GEOMETRIC DECOMPOSABILITY PROPERTIES
OF PROBABILITY MEASURES

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Let \( X \) denote a real separable Banach space with the norm \( \| \cdot \| \) and with the dual space \( X^* \). By a probability measure \( \mu \) on \( X \) we shall understand a countably additive nonnegative set function \( \mu \) on the class of Borel subsets of \( X \) with the property that \( \mu(\emptyset) = 1 \).

The characteristic functional of \( \mu \) is defined on \( X^* \) by the formula

\[
\hat{\mu}(\xi) = \int_X e^{i\langle \xi, x \rangle} \mu(dx) \quad (\xi \in X^*).
\]

It is well known that the characteristic functional uniquely determines the probability measure. Moreover, it is continuous in the \( \mathcal{X} \)-topology of the space \( X^* \) (I), Theorem 6.2.1).

Given a linear operator \( A \) and a probability measure \( \mu \) on \( X \), we denote by \( A\mu \) the probability measure defined by the formula \( A\mu(E) = \mu(A^{-1}(E)) \) for all Borel subsets \( E \) of \( X \). In the study of limit probability distributions [6] I introduced the concept of decomposability semigroup \( D(\mu) \) of linear operators associated with the probability measure \( \mu \). Namely, \( D(\mu) \) consists of all linear operators \( A \) on \( X \) for which the equality

\[
\mu = A\mu^\ast
\]

holds for a certain probability measure \( \nu \). The asterisk denotes here the convolution of measures. Of course, \( D(\mu) \) is the semigroup under composition of operators and \( D(\mu) \) always contains the zero and the unit operators. Moreover, \( D(\mu) \) is closed in the weak* operator topology. It has been shown in [6], [7] and [8] that some purely probabilistic properties of \( \mu \) are equivalent with some algebraic and topological properties of the decomposability semigroup \( D(\mu) \).

In the case of real line \( X = \mathbb{R} \) we can identify operators \( T_x \) with \( x \in \mathbb{R} \).

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by M. Loève in [5] p. 334 is equivalent with the relation \( e \in D(\varphi) \). Recently O. K. Zhe-kuilo extended in [10] these investigations to the case of Euclidean spaces \( X = \mathbb{R}^n \) and obtained a new class of probability measures which in terms of decomposability semigroups can be described by the condition \( A \in D(\varphi) \), where \( A \) is an invertible linear operator with \( ||A|| < 1 \).

The main aim of this paper is to characterize isotropic Gaussian probability measures by a certain geometric property of their decomposability semigroups.

Let \( \mu \) be a probability measure and \( A \) a linear operator on \( X \). Then for every \( y \in X^* \),

\[ \hat{\mu}(y) = \hat{\mu}(A^* y) \]

Consequently, \( A \in D(\varphi) \) if and only if

\[ \hat{\mu}(y) = \hat{\mu}(A^* y) \hat{\nu}(y) \]

for a certain probability measure \( \nu \) on \( X \). By a projector \( P \) on \( X \) we mean a linear operator with the property that \( P^2 = P \). Further, by \( I \) we denote the unit operator on \( X \).

**Lemma 1.** Let \( P \) be a projector on \( X \) and \( P \in D(\varphi) \). Then \( I - P \in D(\varphi) \) and

\[ \mu = P \mu * (I - P) \mu \]

where \( \ast \) is a probability measure. Hence, we get the formula \( P \mu = P \mu * P \mu \). Consequently, \( \hat{P} \mu = \hat{P} \mu * \hat{P} \mu \), which implies the equation \( \hat{P} \mu = 1 \) in a neighborhood of 0 in \( X^* \). But the last condition implies the formula \( \hat{P} \mu = 1 \) for all \( y \in X^* \) (see [2], Proposition 2.3). Thus \( \mu = \delta_0 \), where \( \delta_0 \) is the probability measure concentrated at 0 in \( X \). Hence, in particular, it follows that \( \nu \) itself is concentrated on the subspace \( (I - P)X \). In other words, \( (I - P) \mu = 0 \). Since \( \delta_0 \) is the last equation and (1) yield the formula

\[ (I - P) \mu = (I - P) P \mu = (I - P) \mu \ast (I - P) \mu = \nu \]

Thus \( \mu = P \mu * (I - P) \mu \) and, consequently, \( I - P \in D(\varphi) \) which completes the proof.

