PROBABILITY THEORY
BANACH CENTER PUBLICATIONS, VOLUME 5
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

GEOMETRIC DECOMPOSABILITY PROPERTIES OF PROBABILITY MEASURES

K. URBANIK

Institute of Mathematics, Wroclaw University, Wroclaw, Poland

Let X denote a real separable Banach space with the norm $||\cdot||$ and with the dual space X^*. By a *probability measure* μ on X we shall understand a countably additive nonnegative set function μ on the class of Borel subsets of X with the property that $\mu(X) = 1$.

The *characteristic functional* of μ is defined on X^* by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle y, x\rangle}\mu(dx) \qquad (y \in X^*).$$

It is well known that the characteristic functional uniquely determines the probability measure. Moreover, it is continuous in the X-topology of the space X^* ([1], Theorem 6.2.1).

Given a linear operator A and a probability measure μ on X, we denote by $A\mu$ the probability measure defined by the formula $A\mu(E) = \mu\big((A^{-1}(E)\big)$ for all Borel subsets E of X. In the study of limit probability distributions [6] I introduced the concept of *decomposability semigroup* $D(\mu)$ of linear operators associated with the probability measure μ. Namely, $D(\mu)$ consists of all linear operators A on X for which the equality

$$\mu = A\mu * \nu$$

holds for a certain probability measure ν. The asterisk denotes here the convolution of measures. Of course, $D(\mu)$ is the semigroup under composition of operators and $D(\mu)$ always contains the zero and the unit operators. Moreover, $D(\mu)$ is closed in the weak* operator topology. It has been shown in [6], [7] and [8] that some purely probabilistic properties of μ are equivalent with some algebraic and topological properties of the decomposability semigroup $D(\mu)$.

In the case of real line $X = \boldsymbol{R}$ we can identify operators $T_c x = cx$ $(x \in \boldsymbol{R})$ with the real numbers c so that the decomposability semigroup can be regarded as a subsemigroup of the multiplicative semigroup of real numbers. In terms of decomposibility semigroups the classical result of P. Lévy ([4], Theorem 56) can be formulated as follows: μ belongs to the class L of limit distributions if and only if $[0, 1] \subset D(\mu)$. Further, we note that the concept of c-decomposability introduced by M. Loéve in [5] p. 334 is equivalent with the relation $c \in D(\mu)$. Recently O. K. Zakusilo extended in [10] these investigations to the case of Euclidean spaces $X = R^n$ and obtained a new class of probability measures which in terms of decomposability semigroups can be described by the condition $A \in D(\mu)$, where A is an invertible linear operator with $||A|| < 1$.

The main aim of this paper is to characterize isotropic Gaussian probability measures by a certain geometric property of their decomposability semigroups.

Let μ be a probability measure and A a linear operator on X. Then for every $y \in X^*$,

$$\widehat{A\mu}(y) = \hat{\mu}(A^*y).$$

Consequently, $A \in D(\mu)$ if and only if

$$\hat{\mu}(y) = \hat{\mu}(A^*y)\hat{\nu}(y)$$

for a certain probability measure ν on X. By a *projector* P on X we mean a linear operator with the property that $P^2 = P$. Further, by I we denote the unit operator on X.

LEMMA 1. *Let P be a projector on X and $P \in D(\mu)$. Then $I-P \in D(\mu)$ and*

$$\mu = P\mu*(I-P)\mu.$$

Proof. Since $P \in D(\mu)$, we have the decomposition

$$\mu = P\mu*\nu, \tag{1}$$

where ν is a probability measure. Hence we get the formula $P\mu = P\mu * P\nu$. Consequently, $\widehat{P\mu} = \widehat{P\mu}\cdot\widehat{P\nu}$, which implies the equation $\widehat{P\nu}(y) = 1$ in a neighborhood of 0 in X^*. But the last condition implies the formula $\widehat{P\nu}(y) = 1$ for all $y \in X^*$ (see [2], Proposition 2.3). Thus $P\nu = \delta_0$, where δ_0 is the probability measure concentrated at 0 in X. Hence, in particular, it follows that ν itself is concentrated on the subspace $(I-P)X$. In other words, $(I-P)\nu = \nu$. Since $O\mu = \delta_0$, the last equation and (1) yield the formula

$$(I-P)\mu = (I-P)P\mu*(I-P)\nu = \nu.$$

Thus $\mu = P\mu*(I-P)\mu$ and, consequently, $I-P \in D(\mu)$ which completes the proof.

