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A REPRESENTATION OF THE CHARACTERISTIC FUNCTIONS OF STABLE DISTRIBUTIONS IN HILBERT SPACES

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1

Let H be a separable, real Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. A countable additive and normed measure μ defined on the σ-field $\mathscr{B}$ of Borel subsets of H is called a *probability distribution* in H. By $\mu * \nu$ we denote the convolution of the distributions μ and ν (see [5], p. 57), and by δ_x, where $x \in H$, we denote the one-point distribution concentrated at the point x. The characteristic function $\hat{\mu}$ of a distribution μ is defined by the formula

$$\hat{\mu}(y) = \int_H e^{i(y,x)}\mu(dx),$$

where $y \in H$. A distribution μ is uniquely defined by the characteristic function $\hat{\mu}$ (see [5], p. 152).

For every non-negative number a we define the mapping T_a from H into itself by means of the formula $T_a x = ax$. Further, if μ is a distribution, and a is a positive number, then $T_a\mu$ denotes the distribution defined by the formula

$$(T_a\mu)(E) = \mu(a^{-1}E)$$

for all $E \in \mathscr{B}$. For $a = 0$ we put $T_0\mu = \delta_0$. A distribution μ is said to be *stable* if for every pair of positive numbers a and b there exist a positive number c and an element x of the space H such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x.$$

In [2] the following theorem has been proved:

THEOREM 1. *A function φ defined on H is the characteristic function of a stable distribution in H if and only if either*

$$\varphi(y) = \exp[i(y, x_0) - \tfrac{1}{2}(Dy, y)],$$

where $x_0 \in H$ and D is an S-operator, or

$$\varphi(y) = \exp\left\{i(y, x_0) + \int_{H\setminus\{0\}} \left[e^{i(y,x)} - 1 - \frac{i(y, x)}{1+\|x\|^2}\right] M(dx)\right\},$$

where $x_0 \in H$ and M is a σ-finite measure on H, finite on the complement of every neighbourhood of zero in H and such that

$$\int_{\|x\| \leqslant 1} \|x\|^2 M(dx) < \infty,$$

and there exists a $0 < p < 2$ such that $T_a M = a^p M$ for every positive a.

(For the proof of this theorem see also [3].)

The parameter p we call the *exponent* of the distribution μ.

The aim of this paper is to give the canonical representation of a stable distribution in a Hilbert space, which is a generalization of the classical formula of Lévy-Khintchine (see [1], p. 164). The method of proof, stimulated by the results of Professor K. Urbanik [7], consists in finding the extreme points of a certain convex set formed by Lévy–Khintchine measures corresponding to stable distributions. Once the extreme points are found, one can apply a theorem by Choquet on the representation of the points of a compact convex set as barycenters of the extreme points.

2

Let $\boldsymbol{R}^+$ denote the set of positive real numbers, and $\overline{\boldsymbol{R}^+}$ its compactification: $\overline{\boldsymbol{R}^+} = \boldsymbol{R}^+ \cup \{0\} \cup \{\infty\}$. Let B and S denote the closed unit ball and the unit sphere of H, respectively. Put $Q = B \times \overline{\boldsymbol{R}^+}$, and by $[x, t]$ denote an element of Q. If B is endowed with the relative weak-* topology of H, then B becomes a compact metric space. Thus Q is a compact metric space with the product topology. We define a one-parameter group L_s ($s \in \boldsymbol{R}^+$) of transformations of Q by assuming

$$L_s[x, t] = [x, st]. \tag{1}$$

For every element $[x, t]$ belonging to Q we put

$$|[x, t]| = t. \tag{2}$$

Each element z of $H \setminus \{0\}$ can be uniquely represented in the form $z = T_{\|z\|}\left(\frac{z}{\|z\|}\right)$. Thus, the mapping π from $H \setminus \{0\}$ into Q defined by the formula

$$\pi(z) = \left[\frac{z}{\|z\|}, \|z\|\right] \tag{3}$$

is an embedding of $H \setminus \{0\}$ onto $S \times \boldsymbol{R}^+$. Obviously, we have

$$\|y\| = |\pi(y)|, \tag{4}$$

$$\pi(T_s y) = L_s \pi(y) \tag{5}$$

for all $y \in H \setminus \{0\}$, and $s \in \boldsymbol{R}^+$.