In what follows by \( S_r \) or \( (r > 0) \) we shall denote the sphere of radius \( r \) in the space of linear operators on \( X \), i.e.

\[ S_r = \{ A : ||A|| = r \} \]

**Lemma 2.** If \( S_r \in D(\varphi) \) then for every pair \( y, z \in X^* \) with \( ||y|| = ||z|| \) the inequality

\[ ||\hat{\mu}(y)|| \leq ||\hat{\mu}(z)|| \]

holds.

**Proof.** Let \( y, z \in X^* \) and \( ||y|| = ||z|| = a \). Without loss of generality we may assume that \( a > 0 \) because in the case \( a = 0 \) the inequality is obvious. Taking \( x \in X \) with the properties \( ||x|| = 1 \) and \( y(x) = a \), we define the linear operator \( A \) by assuming

\[ A(x) = \frac{r}{a} y(x) x \quad (x \in X) \]

It is clear that \( ||A|| = r \) and, consequently, \( A \in D(\varphi) \). Moreover, \( A^* y = y r \). Thus there exists a probability measure \( \nu \) such that

\[ \hat{\mu}(y) = \hat{\mu}(r \nu) \hat{\nu}(y) \]

Since \( \hat{\nu}(y) \leq 1 \), we have the assertion of the lemma.

**Theorem 1.** Suppose that \( X \) is infinite-dimensional. Then \( S_r \in D(\varphi) \) if and only if \( \mu \) is concentrated at a single point.

**Proof.** Let \( y \in X^* \). By Wehausen Theorem (9) 0 belongs to the closure of the set \( \{ z \in X^* : ||z|| = \frac{||y||}{r} \} \) in the weak topology of the space \( X^* \). Since \( X^* \) is separable in the X-topology, we can select a sequence \( x_1, x_2, \ldots \) tending to 0 in the X-topology. Consequently, \( \hat{\mu}(x_n) \to 1 \) which, by Lemma 2, yields \( ||\hat{\nu}(y)|| = 1 \) for all \( y \in X^* \). This shows that \( \mu \) is concentrated at a single point.

We call a probability measure isotropic if it is invariant under all linear isometries of \( X \).

**Theorem 2.** Suppose that \( X \) is finite-dimensional, \( \dim X \geq 2 \) and \( \mu \) is not concentrated at a single point. Then \( S_r \in D(\varphi) \) if and only if \( \mu \) is a translate of an isotropic Gaussian measure and \( X \) is a Hilbert space under the norm \( ||\cdot|| \).

**Proof.** First we shall prove the sufficiency of the condition \( S_r \in D(\varphi) \). Let \( Y \) be a subspace of \( X \) with \( \codim Y = 2 \). By \( ||\cdot|| \) we shall denote the norm induced by \( ||\cdot|| \) on \( X / Y \) and by \( p \) the standard mapping from \( X \) onto \( X / Y \). Evidently, \( ||p(x)|| = ||x|| \) for all \( x \in X \). Let \( \mu \) be the probability measure induced by \( \mu \) on \( X / Y \). Then for every \( h \in (X / Y)^* \) the composition \( h \circ p \) belongs to \( X^* \) and

\[ \hat{\mu}(h) = \hat{\mu}(h \circ p) \]

Given \( u \in X / Y \) with \( ||u||_r = 1 \), we denote by \( [u] \) the one-dimensional subspace generated by \( u \).

By Hahn–Banach Theorem there exists a projector \( Q \), from \( X / Y \) onto \( [u] \) with \( ||Q|| = 1 \). Thus for every \( x \in X \) we have the formula \( Q x = c_x x u \), where \( c_x(u) \) is a real number. Selecting an element \( x_u \) in \( X \) with the properties \( ||x_u|| = 1 \) and \( p(x_u) = u \) we define the projector \( P_u \) from \( X \) onto \( [x_u] \) by means of the formula \( P_u x = c_x(x_u) x_u \) \( (x \in X) \). Obviously, \( ||P_u|| = ||Q|| = 1 \) and, consequently, \( P_u \in D(\varphi) \). Thus there exists a probability measure \( \nu \) on \( X \) such that

\[ \hat{\mu}(y) = \hat{\mu}(P_u \nu) \hat{\nu}(y) \quad (y \in X^*) \]