In what follows by S_r $(r > 0)$ we shall denote the sphere of radius r in the space of linear operators on X, i.e.

$$S_r = \{A\colon ||A|| = r\}.$$

LEMMA 2. *If $S_r \subset D(\mu)$, then for every pair $y_1, y_2 \in X^*$ with $||y_1|| = ||y_2||$ the inequality*

$$|\hat{\mu}(y_1)| \leqslant |\hat{\mu}(ry_2)|$$

holds.

Proof. Let $y_1, y_2 \in X^*$ and $||y_1|| = ||y_2|| = a$. Without loss of generality we may assume that $a > 0$ because in the case $a = 0$ the inequality is obvious. Taking $x_j \in X$ with the properties $||x_j|| = 1$ and $y_j(x_j) = a$, we define the linear operator A by assuming

$$A(x) = \frac{r}{a}y_1(x)x_2 \quad (x \in X).$$

It is clear that $||A|| = r$ and, consequently, $A \in D(\mu)$. Moreover, $A^*y_1 = ry_2$. Thus there exists a probability measure ν such that

$$\hat{\mu}(y_1) = \hat{\mu}(ry_2)\hat{\nu}(y_1).$$

Since $|\hat{\nu}(y)| \leqslant 1$, we have the assertion of the lemma.

THEOREM 1. *Suppose that X is infinite-dimensional. Then $S_r \subset D(\mu)$ if and only if μ is concentrated at a single point.*

Proof. Let $y \in X^*$. By Wehausen Theorem ([9]) 0 belongs to the closure of the set $\left\{z\colon ||z|| = \frac{||y||}{r}\right\}$ in the weak topology of the space X^*. Since X^* is separable in the X-topology, we can select a sequence $z_1, z_2, \ldots$ tending to 0 in the X-topology. Consequently, $\hat{\mu}(z_n) \to 1$ which, by Lemma 2, yields $|\hat{\mu}(y)| = 1$ for all $y \in X^*$. This shows that μ is concentrated at a single point.

We call a probability measure *isotropic* if it is invariant under all linear isometries of X.

THEOREM 2. *Suppose that X is finite-dimensional,* $\dim X \geqslant 2$ *and μ is not concentrated at a single point. Then $S_1 \subset D(\mu)$ if and only if μ is a translate of an isotropic Gaussian measure and X is a Hilbert space under the norm $||\cdot||$.*

Proof. First we shall prove the sufficiency of the condition $S_1 \subset D(\mu)$. Let Y be a subspace of X with $\operatorname{codim} Y = 2$. By $||\cdot||_Y$ we shall denote the norm induced by $||\cdot||$ on X/Y and by p the standard mapping from X onto X/Y. Evidently, $||p(x)||_Y \leqslant ||x||$ for all $x \in X$. Let μ_Y be the probability measure induced by μ on X/Y. Then for every $h \in (X/Y)^*$ the composition $h \circ p$ belongs to X^* and

$$\hat{\mu}_Y(h) = \hat{\mu}(h \circ p). \tag{2}$$

Given $u \in X/Y$ with $||u||_Y = 1$, we denote by $[u]$ the one-dimensional subspace generated by u.

By Hahn–Banach Theorem there exists a projector Q_u from X/Y onto $[u]$ with $||Q_u||_Y = 1$. Thus for every $x \in X$ we have the formula $Q_u p(x) = c_u(x)u$, where $c_u(x)$ is a real number. Selecting an element x_u in X with the properties $||x_u|| = 1$ and $p(x_u) = u$ we define the projector P_u from X onto $[x_u]$ by means of the formula $P_u x = c_h(x)x_u$ $(x \in X)$. Obviously, $||P_u|| = ||Q_u||_Y = 1$ and, consequently, $P_u \in D(\mu)$. Thus there exists a probability measure ν on X such that

$$\hat{\mu}(y) = \hat{\mu}(P_u^*y)\hat{\nu}(y) \quad (y \in X^*).$$

Since for every $h \in (X/Y)^*$, $P_u^*(h \circ p) = Q_u^*(h)$, we have, by virtue of (2),

$$\hat{\mu}_Y(h) = \hat{\mu}_Y(Q_u^*h)\hat{\nu}_Y(h),$$

where ν_Y is induced by ν on X/Y. In other words, $Q_u \in D(\mu_Y)$. By Lemma 1 we have also $I-Q_u \in D(\mu_Y)$ and

$$\mu_Y = Q_u\mu_Y*(I-Q_u)\mu_Y. \tag{3}$$

Since $\operatorname{codim} Y = 2$, we can select a pair $u_1, u_2 \in X/Y$ with the properties $||u_1||_Y =$

