Чтобы доказать соотношение (24) заметим, что для любой функции $\varphi \in C_0^2(\Pi_T)$ с вероятностью единица

$$\int_{\Pi_T} L_z\varphi(z)\left\{\int_{\Pi_T} I(z\prec v)R(z,v)\frac{A(v)}{B^2(v)}d\xi_v\right\}dz = \int_{\Pi_T} f(v)\frac{A(v)}{B^2(v)}d\xi_v,$$

где $f(v) = \int I(z\prec v)R(z,v)L_z\varphi(z)dz$ и, как нетрудно проверить, $f(v) = \varphi(v)$. Отсюда следует (24) и, значит, $\bar{\theta}_z$ удовлетворяет уравнению (5) (в обобщенном смысле).

В заключение автор выражает признательность Р. Ш. Липцеру и А. Н. Ширяеву за внимание к работе.

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ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM

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This note studies the rate of convergence in the central limit theorem in terms of Trotter's operators. The obtained results are more general and, in some cases, give sharper estimates than those of [1].

1. Introduction and notations

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with variances

$$0 < \sigma_n^2 = \sigma^2 X_n < \infty, \quad \text{for each } n \in N.$$

Write

$$\alpha_{ij} = EX_i^j, \quad \beta_{ij} = E|X_i|^j, \quad 0 \leqslant j \leqslant r;\ i, j \in N,\ r \geqslant 2.$$

Further on, we assume that

$$\frac{\alpha_{ij}}{\sigma_i^j} = \gamma_j, \quad \text{for} \quad 0 \leqslant j \leqslant r;\ i \in N. \tag{1}$$

Let us observe that (1) takes place if, for instance, we have the pseudomoments

$$\nu_i(j) = \int_R x^j d[F_{X_i}(x) - \Phi(x/\sigma_i)] = 0, \quad \text{for} \quad 0 \leqslant j \leqslant r;\ i \in N,$$

where Φ denotes the standard normal distribution function.

Let us put

$$S_n = \sum_{i=1}^n X_i, \quad s_n^2 = \sum_{i=1}^n \sigma_i^2, \quad X_i^* = X_i/s_n,$$

$$S_n^* = \sum_{i=1}^n X_i^*, \quad a_{nt}^{-1}(\delta) = \min_{1 \leqslant i \leqslant n} \int_{|x| < \delta s_n} |x|^t dF_i(x),$$

where $\delta > 0$, $t > 0$;

$$b_{it} = E\frac{|X_i|^t}{s_n^{2t} + X_n^{2t}}, \quad b'_{it} = E\frac{\sigma_i^t |Y|^t}{s_n^{2t} + (\sigma_i Y)^{2t}},$$

where Y denotes a normally distributed random variable with mean value 0 and variance 1;

$$B_{nt} = \sum_{i=1}^{n} b_{it}, \qquad B'_{nt} = \sum_{i=1}^{n} b'_{it},$$

$$B''_{nt} = \sum_{i=1}^{n} \sigma_i^{s-t} b'_{it}, \qquad B^*_{nt} = \sum_{i=1}^{n} \max_{r+1 \leqslant j \leqslant s} \sigma_i^j E \frac{|Y|^t}{s_n^{2t} + (\sigma_i Y)^{2t}}.$$

We will use

$$F_n^{-1} = \max_{1 \leqslant i \leqslant n} \frac{\sigma_i}{s_n}, \qquad K_n^{(t)}(\delta) = \int_{|x| \geqslant \delta F_n} |x|^t d\Phi(x),$$

$$L_n^{(t)}(\delta) = \frac{1}{s_n^t} \sum_{i=1}^{n} \int_{|x| \geqslant \delta s_n} |x|^t dF_i(x), \qquad D_{nm}^{(t)} = \frac{\sum_{i=1}^{n} \sigma_i^m}{s_n^t}, \qquad m \in N.$$

If $\lim_{n\to\infty} F_n^{-1} = 0$ and $\lim_{n\to\infty} L_n^{(t)}(\delta) = 0$, $\delta > 0$, then we say that *Feller's condition* and a *generalized Lindeberg condition* are satisfied, respectively. Since $L_n^{(t)}(\delta) \leqslant L_n^{(t+1)}(\delta)/\delta$, the condition $L_n^{(t+1)}(\delta) \to 0$ implies $L_n^{(t)}(\delta) \to 0$. Moreover, we see that Lapunov's condition implies $\lim_{n\to\infty} L_n^{(3)}(\delta) = 0$.

