems in [2]. We also give some applications to stochastic delay systems.

In the Appendix we prove a general theorem on the spectral radius of a monotonic linear transformation, which plays an important role in this note, and which for matrices was proved in [7].

The results given here were only sketched in a previous preprint [9].

2. A theorem of A. M. Liapunov for a stochastic system in Hilbert space

It is well known, see Datko [3], that:

The system $\dot{x} = Ax$ is stable, or equivalently: the semigroup $(T_t)_{t\geq 0}$ is stable, if and only if there exists a bounded positive semidefinite operator P on H such that

$$2(PAx, x)+(Qx, x) = 0$$

for all $x \in D(A)$ and for some invertible positive operator Q.

The following theorem is a generalization of Datko's result to stochastic systems.

THEOREM 1. *System* (1) *is stable if and only if there exists a bounded operator* P *on* H *positive semidefinite such that*

$$2(PAx, x)+\sum_{j=1}^{p}(B_j^* PB_j x, x)+(x, x) = 0 \tag{3}$$

for all $x \in D(A)$.

To prove the theorem we need the following lemma, which was proved in a more general setting by Akira Ichikawa [5]:

LEMMA 1. *There exists exactly one strongly continuous family* $(Q(u))_{u\geq 0}$ *of non-negative operators which satisfies the equation*

$$Q(t) = \int_0^t T^*(t-v)\Big[S+\sum_{j=1}^{p} B_j^* Q(v) B_j\Big] T(t-v)\,dv, \qquad t \geq 0, \tag{4}$$

where S *is a given non-negative operator on* H *and* $(T(t))_{t\geq 0}$ *is the semigroup generated by* A. *Moreover,*

$$(Q(t)x, x) = E\int_0^t (Sx_s^x, x_s^x)\,ds. \tag{5}$$

Proof of Theorem 1. Taking S equal to the identity operator I, we infer from Lemma 1 that system (1) is stable iff the family $(Q(t))_{t\geq 0}$ is bounded. Let us assume that system (1) is stable, then the operator $\bar{Q} = \lim_{t\to+\infty} Q(t)$ is well defined and also $\int_0^{+\infty} (T_t x, T_t x)\,dt \leq (\bar{Q}x, x) < +\infty$ for all $x \in H$. Therefore A is a stable generator and equation (3) is equivalent to

$$P = \int_0^{+\infty} T_t^* T_t\,dt+\sum_{j=1}^{p}\int_0^{+\infty} T_t^* B_j^* PB_j T_t\,dt, \qquad P \geq 0. \tag{6}$$

Putting in (4) $t \to +\infty$ and taking into account the monotonicity of $(Q(t))_{t\geq 0}$ and the exponential decay of $(T_t)_{t\geq 0}$, we obtain

$$\bar{Q} = \int_0^{+\infty} T_t^* T_t\,dt+\sum_{j=1}^{p}\int_0^{+\infty} T_t^* B_j^* \bar{Q} B_j T_t\,dt.$$

Thus $\bar{Q}$ satisfies the conditions required in the theorem. To prove the opposite implication assume that an operator $\bar{P} \geq 0$ satisfies equation (3). Then by quoted Datko's result the operator A is stable. Consequently $\bar{P}$ also satisfies (6). Let us define

$$Q_0(t) \equiv 0,\ t \geq 0 \quad \text{and} \quad Q_{n+1}(t) = \int_0^t T^*(t-v)\Big[S+\sum_{j=1}^{p} B_j^* Q_n(v) B_j\Big] T(t-v)\,dv.$$

Then we show by induction that $\bar{Q} \geq Q_{n+1}(t) \geq Q_n(t) \geq 0, n = 0, 1, 2 \ldots$ But $Q_\infty(t) = \lim_n Q_n(t)$ is a unique solution of (4); therefore this unique solution is bounded, and thus (1) is a stable stochastic system. This completes the proof.

COROLLARY 1. *From the proof it follows that if system* (1) *is stable, then the semigroup* $(T_t)_{t\geq 0}$ *generated by the operator* A *is stable. Consequently* (*see* [3]),

$$|T_t| \leq Me^{-\omega t}, \qquad t \geq 0, \tag{7}$$

for some positive M *and* ω.

3. Formulation of the main result

The main result of the paper can be formulated as follows:

THEOREM 2. *Assume that* $B_j = b_j|c_j$, $j = 1, 2, \ldots, p$; *then system* (1) *is stable if and only if*

(1) *the semigroup* $(T_t)_{t\geq 0}$ *is stable,*

(2) *the spectral radius of the matrix* $D = (d_{k,j})_{k,j=1,\ldots,p}$,

$$d_{k,j} = \int_0^{+\infty} |(T_t b_k, c_j)|^2 dt, \tag{8}$$

is less than 1.