We say that a subset E of Q is *bounded from below* if $\inf\{|a|: a \in E\}$ is positive. Let λ be a finite Borel measure on Q. For any Borel subset E of Q bounded from below we put

$$I_\lambda(E) = \int_E (1 + |u|^{-2}) \lambda(du), \tag{6}$$

where the integrand is assumed to be 1 if $|u| = \infty$. Let M_p, where $0 < p < 2$, be the set of all finite Borel measures λ on Q satisfying the condition

$$a^p I_\lambda(E) = L_a I_\lambda(E) \tag{7}$$

for all positive a and all Borel subsets E bounded from below. By K_p we denote the subset of M_p consisting of probability measures. Recall that the space $\mathscr{M}(Q)$ of all probability measures on Q is compact and metrizable in the topology of weak convergence (see [5], pp. 45–46).

LEMMA 1. *The set K_p is convex and compact.*

Proof. It is clear that K_p is convex. Because K_p is the subset of $\mathscr{M}(Q)$ it is only necessary to show that K is closed. Let $\lambda_n \in K_p$ and let λ be the weak limit of the sequence $\{\lambda_n\}$. If $E \subset Q$ is bounded from below and $\lambda(\partial E) = 0$, where ∂E denotes the boundary of E, then $I_{\lambda_n}(E) \to I(E)$, because the function $1 + |u|^{-2}$ is continuous and bounded on E. Hence, in view of the continuity of the mappings L_a, we have $L_a I_{\lambda_n}(E) \to L_a I_\lambda(E)$. Thus the lemma is proved.

Suppose that a Borel subset F of Q is L_a-invariant (i.e. $L_a F \subset F$) for all positive a, and $\lambda \in M_p$. Then the restriction $\lambda|F$ belongs to M_p because of the equations

$$I_{\lambda|F}(E) = I_\lambda(E \cap F) \quad \text{and} \quad L_a I_{\lambda|F}(E) = L_a I_\lambda(E \cap F).$$

LEMMA 2. *The extreme points of K_p are measures concentrated on one of the ollowing sets: $\{[x, 0]\}$, $\{[x, \infty]\}$, and $\{[x, t]: t \in \boldsymbol{R}^+\}$, where $x \in B$.*

Proof. It is easy to see that the sets described in Lemma 2 are L_a-invariant for all positive a, which completes the proof.

LEMMA 3. *If λ is a measure from K_p concentrated on $F_x = \{[x, t]: t \in \boldsymbol{R}^+\}$, then for every continuous and bounded function f on F_x the following equation holds:*

$$\int_{F_x} f(z) \lambda(dz) = c_p \int_0^\infty f([x, t]) \frac{t^{1-p}}{1+t^2} dt,$$

where $c_p = \frac{2}{\pi} \sin\left(\frac{p\pi}{2}\right)$.

Proof. Let λ be a probability measure concentrated on F_x. Put

$$J_\lambda(u) = I_\lambda(\{[x, t]: t \geqslant u\}), \quad u \in \boldsymbol{R}^+. \tag{8}$$

Hence, and from (7) we have $\lambda \in K_p$ if and only if for every positive numbers a and u

$$a^p J_\lambda(u) = J_\lambda(u/a).$$

Setting $a = u$, we have

$$J_\lambda(u) = u^{-p} J_\lambda(1) \tag{9}$$

for all positive u. The measure λ is concentrated on F_x. Thus the constant $J_\lambda(1)$ is positive. Taking into account (6), (8) and (9), we get the formula

$$\int_{F_x} f(z)\lambda(dz) = pJ_\lambda(1)\int_0^\infty f([x,t])\frac{t^{1-p}}{1+t^2}dt$$

for every continuous, bounded function on F_x. Setting $f\equiv 1$, we obtain $pJ_\lambda(1) = \frac{2}{\pi}\sin\frac{\pi p}{2}$. Thus the lemma is proved.