Let us set

$$A_{nm}^{(t)}(\delta) = L_n^{(t)}(\delta) + D_{nm}^{(t)} K_n^{(t)}(\delta),$$

$$\tilde{A}_{nm}^{(t)}(\delta) = s_n L_n^{(t+1)}(\delta) + D_{nm}^{(t)} K_n^{(t+1)}(\delta),$$

and

$$\bar{A}_n^{(t)}(\delta) = \int_{|x| \geqslant \delta\sqrt{n}} |x|^t dF(x) + \int_{|x| \geqslant \delta\sqrt{n}} |x|^t d\Phi(x).$$

Let $C_B^{(r)}(R) = \{f \in C_B(R) \colon f^{(j)} \in C_B(R),\ 1 \leqslant j \leqslant r\}$, where $C_B(R)$ stands for the class of bounded, uniformly continuous functions defined on R. By $\omega(f; \delta) = \sup_{|h|<\delta} |f(x+h) - f(x)|$, for $f \in C_B(R)$, $\delta > 0$, we define the modulus of continuity. A function $f \in C_B(R)$ is said to satisfy a *Lipschitz condition of order* α, $0 < \alpha \leqslant 1$, in symbols $f \in \operatorname{Lip}\alpha$, if $\omega(f; \delta) = O(\delta^\alpha)$. One can prove that

(2) $$\omega(f; \lambda\delta) \leqslant (1+\lambda)\omega(f; \delta).$$

The operator $T_X \colon C_B(R) \to C_B(R)$ defined by

$$T_X f(y) = Ef(X+y) = \int_R f(x+y) dF_X(x)$$

will be called *Trotter's* operator. It is well known [3] ([2], p. 516) that

(3) $$\lim_{n\to\infty} P[S_n - ES_n < xs_n] = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$$

holds if there exists an $r \in N$ such that for each $f \in C_B^{(r)}(R)$

(4) $$\lim_{n\to\infty} \|T_{\frac{S_n - ES_n}{s_n}} f - T_Y f\| = 0.$$

In [1] has been established the rate of convergence in (4). The aim of this note is to extend the results of [1] to a larger class of independent random variables and to give more precise estimates in some particular cases.

2. Results

We shall first give the estimates in the case of non-identically distributed random variables. Without loss of generality we can assume that $EX_i = 0$, $i \in N$.

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that (1) holds with γ_j, $0 \leqslant j \leqslant r$, as the moments of a normally distributed random variable.*

Suppose that

(5) $$\beta_{si} = E|X_i|^s < \infty \quad \textit{for some } s > 0,\ i \in N.$$

A. *If (5) holds with $s \geqslant r$, then for any $f \in C_B^{(r)}(R)$*

(6) $$\|T_{S_n^*} f - T_Y f\| \leqslant \frac{2\omega(f^{(r)}; s_n^{-1})}{r!} s_n^{2s-r} \{a_{ns}^{1-r/s}(1) B_{ns} + a_{n(s+1)}^{1-(r+1)/(s+1)}(1) s_n^2 B_{n(s+1)} + B'_{ns} + s_n^2 B'_{n(s+1)}\} + \frac{2}{r!} \min[\omega(f^{(r)}; s_n^{-1}) \tilde{A}_{ns}^{(s)}(1);\ \|f^{(r)}\| A_{ns}^{(s)}(1)],$$

and moreover, for $s > r$,

(7) $$\|T_{S_n^*} f - T_Y f\| \leqslant \frac{2\omega(f^{(r)}; s_n^{-1})}{r!} s_n^{2s-r} (a_{ns}^{1-r/s}(1) B_{ns} + B'_{ns}) + \frac{2}{r!} \min[\omega(f^{(r)}; s_n^{-1}) \tilde{A}_{ns}^{(s)}(1);\ \|f^{(r)}\| A_{ns}^{(s)}(1)].$$