The proof of Theorem 2 will be given is Section 4.

We recall that a semigroup $(T_t)_{t\geq 0}$ is stable if (7) holds, and that the spectral radius of D is equal to

$$r(D) = \lim_n \sqrt[n]{\|D^n\|} = \inf_n \sqrt[n]{\|D^n\|}\,.$$

Remark 1. By Parseval's equality

$$d_{k,j} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |(R_\lambda b_k, c_j)|^2 d\lambda, \quad k,j = 1,\ldots,p.$$

Here (R_λ) denotes the resolvent of the semigroup $(T_t)_{t\geq 0}$. In particular, for stable semigroups

$$R_\lambda = \int_0^{+\infty} e^{-\lambda t} T_t dt \quad \text{if} \quad \operatorname{Re}\lambda \geq 0.$$

Thus we can reformulate Theorem 2 in an obvious way, using the "frequency-domain" framework as in [7].

PROPOSITION 1. *If the semigroup* $(T_t)_{t\geq 0}$ *is stable and*

$$\int_0^{+\infty} \Big(\sum_{j=1}^{p} |(T_t b_k, c_j)|^2\Big) dt < 1 \quad \text{for} \quad k = 1,2,\ldots,p,$$

then system (1) *with* $B_j = b_j|c_j$ *is stable.*

Proof. Let us introduce in R^p the norm: $|x| = \max(|x_1|,\ldots,|x_p|)$, $x = (x_1,\ldots,x_p) \in R^p$. Then the linear operator corresponding to the matrix $D = (d_{k,j})_{k,j=1,\ldots,p}$ has the norm

$$|D| = \max_k \sum_{j=1}^{p} |d_{k,j}|.$$

Thus, if $|D| = \max_k \sum_{j=1}^{p} |d_{k,j}| < 1$, then the spectral radius of the matrix D is less than 1. Obviously the condition $|D| < 1$ is equivalent to $\sum_{j=1}^{p} |d_{k,j}| < 1$ for $k = 1,2,\ldots,p$. This completes the proof.

COROLLARY 2. *If for some positive numbers* $M, \omega > 0$ *and all* $t \geq 0$

$$|T_t| \leq Me^{-\omega t},$$

and if for every $k = 1,2,\ldots,p$

$$|b_k| \leq \frac{\sqrt{2\omega}}{M\sqrt{\sum_{j=1}^{p}|c_j|^2}},$$

then system (1) *with* $B_j = b_j|c_j$ *is stable.*

4. Proof of Theorem 2

The proof is a modification of the proof by M. V. Levit and V. A. Yakubovich [7]. Let us assume that system (1) with $B_j = b_j|c_j$, $j = 1,2,\ldots,p$ is stable. By Corollary 1, the semigroup $(T_t)_{t\geq 0}$ generated by the operator A is also stable. Consequently, for any bounded operator $R \geq 0$, the equation

$$2(PAx, x)+(Rx, x) = 0, \quad x \in D(A)$$

has a unique solution given by the formula

$$P = \int_0^{+\infty} T_t^* R T_t\, dt.$$

Thus equation (3) is equivalent to the equation

$$P = \int_0^{+\infty} T_t^*\Big[\sum_{j=1}^{p} B_j^* P B_j + I\Big] T_t dt, \quad P \geq 0 \tag{9}$$

or, by using the special form of the operators B_j, to

$$P = Q + \sum_{j=1}^{p} (b_j, Pb_j) \int_0^{+\infty} T_t^* c_j|c_j T_t dt, \quad P \geq 0; \tag{10}$$

Q denotes here the operator $\int_0^{+\infty} T_t^* T_t dt$.

From (10) we infer that for $k = 1,\ldots,p$

$$(b_k, Pb_k) = (b_k, Qb_k) + \sum_{j=1}^{p} (b_j, Pb_j) \int_0^{+\infty} |(T_t b_k, c_j)|^2 dt. \tag{11}$$

Thus the sequence $x = \big((b_1, Pb_1),\ldots,(b_p, Pb_p)\big)$ is a nonnegative solution of the equation

$$x = y + Dx, \tag{12}$$

where $y = \big((b_1, Qb_1),\ldots,(b_p, Qb_p)\big)$ is a vector with positive coordinates and D is the matrix defined in Theorem 2. By Corollary 3 of the Appendix, D is a stable matrix. This completes the proof of Theorem 2 in one direction. To prove the theorem in the opposite direction let us remark that if the semigroup $(T_t)_{t\geq 0}$ is stable then the matrix D is well defined. If, in addition, D is a stable matrix, then equation (12), with $y = \big((b_1, Qb_1),\ldots,(b_p, Qb_p)\big)$, has a unique nonnegative solution, which we denote by $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)$.