LEMMA 4. *For each $x\in B$, there exists a measure from K_p concentrated on F_x.*

Proof. We define a measure λ on sets E of the form $\{[x,t]: c\leqslant t\leqslant d\}$, where $0<c<d<\infty$, by the formula

$$\lambda(E) = c_p\int_c^d \frac{t^{1-p}}{1+t^2}dt.$$

It is easy to see that λ is uniquely defined on all Borel subsets of F_x. Moreover, λ is a probability measure. For all sets E and distribution λ condition (7) is fulfilled. Thus λ belongs to K_p, which completes the proof.

By Lemmas 3 and 4 we have the following

COROLLARY. *For each $x\in B$, there exists exactly one measure from K_p concentrated on F_x.*

Let $m_{[x,1]}$, where $x\in B$, denote the unique probability measure concentrated on F_x, and let $m_{[x,a]} = \delta_{[x,a]}$ if either $a=0$ or $a=\infty$.

By (1), (3) and Lemma 3, for any function f continuous on Q we have the formula

$$\int_Q f(z)m_{[x,1]}(dz) = c_p\int_0^\infty f(L_s[x,1])\frac{s^{1-p}}{1+s^2}ds. \tag{10}$$

Further, if y belongs to the unit sphere S in H, then we have the equality

$$\int_Q f(z)m_{\pi(y)}(dz) = c_p\int_0^\infty f(L_s\pi(y))\frac{s^{1-p}}{1+s^2}ds. \tag{11}$$

Let us consider the following subsets of Q:

$$\begin{aligned} A_1 &= \{[x,1]: x\in B \text{ and } \|x\|<1\},\\ A_2 &= \{[x,a]: x\in B \text{ and either } a=0 \text{ or } a=\infty\},\\ A_3 &= S\times\{1\}\end{aligned} \tag{12}$$

and

$$A = A_1\cup A_2\cup A_3. \tag{13}$$

The set A is compact, as a closed subset of Q. We note that the mapping $z\to m_z$

from A onto the set $\mathrm{ex}(K_p)$ of extreme points of K_p is one-to-one and continuous. Hence by a well-known theorem (see [4], p. 11) we conclude that this mapping is a homeomorphism between A and $\mathrm{ex}(K_p)$. Once the extreme points of K_p are found, we can apply a theorem by Choquet (see [6], Chapter 3). Since each element of M_p is of the form $c\nu$, where $c\geqslant 0$ and $\nu\in K_p$, we get the following

LEMMA 5. *A measure μ belongs to M_p if and only if there exists a finite Borel measure ω on A such that for each continuous function f on Q the following equation holds:*

$$\int_Q f(u)\mu(du) = \int_A\int_Q f(u)m_z(du)\omega(dz),$$

where set A is defined by (13).

3

In the sequel we shall use the following representation. The characteristic function $\hat{\mu}$ is infinitely divisible if and only if it is of the form

$$\hat{\mu}(y) = \exp\left\{i(a,y)-\tfrac{1}{2}(Dy,y)+\int_{H\setminus\{0\}}\left[e^{i(y,x)}-1-\frac{i(y,x)}{1+\|x\|^2}\right]\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx)\right\}, \tag{14}$$

where a is an element from H, D is an S-operator, and γ is a finite Borel measure on $H\setminus\{0\}$. The triplet a, D and γ is uniquely determined by μ. The measure γ will be called the *Lévy–Khintchine* measure.

LEMMA 6. *The characteristic function of the form* (14), *where $D\equiv 0$, is stable if and only if there exists a $0<p<2$ such that the following equation holds*

$$a^p\gamma(E) = \int_E \frac{\|x\|^2}{1+\|x\|^2}\cdot\frac{1+\|T_a^{-1}x\|^2}{\|T_a^{-1}x\|^2}(T_a\gamma)(dx) \tag{15}$$

for all positive a, and all Borel subsets E of $H\setminus\{0\}$.

Proof. The measures M in Theorem 1, and γ in (14) are uniquely determined, whence follows the equality

$$\gamma(E) = \int_E\frac{\|x\|^2}{1+\|x\|^2}M(dx).$$

In view of Theorem 1, by a simple computation we obtain formula (15). Thus the lemma is proved.