B. *If (5) holds with $s \geqslant r$, then for any $f^{(r)} \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$,*

(8) $$\|T_{S_n^*} f - T_Y f\| = O\big(s_n^{2s-r+\alpha} [a_{n(s+\alpha)}^{1-(r+\alpha)/(s+\alpha)}(1) B_{n(s+\alpha)} + B'_{n(s+\alpha)}] + A_{n(s+\alpha)}^{(s+\alpha)}(1)\big),$$

and moreover, for $s > r$,

(9) $$\|T_{S_n^*} f - T_Y f\| = O\big(s_n^{2s-r-\alpha} [a_{ns}^{1-(r+\alpha)/s}(1) B_{ns} + B'_{ns}] + A_{ns}^{(s)}(1)\big).$$

C. *If (5) holds with $s > r$, then for any $f \in C_B^{(s)}(R)$*

(10) $$\|T_{S_n^*} f - T_Y f\| = O\Big(s_n^{2s-r-\alpha} \Big[\sum_{j=r+1}^{s} a_{ns}^{1-j/s}(\delta) B_{ns} + B^*_{ns}\Big] + A_{ns}^{(s)}(\delta)\Big),$$

and moreover, if r is even,

(11) $$\|T_{S_n^*} f - T_Y f\| = O\Big(s_n^{2s-r-1} \Big[\sum_{j=r+1}^{s} a_{ns}^{1-j/s}(\delta) B_{ns} + s_n^{-1} B^*_{ns}\Big] + A_{ns}^{(s)}(\delta)\Big)$$

holds.

One can observe that, in the case $s = r$, the estimates (6) and (8) concern a larger class of independent random variables than those of [1] (Theorem 1), and they are, in general, more precise.

If $\{X_n, n \geqslant 1\}$ is a sequence of independent random variables such that $a_{ns}^{-1}(\delta) = a > 0$ for $n \geqslant n_0$, then we can obtain more convenient bounds of the type in question. For example, Theorem 1 implies

THEOREM 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables such that $EX_1 = 0$, $EX_1^2 = 1$ and that*

(12) $$\alpha_j = \gamma_j, \quad 0 \leqslant j \leqslant r, \; r \in N,$$

where γ_j, $0 \leqslant j \leqslant r$, are the moments of the standard normal distribution function and

(13) $$\beta_s = E|X_1|^s < \infty, \quad \text{for some } s > 0.$$

A′. *If* (13) *holds with $s \geqslant r$, then for any $f \in C_B^{(r)}(R)$*

(6′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(\omega(f^{(r)}; n^{-1/2}) n^{s-(r-2)/2}\Big[E\frac{|X|^s}{n^s + X^{2s}} + nE\frac{|X|^{s+1}}{n^{s+1} + X^{2(s+1)}} + E\frac{|Y|^s}{n^s + Y^{2s}} + nE\frac{|Y|^{s+1}}{n^{s+1} + Y^{2(s+1)}}\Big] + n^{-(s-2)/2}\min[\omega(f^{(r)}; n^{-1})\bar{A}_n^{(s+1)}(1); \bar{A}_n^{(s)}(1)]\Big)$$

and moreover, for $s > r$,

(7′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(\omega(f^{(r)}; n^{-1/2}) n^{s-(r-2)/2}\Big[E\frac{|X|^s}{n^s + X^{2s}} + E\frac{|Y|^s}{n^s + Y^{2s}}\Big] + n^{-(s-2)/2}\min[\omega(f^{(r)}; n^{-1/2})\bar{A}_n^{(s+1)}(1); \bar{A}_n^{(s)}(1)]\Big).$$

B′. *If* (13) *holds with $s \geqslant r$, then for $f^{(r)} \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$,*