$||u_2||_Y = 1$ and $u_2 \in (I-Q_{u_1}) X/Y$. Evidently,

$$X/Y = [u_1] \oplus [u_2].$$

Let ζ be an X/Y-valued random variable with the probability distribution μ_Y. Then

$$\zeta = \xi_1 u_1 + \xi_2 u_2, \tag{4}$$

where ξ_1 and ξ_2 are real-valued random variables. For every pair t_1, t_2 of real numbers we define the functional g on X/Y by assuming $g(c_1 u_1 + c_2 u_2) = t_1 c_1 + t_2 c_2$ $(c_1, c_2 \in \boldsymbol{R})$. Then, in view of (3),

$$Ee^{i\langle g, \zeta\rangle} = Ee^{i(t_1\xi_1 + t_2\xi_2)} = Ee^{it_1\xi_1} \cdot Ee^{it_2\xi_2},$$

which implies the independence of the random variables ξ_1 and ξ_2. Setting $a^{-1} = ||u_1 - u_2||_Y$ and

$$v_1 = a(u_1 - u_2),$$

and selecting an element v_2 from $(I-Q_{v_1}^{*}) X/Y$ with $||v_2||_Y = 1$, we have

$$X/Y = [v_1] \oplus [v_2].$$

Consequently, the random variable ζ can be written in the form

$$\zeta = \eta_1 v_1 + \eta_2 v_2, \tag{5}$$

where η_1 and η_2 are real-valued random variables. The same arguments as in the case of the pair u_1, u_2 yield the independence of the random variables η_1 and η_2. Further, by the linear independence of v_1 and v_2 the coefficients a_1, a_2 in the expansion $v_2 = a_1 u_1 + a_2 u_2$ satisfy the condition

$$a_1 + a_2 \neq 0. \tag{6}$$

Moreover, from (4) and (5) we get the formulas

$$\eta_1 = \frac{a_2\xi_1 - a_1\xi_2}{a(a_1+a_2)}, \quad \eta_2 = \frac{\xi_1 + \xi_2}{a_1 + a_2}.$$

Hence it follows that two linear forms $\xi_1 + \xi_2$ and $a_2\xi_1 - a_1\xi_2$ of independent random variables ξ_1 and ξ_2 are independently distributed. By Skitovich–Darmois Theorem ([3], Theorem 5.1.1) each random variable ξ_j, which has non-zero coefficients in both forms, is Gaussian. By (6) we conclude that at least one random variable, say ξ_k $(k = 1$ or $2)$ is Gaussian. Setting $y_k(x) = c_{u_k}(p(x))$ for $x \in X$, we infer that $y_k \in X^*$, $||y_k|| = 1$ and $\hat{\mu}(ty_k) = Ee^{it\xi_k}$ for every $t \in \boldsymbol{R}$. Thus $\hat{\mu}(ty_k) = e^{imt - \frac{\sigma^2}{2}t^2}$, where $m, \sigma \in \boldsymbol{R}$. Applying Lemma 2 for $r = 1$, we infer that $|\hat{\mu}(y)| = |\hat{\mu}(ty_k)|$ whenever $||y|| = ||ty_k|| = |t|$. Consequently,

$$|\hat{\mu}(y)|^2 = e^{-\sigma^2||y||^2}$$

for all $y \in X^*$. Thus for every $t \in \boldsymbol{R}$ and $y \in X^*$

$$|\hat{\mu}(ty)|^2 = e^{-\sigma^2||y||^2 t^2}$$

which implies, by the Cramér Decomposition Theorem ([5], p. 271),

$$\hat{\mu}(ty) = e^{im(y)t - \frac{\sigma^2}{2}||y||^2 t^2}. \tag{7}$$

In other words, the probability measure μ is Gaussian. Regarding X in the equivalent Euclidean metric, we have the standard representation of the characteristic functional $\hat{\mu}$:

$$\hat{\mu}(y) = e^{i\langle y, x_0\rangle - \frac{1}{2}(By, y)}, \tag{8}$$

where $(\cdot, \cdot)$ is the Euclidean inner product, $x_0 \in X$ and B is the covariance operator, i.e. a non-negative self-adjoint operator. We have assumed that μ is not concentrated at a single point. Consequently, $\sigma^2 > 0$ in (7) and, by (8), $||y||^2 = \sigma^{-2}(By, y)$ which shows that the norm $||\cdot||$ on X^* is induced by an inner product. Hence it follows that the norm $||\cdot||$ on X has the same property. Thus X is Hilbert space under this norm. Further, setting $\hat{\mu}_0(y) = e^{-\frac{\sigma^2}{2}||y||^2}$ we get an isotropic Gaussian measure on X. By (7) μ is a translate of μ_0 which completes the proof of the sufficiency of the condition $S_1 \subset D(\mu)$.