Since for every \( h \in (X / Y)^* \), \( P_u \nu(h) = \nu(Q^* h \nu y) \), we have, by virtue of (2),

\[ \hat{\mu}(h) = \hat{\nu}(Q^* h \nu y) \]

where \( \nu \) is induced by \( \nu \) on \( X / Y \). In other words, \( \nu \in D(\mu) \). By Lemma 1 we have also \( I - Q \in D(\mu) \) and

\[ \mu_r = Q \mu r + (I - Q) \mu_r \]

Since \( \codim Y = 2 \), we can select a pair \( u_1, u_2 \in X / Y \) with the properties \( ||u_1||_r = ||u_2||_r = 1 \) and \( u_1 \perp u_2 \). Therefore, \( \mu_r = Q \mu r + (I - Q) \mu_r \)].
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\[ ||u||_Y = 1 \quad \text{and} \quad u \in (I - Q_+^{*}) X/Y. \] Evidently,
\[ X/Y = [u]_Y \oplus [u]_2. \]

Let \( \zeta \) be an \( X/Y \)-valued random variable with the probability distribution \( \mu_Y \).

Then
\[ \zeta = \zeta_1 \oplus \zeta_2, \]
where \( \zeta_1 \) and \( \zeta_2 \) are real-valued random variables. For every pair \( t_1, t_2 \) of real numbers we define the functional \( g \) on \( X/Y \) by assuming \( g(c_1 u_1 + c_2 u_2) = t_1 c_1 + t_2 c_2 \) for every \( c_1, c_2 \in \mathbb{R} \). Then, in view of (4),
\[ E_{(x \in Y)} g(\xi_1 \oplus \xi_2) = E_{(y \in Y)} h_{\xi_1} \oplus E_{(y \in Y)} h_{\xi_2}, \]
which implies the independence of the random variables \( \xi_1 \) and \( \xi_2 \). Setting \( g^{-1} \)
\[ = ||u_1 - u_2||_Y \quad \text{and} \]
\[ v_1 = a(u_1 - u_2), \]
and selecting an element \( v_2 \) from \( (I - Q_+^{*}) X/Y \) with \( ||v_2||_Y = 1 \), we have
\[ X/Y = [v_1]_Y \oplus [v_2]. \]

Consequently, the random variable \( \zeta \) can be written in the form
\[ \zeta = \eta_1 \oplus \eta_2, \]
where \( \eta_1 \) and \( \eta_2 \) are real-valued random variables. The same arguments as in the case of the pair \( u_1, u_2 \) yield the independence of the random variables \( \eta_1 \) and \( \eta_2 \). Further, by the linear independence of \( v_1 \) and \( v_2 \), the coefficients \( a_1, a_2 \) in the expansion \( v_1 = a_1 u_1 + a_2 u_2 \) satisfy the condition
\[ a_1 + a_2 \neq 0. \]

Moreover, from (4) and (5) we get the formulas
\[ \eta_1 = \frac{a_2 \xi_1 - a_1 \xi_2}{a_1 + a_2}, \quad \eta_2 = \frac{a_1 \xi_1 + a_2 \xi_2}{a_1 + a_2}. \]

Hence it follows that two linear forms \( \xi_1 \pm \xi_2 \) and \( a_2 \xi_1 - a_1 \xi_2 \) of independent random variables \( \xi_1 \) and \( \xi_2 \) are independently distributed. By Skitovich–Darmois Theorem (3, Theorem 5.1) each random variable \( \xi_1 \), which has non-zero coefficients in both forms, is Gaussian. By (6) we conclude that at least one random variable, say \( \xi_2 \) (\( k = 1 \) or 2) is Gaussian. Setting \( y_k(x) = c_k(\phi(x)) \) for \( x \in X \), we infer that
\[ y_k(x) \in X^*, ||y_k|| = 1 \quad \text{and} \quad \hat{\mu}_k(y_k) = e^{im - \frac{1}{2} ||y_k||^2}, \]
where \( m, \sigma \in \mathbb{R} \). Applying Lemma 2 for \( r = 1 \), we infer that \( ||y|| = ||y_k|| = ||y_k|| \) whenever \( ||y|| = ||y_k|| = ||y_k|| \). Consequently,
\[ ||y|| = e^{im - \frac{1}{2} ||y_k||^2} \]
for all \( y \in X^* \). Thus for every \( r \in \mathbb{R} \) and \( y \in X^* \)
\[ ||\hat{\mu}(y)||^2 = e^{-m ||y||^2}. \]

which implies, by the Cramér Decomposition Theorem (5), p. 271,
\[ \hat{\mu}(y) = e^{imy - \frac{1}{2} ||y||^2}. \]