(8′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(n^{s-(r-2-\alpha)/2}\Big[E\frac{|X|^{s+\alpha}}{n^{s+\alpha} + X^{2(s+\alpha)}} + E\frac{|Y|^{s+\alpha}}{n^{s+\alpha} + Y^{2(s+\alpha)}}\Big] + n^{-(s-2+\alpha)/2}\bar{A}_n^{(s+\alpha)}(1)\Big),$$

and moreover, for $s > r$,

(9′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(n^{s-(r-2+\alpha)/2}\Big[E\frac{|X|^s}{n^s + X^{2s}} + E\frac{|Y|^s}{n^s + Y^{2s}}\Big] + n^{-(s-2)/2}\bar{A}_n^{(s)}(1)\Big).$$

C′. *If* (13) *holds with $s > r$, then for $f \in C_B^{(s)}(R)$,*

(10′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(n^{s-(r-1)/2}\Big[E\frac{|X|^s}{n^s + X^{2s}} + E\frac{|Y|^s}{n^s + Y^{2s}}\Big] + n^{-(s-2)/2}\bar{A}_n^{(s)}(\delta)\Big),$$

and moreover, if r is even,

(11′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(n^{s-(r-1)/2}\Big[E\frac{|X|^s}{n^s + X^{2s}} + n^{-1/2}E\frac{|Y|^s}{n^s + Y^{2s}}\Big] + n^{-(s-2)/2}\bar{A}_n^{(s)}(\delta)\Big).$$

It can be observed that estimates (6′)–(9′) can be used in more general situations than those of [1] (Theorems 1 and 2).

COROLLARY. *Under the assumptions of Theorem 2 (the point C′):*

(14) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O(n^{-(r-1)/2})$$

and in the case where r is even

(14′) $$\|T_{S_n/\sqrt{n}}f - T_Y f\| = O\Big(n^{s-(r-1)/2}E\frac{|X|^s}{n^s + X^{2s}}\Big).$$

Proof of Theorem 1. By the definition of Trotter's operator and the assumptions $f \in C_B^{(r)}(R)$ and (5) we have

(15) $$T_{X_i^*}f(y) = Ef(X_i/s_n + y) = \int_R f(x/s_n + y)\,dF_i(x)$$
$$= \sum_{j=0}^{r} f^{(j)}(y)\alpha_{ij}/j!s_n^j + \frac{1}{r!s_n^r}\int_{|x|<s_n} x^r[f^{(r)}(\theta x/s_n + y) - f^{(r)}(y)]\,dF_i(x) + \frac{1}{r!s_n^r}\int_{|x|\geqslant s_n} x^r[f^{(r)}(\theta x/s_n + y) - f^{(r)}(y)]\,dF_i(x),$$

where θ is some number between 0 and 1.

Now, using (2), the inequality $E|X| \leqslant (E|X|^t)^{1/t}$, and by simple evaluations, we have

(16) $$\Big|\frac{1}{r!s_n^r}\int_{|x|<s_n} x^r[f^{(r)}(\theta x/s_n + y) - f^{(r)}(y)]\,dF_i(x)\Big|$$
$$\leqslant \frac{1}{r!s_n^r}\int_{|x|<s_n} |x|^r\omega(f^{(r)}; |\theta x/s_n|)\,dF_i(x)$$
$$\leqslant \frac{\omega(f^{(r)}; s_n^{-1})}{r!s_n^r}\int_{|x|<s_n}|x|^r dF_i(x) + \frac{\omega(f^{(r)}; s_n^{-1})}{r!s_n^r}\int_{|x|<s_n}|x|^{r+1}dF_i(x)$$
$$\leqslant \frac{\omega(f^{(r)}; s_n^{-1})}{r!s_n^r}a_{ns}^{1-r/s}(1)\int_{|x|<s_n}|x|^s dF_i(x) + \frac{\omega(f^{(r)}; s_n^{-1})}{r!s_n^r}a_{n(s+1)}^{1-(r+1)/(s+1)}(1)\int_{|x|<s_n}|x|^{s+1}dF_i(x)$$
$$\leqslant 2\frac{\omega(f^{(r)}; s_n^{-1})}{r!}a_{ns}^{1-r/s}(1)s_n^{2s-r}b_{is} + 2\frac{\omega(f^{(r)}; s_n^{-1})}{r!}a_{n(s+1)}^{1-(r+1)/(s+1)}(1)s_n^{2(s+1)-r}b_{i(s+1)}.$$