LEMMA 7. *A finite Borel measure γ on $H\setminus\{0\}$ is a Lévy–Khintchine measure of a stable distribution if and only if $\pi\gamma$ belongs to M_p for a certain $0<p<2$.*

Proof. Let E be a Borel subset of Q bounded from below. Taking into account (4), (5), (6) and Lemma, 6, we have

$$\int_{\pi^{-1}E}\frac{1+\|u\|^2}{\|u\|^2}\gamma(du) = \int_E\frac{1+|u|^2}{|u|^2}\pi\gamma(du) = I_{\pi\gamma}(E)$$

and

$$\int_{\pi^{-1}E} \frac{1+\|u\|^2}{\|u\|^2}\,\frac{\|u\|^2}{1+\|u\|^2}\,\frac{1+\|T_a^{-1}u\|^2}{\|T_a^{-1}u\|^2}\,T_a\gamma(du) = \int_{L_a^{-1}E} \frac{1+|u|^2}{|u|^2}\,\pi\gamma(du) = L_a I_{\pi\gamma}(E).$$

Consequently, by (15), γ is the Lévy–Khintchine measure of the stable distribution if and only if $\pi\gamma \in M_p$.

THEOREM 2. *A function φ on H is a characteristic function of a stable distribution with exponent $0 < p < 2$ if and only if*

$$(16)\qquad \varphi(y) = \exp\left\{i(y, x_0)+\int_S\left[\int_0^\infty\left(e^{i(y,x)s}-1-\frac{i(y,x)s}{1+s^2}\right)\frac{ds}{s^{p+1}}\right]\nu(dx)\right\},$$

where x_0 is a vector from H and ν is a finite Borel measure on the unit sphere S of H.

Proof. Necessity. Suppose that the distribution μ is stable with exponent p, where $0 < p < 2$. Then its Lévy–Khintchine measure γ is finite on $H\setminus\{0\}$, and by Lemma 7 the measure $\pi\gamma$ on Q belongs to M_p (with the same p). By Lemma 5, there exists a finite Borel measure ω on the set A (see (13)) such that for every continuous function f on Q we have the equation

$$(17)\qquad \int_Q f(u)\pi\gamma(du) = \int_A\int_Q f(u)m_z(du)\omega(dz).$$

It is clear that the measure $\pi\gamma$ is concentrated on the set $\pi(H\setminus\{0\}) = S\times \boldsymbol{R}^+$. Consequently, by (17) the measure ω is concentrated on the set A_3 (see (12)). Since for $z \in A_3$ the measures m_z are concentrated on $F_{\pi^{-1}z}$, formula (17) can be rewritten in the form

$$(18)\qquad \int_{S\times R^+} f(u)\pi\gamma(du) = \int_{A_3}\int_{F_{\pi^{-1}z}} f(u)m_z(du)\omega(dz)$$

for any continuous and bounded function on $S\times\boldsymbol{R}^+$. We introduce a finite, non-negative measure $\nu = c_p\pi^{-1}\omega$ on the unit sphere S in H. Then for every continuous and bounded function g on $H\setminus\{0\}$, in view of (3), (5), (11), (18) and Lemma 3, we get the formula

$$(19)\qquad \int_{H\setminus\{0\}} g(x)\gamma(dx) = \int_S\left[\int_0^\infty g(T_s x)\frac{s^{1-p}}{1+s^2}\,ds\right]\nu(dx).$$

Setting

$$g(x) = \left[e^{i(y,x)}-1-\frac{i(y,x)}{1+\|x\|^2}\right]\frac{1+\|x\|^2}{\|x\|^2},$$

where $y \in H$, into the last formula, and then setting $T_s x = sx$, we get representation (16). Thus the necessity of the condition is proved.