Let $\bar{P}$ be a positive definite operator defined as

$$\bar{P} = Q + \sum_{j=1}^{p} \bar{x}_j \int_0^{+\infty} T_t^* c_j|c_j T_t\, dt;$$

then

$$\big((b_1, \bar{P}b_1),\ldots,(b_p, \bar{P}b_p)\big) = y + D\bar{x} = \bar{x},$$

and therefore

$$\bar{P} = Q + \sum_{j=1}^{p} (b_j, \bar{P} b_j) \int_0^{+\infty} T_t^* c_j | c_j T_t \, dt.$$

Thus the operator $\bar{P}$ satisfies equation (10) and also the equivalent equation (3). An application of Theorem 1 completes the proof.

5. An application to delay systems

In this section we apply Theorem 2 to systems described by a linear stochastic equation with delay of the form:

$$(13) \qquad dx(t) = \int_{-h}^{0} dN(s) x(t+s) \, dt + \sum_{j=1}^{p} b_j (c_j, x(t)) \, dw_t^j.$$

Here $N(\cdot)$ is a function of bounded variation from the interval $[-h, 0]$ into the space of $n \times n$-matrices; $b_j, c_j, j = 1, \ldots, p$ are n-vectors and $(w_t^1)_{t \geq 0}, \ldots, (w_t^p)_{t \geq 0}$ are independent, real-valued Wiener processes. Under a very weak assumption equation (13) can be treated as an equation of type (1) in the Hilbert space $\mathfrak{M}^2 = \boldsymbol{R}^n \times L^2(-h, 0; \boldsymbol{R}^n)$. Namely, let us introduce the following assumption:

ASSUMPTION 1. *The operator A defined on*

$$D(A) = \{(a, \varphi);\ \varphi \in W_2^1(-h, 0; \boldsymbol{R}^n),\ a = \varphi(0)\},$$

by the formula

$$(14) \qquad A \begin{bmatrix} \varphi(0) \\ \varphi \end{bmatrix} = \begin{bmatrix} \int_{-h}^{0} dN(s) \varphi(s) \\ \dfrac{d\varphi}{ds} \end{bmatrix}$$

is the infinitesimal generator of a C_0-semigroup on $\mathfrak{M}^2$ and the spectrum of A, $\sigma(A) = \left\{\lambda:\ \det\left(\int_{-h}^{0} e^{\lambda s} dN(s) - \lambda I\right) = 0\right\}$.

The following theorem is a special case of a result proved by Chojnowska-Michalik [2]:

THEOREM 3. *Make Assumption* 1 *and define $\bar{X}(t) = (x(t),\ x_t) \in \mathfrak{M}^2$, where $x(\cdot)$ is a solution of* (13) *and $x_t(\theta) = x(t+\theta)$ for all $t \geq 0$ and $\theta \in [-h, 0]$. Then $(\bar{X}(t))_{t \geq 0}$ is the unique solution of the equation*

$$\bar{X}(t) = T_t \bar{X}(0) + \sum_{j=1}^{p} \int_0^t T_{t-s} \bar{B}_j \bar{X}(s) \, dw_s^j,$$

where $(T_t)_{t \geq 0}$ is a semigroup generated by the operator A given by (14) *and $\bar{B}_j(a, \varphi) = (b_j(c_j, a), 0)$.*

This theorem shows that the following definition is quite natural:

System (13) is said to be *stable* if and only if for every initial condition $\bar{X}(0) \in \mathfrak{M}^2$ the corresponding solution $\bar{X}(t)$, $t \geq 0$ satisfies $E \int_0^{+\infty} |\bar{X}(t)|^2 dt < +\infty$.

Let us remark that the inequality $E \int_0^{+\infty} |\bar{X}_t|^2 dt < +\infty$ is equivalent to $E \int_0^{+\infty} |x(t)|^2 dt < +\infty$.

As a corollary from Theorem 2 we obtain the following

THEOREM 4. *Let us assume that Assumption* 1 *holds; then system* (13) *is stable if and only if*

(1) $\sup\left\{\operatorname{Re}\lambda;\ \det\left(\int_{-h}^{0} e^{\lambda s} dN(s) - \lambda I\right) = 0\right\} < 0$,

(2) *absolute values of the eigenvalues of the matrix $D = (d_{k,j})_{k,j=1,\ldots,n}$, with the components*

$$d_{k,j} = \int_0^{+\infty} |(x_t^{b_k}, c_j)|^2 dt$$

are strictly less than 1. *Here $(x_t^b)_{t \geq 0}$ denotes the solution of the deterministic delay equation*

$$\dot{x} = \int_{-h}^{0} dN(s) x(t+s)$$

with the initial conditions

$$x(u) = 0, \quad \text{for} \quad u \in [-h, 0),$$
$$x(0) = b.$$

Remark 2. Assumption 1 is satisfied for many interesting delay systems (see, for instance [4]). More explicit sufficient conditions can also be obtained by the application of Corollary 2.