Analogously, we obtain

$$(17)\qquad \frac{1}{r!\,s_n^r}\Big|\int\limits_{|x|\geqslant s_n} x^r[f^{(r)}(\theta x/s_n+y)-f^{(r)}(y)]\,dF_i(x)\Big|$$
$$\leqslant \frac{\omega(f^{(r)};s_n^{-1})}{r!\,s_n^s}\int\limits_{|x|\geqslant s_n}|x|^{s+1}dF_i(x)+\frac{\omega(f^{(r)};s_n^{-1})}{r!\,s_n^s}\int\limits_{|x|\geqslant s_n}|x|^s dF_i(x)$$
$$\leqslant 2\,\frac{\omega(f^{(r)};s_n^{-1})}{r!\,s_n^s}\int\limits_{|x|\geqslant s_n}|x|^{s+1}dF_i(x),$$

and also

$$(17')\qquad \frac{1}{r!\,s_n^r}\Big|\int\limits_{|x|\geqslant s_n} x^r[f^{(r)}(\theta x/s_n+y)-f^{(r)}(y)]\,dF_i(x)\Big| \leqslant \frac{2\|f^{(r)}\|}{r!\,s_n^s}\int\limits_{|x|\geqslant s_n}|x|^s dF_i(x).$$

Hence, by (15)–(17), and (17′), we have

$$(18)\qquad \Big\|T_{X_i^*}f-\sum_{j=0}^{r}\frac{f^{(j)}}{j!\,s_n^j}\alpha_{ij}\Big\|$$
$$\leqslant \frac{2\omega(f^{(r)};s_n^{-1})}{r!}\,[a_n^{1-r/s}s_n^{2s-r}b_{is}+a_{n(s+1)}^{1-(r+1)/(s+1)}(1)\cdot s_n^{2(s+1)-r}b_{i(s+1)}]+$$
$$+\frac{2}{r!\,s_n^s}\min\Big[\omega(f^{(r)};s_n^{-1})\int\limits_{|x|\geqslant s_n}|x|^{s+1}dF_i(x);\ \|f^{(r)}\|\int\limits_{|x|\geqslant s_n}|x|^s dF_i(x)\Big].$$

Taking into account that for Y there are moments of arbitrary order, we can obtain by the same methods

$$(19)\qquad \Big\|T_{\sigma_i Y/s_n}f-\sum_{j=0}^{r}\frac{f^{(j)}}{j!\,s_n^j}\sigma_i^j\gamma_j\Big\|\leqslant \frac{2\omega(f^{(r)};s_n^{-1})}{r!}\,s_n^{2s-r}[b'_{is}+s_n^2 b'_{s(s+1)}]+$$
$$+\frac{2}{r!}\min\Big[\omega(f^{(r)};s_n^{-1})\frac{\sigma_i^s}{s_n^s}\int\limits_{|x|\geqslant F_n}|x|^{s+1}d\Phi(x);\ \|f^{(r)}\|\frac{\sigma_i^s}{s_n^s}\int\limits_{|x|\geqslant F_n}|x|^s d\Phi(x)\Big].$$

By (18) and (19), and using the triangle inequality, we get (6). Estimate (7) can be obtained in the same way.

The assertions of B follow from analogous considerations, by using the following estimates: for $s\geqslant r$

$$\frac{1}{r!\,s_n^s}\Big|\int\limits_{|x|<s_n} x^r[f^{(r)}(\theta x/s_n+y)-f^{(r)}(y)]\,dF_i(y)\Big|$$
$$\leqslant \frac{1}{r!\,s_n^{r+\alpha}}\int\limits_{|x|<s_n}|x|^{r+\alpha}dF_i(x)\leqslant \frac{1}{r!\,s_n^{r+\alpha}}\,a_{n(s+\alpha)}^{1-(r+\alpha)/(s+\alpha)}\int\limits_{|x|<s_n}|x|^{s+\alpha}dF_i(x)$$
$$\leqslant \frac{2}{r!}\,s_n^{2s-r+\alpha}a_{n(s+\alpha)}^{1-(r+\alpha)/(s+\alpha)}(1)\,b_{i(s+\alpha)},$$