To prove the necessity of this condition let us assume that X is a Hilbert space and μ is a translate of an isotropic Gaussian measure. Then the covariance operator for μ is of the form bI where b is a positive number. For any operator $A \in S_1$, the operator $b(I - AA^*)$ is non-negative self-adjoint and, consequently, can be regarded as the covariance operator for a Gaussian measure, say ν_0. It is easy to verify the equation

$$\mu = A\mu * \nu$$

where ν is a translate of ν_0. Thus $S_1 \subset D(\mu)$ which completes the proof of the theorem.

It should be noted that the assumption $\dim X \geqslant 2$ in Theorem 2 is essential. In fact, the condition $S_1 \subset D(\mu)$ in the one-dimensional case characterizes translates of symmetric probability measures. It is also evident that for any Banach space X and for $r > 1$ the condition $S_r \subset D(\mu)$ characterizes probability measures concentrated at a single point. It would be interesting to characterize all probability measures μ on a finite-dimensional Banach space X for which the inclusion $S_r \subset D(\mu)$ holds for a certain number $r \in (0,1)$.

References

[1] U. Grenander, *Probabilities on algebraic structures*, New York, London 1963.

[2] K. Itô and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), pp. 35–48.

[3] R. G. Laha and E. Lukacs, *Applications of characteristic functions*, London 1964.

[4] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1954.

[5] M. Loève, *Probability theory*, New York 1950.

[6] K. Urbanik, *Lévy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), pp. 119–148.

[7] —, *Operator semigroups associated with probability measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom. Phys. 23 (1975), pp. 75–76.

[8] —, *A characterization of Gaussian measures on Banach spaces*, Studia Math. 59 (1977), pp. 275–281.
[9] J. V. Wehausen, *Transformations in linear topological spaces*, Duke Math. J. 4 (1938), pp. 157–168.
[10] O. K. Zakusilo, *Some properties of random* $\sum_{i=0}^{\infty} A^i\xi_i$*-type vectors*, Probability Theory and Math. Statistics, 13 (1975), pp. 59–62 (in Russian).

Presented to the Semester
Probability Theory
February 11–June, 1976

PROBABILITY THEORY
BANACH CENTER PUBLICATIONS, VOLUME 5
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

CONSTRUCTION OF DIFFUSION PROCESSES WITH WENTZELL'S BOUNDARY CONDITIONS BY MEANS OF POISSON POINT PROCESSES OF BROWNIAN EXCURSIONS

SHINZO WATANABE

Department of Mathematics, Kyoto University, Kyoto, Japan

1. Introduction

It is well known that, under certain regularity conditions, a diffusion process on a manifold with boundary is determined by a second order differential operator of elliptic type (possibly degenerate) plus a Wentzell's boundary condition (cf. Wentzell [11]). The problem of constructing the diffusion from a given pair of such analytic data has been discussed so far by many authors. K. Sato and T. Ueno [7] laid a fundamental route of construction in an analytical way and following it, J. M. Bony, Ph. Courrege and P. Priouret [1] succeeded in constructing diffusions in very general cases. In a probabilistic way, N. Ikeda [4] applied Itô's stochastic differential equations to this problem and S. Watanabe [8] extended his idea so as to cover more general cases.

In this paper, we will propose still another approach to this problem by constructing directly the *excursions* of the diffusion. Our plan is as follows: we prepare two kinds of Poisson point processes on function spaces which we call *Poisson point processes of the Brownian excursions of the first and second kinds*. Each point of these point processes represents a Brownian excursion and by solving a stochastic differential equation based on this excursion (which is an absorbing barrier Brownian motion with an infinite entrance law) we can associate an excursion of the diffusion to be constructed to each of Brownian excursions. Here, we make use of the space-time relation of the Brownian excursion to produce the frequency of excursions in proportion to the coefficient of the corresponding terms in the given boundary condition.

The path functions of the diffusion to be constructed will be defined by gluing the excursion thus constructed and, in doing this, we need the so-called *process on the boundary*. This is constructed by solving a stochastic differential equation of jump type based on the Poisson point processes of Brownian excursions. Usually stochastic differential equations of jump type, as discussed in e.g. K. Itô [5] and