Remark 3. With an obvious modification Theorem 4 is valid for more general delay systems described by the equation

$$dx(t) = \left(\int_{-h}^{0} x(t+s) N(ds)\right) dt + \sum_{j=1}^{p} b_j (c_j, x(t)) \, dw_t^j + \sum_{j=p+1}^{p+r} b_j \langle c_j, x_t \rangle \, dw_t^j,$$

where $c_j \in L^2[-h, 0; \boldsymbol{R}^n]$ for $j = p+1, \ldots, p+r$ and $\langle c_j, x_t \rangle = \int_{-h}^{0} (c_j(s), x(t+s)) \, ds$. Theorem 3 is also valid in this more general case.

Remark 4. Theorem 4 can be interpreted as a generalization of the results from [7] to linear finite-dimensional delay systems.

6. Appendix. A lemma on monotonic transformation

In the proof of Theorem 2 we have used a very special case of the following theorem (see Corollary 3):

THEOREM 5. *Assume that D is a linear transformation on a Banach space E, monotonic with respect to a normal cone $K \subset E$. Let y be an element of K such that for a certain $\alpha > 0$*

$$\{x \in E;\ |x| \leqslant 1\} \subset \{x \in E;\ -\alpha y \leqslant x \leqslant \alpha y\}.$$

Then the equation

$$x = Dx + y \tag{15}$$

has a solution $x \in K$ if and only if the spectral radius of D is less than 1; $r(D) < 1$.

We recall that a cone K is normal if and only if there exists a constant $\beta > 0$ such that $u, v \in K$, $u \leqslant v$, implies $|u| \leqslant \beta |v|$ (see [4], Theorem 1.2, p. 24).

Proof of the theorem. If $r(D) < 1$, then the series $\sum_{n=0}^{+\infty} D^n y$ converges in E and defines the unique solution of (15). To prove the theorem in the opposite direction we show first that the transformation, $\mathscr{D}(z) = Dz + y$, $z \in K$, is y-concave, i.e. that

(1) for every $z \in K$ there exist positive numbers γ and δ such that

$$\gamma y \leqslant \mathscr{D}(z) \leqslant \delta y,$$

(2) for every $z \in K$ and $t \in (0, 1)$ there exists an $\varepsilon > 0$ such that

$$\mathscr{D}(tz) \geqslant (t+\varepsilon)\mathscr{D}(z).$$

We check, for instance, property (2), which is equivalent to the inequality $D(z) \leqslant \frac{(1-t)-\varepsilon}{\varepsilon} y$. But if $|\eta D(z)| \leqslant 1$, $\eta > 0$, then $\eta D(z) \leqslant \gamma y$. Therefore it is sufficient to take $\varepsilon > 0$, which satisfies $\frac{\gamma}{\eta} \leqslant \frac{(1-t)-\varepsilon}{\varepsilon}$.

From Theorem 6.7 of [6], p. 193 we thus infer that the iterates

$$\mathscr{D}^n(0) = \sum_{i=0}^{n-1} D^i y$$

tend to x, the solution of (15), in the sense of the norm on E. Consequently $|D^n y| \to 0$ as $n \to +\infty$. By virtue of our assumptions, if $x \in E$, $|x| \leqslant 1$, then $-\gamma y \leqslant x \leqslant \gamma y$, and thus $-\gamma D^n y \leqslant D^n x \leqslant \gamma D^n y$. Therefore $0 \leqslant D^n x + \gamma D^n y \leqslant 2\gamma D^n y$. The normality of the cone K implies

$$\begin{aligned} |D^n x| &\leqslant |D^n x + \gamma D^n y| + |\gamma D^n y| \\ &\leqslant 2\gamma\beta |D^n y| + \gamma |D^n y| \\ &\leqslant (2\beta+1)\gamma |D^n y|, \end{aligned}$$

and we obtain

$$|D^n| \leqslant (2\beta+1)\gamma |D^n y| \to 0.$$

This completes the proof of the theorem.

COROLLARY 3. *If $E = \mathbf{R}^p$, D is a matrix with nonnegative elements and y is a vector from $\mathbf{R}^p$ with strictly positive coordinates, then the assumptions of Theorem 4 are satisfied (here $K = \mathbf{R}^p_+$). Therefore the equation*

$$x = Dx + y$$

has a solution $x \in \mathbf{R}^p_+$ if and only if the matrix D is stable.

Remark 5. Corollary 3 was proved in [7] by a different method, based on special properties of matrices.

Remark 6. Theorem 5 is also a generalization of Lemma 3.2 in [8].

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