and for $s>r$

$$\frac{1}{r!\,s_n^r}\Big|\int\limits_{|x|<s_n} x^r[f^{(r)}(\theta x/s_n+y)-f^{(r)}(y)]\,dF_i(y)\Big|\leqslant \frac{2}{r!}\,s_n^{2s-r-\alpha}a_{ns}^{1-(r+\alpha)/s}(1)b_{is}$$

and similar estimates if we consider the random variable $\sigma_i Y$.

Now we are going to prove C. Assuming that $EX_i=0$, $i\geqslant 1$, we have

$$(20)\qquad T_{X_i^*}f(y)=\sum_{j=0}^{r}\frac{f^{(j)}(y)}{j!\,s_n^j}\alpha_{ij}+\sum_{j=r+1}^{s}\frac{f^{(j)}(y)}{j!\,s_n^j}\int\limits_{|x|<\delta s_n}x^j dF_i(x)+$$
$$+\frac{1}{s!\,s_n^s}\int\limits_{|x|<\delta s_n}x^s[f^{(s)}(\theta x/s_n+y)-f^{(s)}(y)]\,dF_i(x)+$$
$$+\sum_{j=r+1}^{s}\frac{f^{(j)}(y)}{j!\,s_n^j}\int\limits_{|x|\geqslant\delta s_n}x^j dF_i(x)+$$
$$+\frac{1}{s!\,s_n^s}\int\limits_{|x|\geqslant\delta s_n}x^s[f^{(s)}(\theta x/s_n+y)-f^{(s)}(y)]\,dF_i(x),$$

where $\delta>0$ is any given number (which will be fixed later). But for $s_n\to\infty$

$$(21)\qquad \Big|\sum_{j=r+1}^{s}\frac{f^{(j)}(y)}{j!\,s_n^j}\int\limits_{|x|<\delta s_n}x^j dF_i(x)\Big|\leqslant C\frac{1}{s_n^s}\sum_{j=r+1}^{s}(s_n a_{ns}(\delta))^{1-j/s}\int\limits_{|x|<\delta s_n}|x|^s dF_i(x)$$
$$\leqslant 2Cs_n^{2s-r-1}\sum_{j=r+1}^{s}a_{ns}^{1-j/s}(\delta)\,E\frac{|X_i|^s}{s_n^{2s}+X_i^{2s}}.$$

Since $f\in C_B^{(s)}(R)$, for each $\varepsilon>0$ there exists a $\delta>0$ (previously introduced) such that $|\theta x/s_n|<\delta$ implies $|f^{(s)}(\theta x/s_n+y)-f^{(s)}(y)|<\varepsilon$. Therefore,

$$(22)\qquad \frac{1}{s!\,s_n^s}\Big|\int\limits_{|x|<\delta s_n}x^s[f^{(s)}(\theta x/s_n+y)-f^{(s)}(y)]\,dF_i(x)\Big|\leqslant 2\varepsilon s_n^s E\frac{|X_i|^s}{s_n^{2s}+X_i^{2s}}.$$

Now we observe that

$$(23)\qquad \Big|\sum_{j=r+1}^{s}\frac{f^{(j)}(y)}{j!\,s_n^j}\int\limits_{|x|\geqslant\delta s_n}x^j dF_i(x)\Big|\leqslant C\frac{1}{s_n^s}\int\limits_{|x|\geqslant\delta s_n}|x|^s dF_i(x),$$

and

$$(24)\qquad \frac{1}{s!\,s_n^s}\Big|\int\limits_{|x|\geqslant\delta s_n}x^s[f^{(s)}(\theta x/s_n+y)-f^{(s)}(y)]\,dF_i(x)\Big|\leqslant \frac{2\|f^{(s)}\|}{s!\,s_n^s}\int\limits_{|x|\geqslant\delta s_n}|x|^s dF_i(x).$$