In other words, the probability measure \( \mu \) is Gaussian. Regarding \( X \) in the equivalent Euclidean metric, we have the standard parameterization of the characteristic function \( \hat{\mu} \):
\[ \hat{\mu}(y) = e^{imy - \frac{1}{2} ||y||^2}, \]
where \(( \cdot, \cdot)\) is the Euclidean inner product, \( x_0 \in X \) and \( \beta \) is the covariance operator, i.e. a non-negative self-adjoint operator. We have assumed that \( \mu \) is not concentrated at a single point. Consequently, \( \sigma^2 > 0 \) in (7) and, by (8), \( ||y|| = \sigma^{-2} (By, y) \) which shows that the norm \( ||\cdot|| \) on \( X^* \) is induced by an inner product. Hence it follows that the norm \( ||\cdot|| \) on \( X \) has the same property. Thus \( X \) is Hilbert space under this norm. Further, setting \( \mu(\beta) = e^{-\frac{1}{2} \beta ||y||^2} \) we get an isotropic Gaussian measure on \( X \). By (7) \( \mu \) is a translate of \( \mu_0 \) which completes the proof of the sufficiency of the condition \( S_1 \subseteq D(\mu) \).

To prove the necessity of this condition let us assume that \( X \) is a Hilbert space and \( \mu \) is a translate of an isotropic Gaussian measure. Then the covariance operator for \( \mu \) is of the form \( \beta h \) where \( h \) is a positive number. For any operator \( A \in S_1 \), the operator \( B \in A - \beta h \) is a non-negative self-adjoint and, consequently, can be regarded as the covariance operator for a Gaussian measure, say \( \nu_0 \). It is easy to verify the equation
\[ \mu = A_0 \nu_0 \]
where \( \nu_0 \) is a translate of \( \nu_0 \). Thus \( S_1 \subseteq D(\mu) \) which completes the proof of the theorem.

It should be noted that the assumption \( \dim X \geq 2 \) in Theorem 2 is essential. In fact, the condition \( S_1 \subseteq D(\mu) \) in the one-dimensional case characterizes translates of symmetric probability measures. It is also evident that for any Banach space \( X \) and for \( r > 1 \) the condition \( S_1 \subseteq D(\mu) \) characterizes probability measures concentrated at a single point. It would be interesting to characterize all probability measures \( \mu \) on a finite-dimensional Banach space \( X \) for which the inclusion \( S_1 \subseteq D(\mu) \) holds for a certain number \( r \in (0, 1) \).

References

CONSTRUCTION OF DIFFUSION PROCESSES WITH WENTZELL’S BOUNDARY CONDITIONS
BY MEANS OF POISSON POINT PROCESSES
OF BROWNIAN EXCURSIONS

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1. Introduction

It is well known that, under certain regularity conditions, a diffusion process on a manifold with boundary is determined by a second order differential operator of elliptic type (possibly degenerate) plus a Wentzell’s boundary condition (cf. Wentzell [1]). The problem of constructing the diffusion from a given pair of such analytic data has been discussed so far by many authors. K. Sato and T. Usano [7] laid a fundamental route of construction in an analytical way and following it, J. M. Bony, Ph. Courrege and P. Priouret [1] succeeded in constructing diffusions in very general cases. In a probabilistic way, N. Ikeda [4] applied Itô’s stochastic differential equations to this problem and S. Watanabe [8] extended his idea so as to cover more general cases.

In this paper, we will propose still another approach to this problem by constructing directly the excursions of the diffusion. Our plan is as follows: we prepare two kinds of Poisson point processes on function spaces which we call Poisson point processes of the Brownian excursions of the first and second kinds. Each point of these point processes represents a Brownian excursion and by solving a stochastic differential equation based on this excursion (which is an absorbing barrier Brownian motion with an infinite entrance law) we can associate an excursion of the diffusion to be constructed to each of Brownian excursions. Here, we make use of the space-time relation of the Brownian excursion to produce the frequency of excursions in proportion to the coefficient of the corresponding terms in the given boundary condition.

The path functions of the diffusion to be constructed will be defined by gluing the excursion thus constructed and, in doing this, we need the so-called process on the boundary. This is constructed by solving a stochastic differential equation of jump type based on the Poisson point processes of Brownian excursions. Usually stochastic differential equations of jump type, as discussed in e.g. K. Itô [5] and

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