Hence, by (20)–(24), we get for sufficiently large n

$$(25)\quad \left\| T_{X_i^*} f - \sum_{j=0}^{r} \frac{f^{(j)}}{j!}\alpha_{ij} \right\|$$

$$\leqslant 2C \sum_{j=r+1}^{s} a_{ns}^{1-j/s}(\delta) s_n^{2s-r-1} E\frac{|X_i|^s}{s_n^{2s}+X_i^{2s}} + \frac{2\|f^{(s)}\|}{s!\, s_n^s} \int\limits_{|x|\geqslant \delta s_n} |x|^s dF_i(x).$$

In a similar way one can obtain for $n \geqslant n_0$

$$(26)\quad \left\| T_{\sigma_i Y/s_n} - \sum_{j=0}^{r} \frac{f^{(j)}}{j!}\sigma_i^j \gamma_j \right\|$$

$$\leqslant 2Cs_n^{2s} \sum_{j=r+1}^{s} s_n^{-j} \max_{r+1\leqslant j\leqslant s} \sigma_i^j E\frac{|Y|^s}{s_n^{2s}+(\sigma_i Y)^{2s}} + 2C\frac{\sigma_i^s}{s_n^s} \int\limits_{|x|\geqslant \delta F_n} |x|^s d\Phi(x).$$

Combining (25) and (26), by the triangle inequality, we get (10).

To prove (11) it is enough to remark that $\int\limits_{-\infty}^{\infty} x^r\, d\Phi(x) = 0$ if r is odd.

The statements of Theorem 2 follow immediately from Theorem 1.

Estimates of the type "o-small" give the following theorems:

THEOREM 3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that* (1) *holds with γ_j, $0 \leqslant j \leqslant r$, as the moments of the standard normally distributed random variable.*

If (5) *holds with $s \geqslant r$, then for any $f \in C_B^{(r)}(\boldsymbol{R})$ and an arbitrary $\varepsilon > 0$*

$$(27)\qquad \|T_{S_n^*} f - T_Y f\| \leqslant 2\varepsilon s_n^{2s-r}(a_{ns}^{1-r/s}(\delta) B_{ns} + B_{ns}'') + \frac{2\|f^{(r)}\|}{r!} A_{ns}^{(s)}(\delta),$$

where δ is the number introduced above.

THEOREM 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that* (12) *holds with γ_j, $0 \leqslant j \leqslant r$, as the moments of the standard normally distributed random variable.*

If (13) *holds with $s \geqslant r$, then for any $f \in C_B^{(r)}(\boldsymbol{R})$ and an arbitrary $\varepsilon > 0$*

$$(28)\quad \|T_{S_n/\sqrt{n}} f - T_Y f\|$$

$$\leqslant 2n^{s-(r-2)/2}\left[E\frac{|X|^s}{n^s+X^{2s}} + E\frac{|Y|^s}{n^s+Y^{2s}}\right] + 2Cn^{-(s-2)/2}\overline{A}_n^{(s)}(\delta).$$

To prove (27) it is enough to use for any $\varepsilon > 0$ the decomposition

$$\int\limits_R x^r[f^{(r)}(\theta x/s_n + y) - f^{(r)}(y)]\, dF_i(x)$$

$$= \int\limits_{|x|<\varepsilon s_n} x^r[f^{(r)}(\theta x/s_n+y) - f^{(r)}(y)]\, dF_i(x) + \int\limits_{|x|\geqslant \varepsilon s_n} x^r[f^{(r)}(\theta x/s_n+y) - f^{(r)}(y)]\, dF_i(x),$$

and then to follow the considerations given in the proof of Theorem 1.

Estimate (28) follows from (27).

COROLLARY. *Under the assumptions of Theorem* 4 *for $s > r$*

$$\|T_{S_n/\sqrt{n}} f - T_Y f\| = o\left(n^{s-(r-2)/2}\left[E\frac{|X|^s}{n^s+X^{2s}} + E\frac{|Y|^s}{n^s+Y^{2s}}\right]\right).$$

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