Sufficiency. Suppose that the function φ is given by formula (16). First we note that φ is a limit of products of Poissonian characteristic functions of the form

$$\exp c\left[e^{i(y,b)}-1-\frac{i(y,b)}{1+\|b\|^2}\right]e^{i(y,x_0)},$$

where $c \geqslant 0$ and $b \in H\setminus\{0\}$. Thus φ is the characteristic function of an infinitely divisible distribution (see [5], Theorems 4.1 and 4.10, p.110). We note that the integrals

$$\int_0^\infty \frac{u^{2-p}}{(1+u^2)(1+a^2u^2)}\,du$$

are finite for $0<p<2$, $a>0$. By a simple computation, we show that, for every pair of a positive numbers a and b, there exist a positive number c ($c = (a^p+b^p)^{1/p}$) and an element x of H such that $\varphi(ay)\cdot\varphi(by) = \varphi(cy)\cdot e^{i(y,x)}$ ($y \in H$). Thus φ is the characteristic function of a stable distribution, which completes the proof of Theorem 2.

THEOREM 3. (Canonical representation of stable distribution.) *A function φ on H is the characteristic function of a stable distribution in H if and only if*

$$(20)\qquad \varphi(y) = \exp\left\{i(y, a)-\int_S\left[1+i\frac{(y,x)}{|(y,x)|}\,\omega(y,x,p)\right]|(y,x)|^p\varrho(dx)\right\},$$

where a is a vector from H, ϱ is a finite Borel measure on the unit sphere S in H, parameter p belongs to the interval $(0, 2]$ and

$$(21)\qquad \omega(y,x,p) = \begin{cases} \operatorname{tg}\left(\dfrac{\pi p}{2}\right) & \text{if}\quad 0<p\leqslant 2 \text{ and } p\neq 1,\\[2mm] \dfrac{2}{\pi}\log|(y,x)| & \text{if}\quad p = 1.\end{cases}$$

Proof. Let $0 < p < 2$. By the same computation as in the real case (see [1], pp. 168–171) we get the formula

$$\int_0^\infty\left[e^{i(y,x)s}-1-\frac{i(y,x)s}{1+s^2}\right]\frac{ds}{s^{p+1}} = -c\left[1+i\frac{(y,x)}{|(y,x)|}\omega(y,x,p)\right]|(y,x)|^p+i(y,x_1),$$

where c is a nonnegative constant, x_1 is a vector from H, and $\omega(y,x,p)$ is described by (21). Setting it into (16), we get formula (20).

Let $p = 2$. The function $\psi(y) = \exp[-\frac{1}{2}(Dy,y)]$, where D is an S-operator, is the characteristic function of a distribution with the property: $\int_H \|x\|^2\mu(dx) < \infty$, and $(Dy,y) = \int_H (y,x)^2\mu(dx)$ (see [5], p. 164). Let us introduce a finite Borel measure ν on H by the formula $\nu(E) = \int_E \|x\|^2\mu(dx)$, and a mapping f from $H\setminus\{0\}$ onto S by the equality $f(x) = \dfrac{x}{\|x\|}$. Then we get the formula.

$$(Dy,y) = \int_S (y,x)^2\nu f^{-1}(dx).$$

Thus the stable distribution with the exponent $p = 2$ also has the characteristic function of form (20), which completes the proof.

Remark. The function φ in (20) does not determine the measure ϱ. In fact, consider $H = R$ and $p = 2$. Then

$$\varphi(t) = \exp\{-c|t|^2\},$$

where $c \geqslant 0$, is the characteristic function of a symmetric normal distribution. It is easy to verify that for

$$\varrho_1(E) = c\delta_1(E)$$

and

$$\varrho_2(E) = \frac{c}{2}\delta_{-1}(E)+\frac{c}{2}\delta_1(E)$$

we have the formula

$$\int_S |(y, x)|^2 \varrho_i(dx) = c|y|^2 \quad (i = 1,2).$$

Thus, the measure ϱ is not uniquely determined by φ.

Added in proof

After this work was completed I found a paper of J. Kuelbs in Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 26 (1973), pp. 259–271, where the canonical representation of stable measures was also obtained. However we note that our methods are enterily different.

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ON THE ADAPTIVE CONTROL OF COUNTABLE MARKOV CHAINS

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1. Introduction

The paper treats the same subject as paper [4], namely the dependence of the asymptotic behaviour of the criterion functional (the reward) on the asymptotic behaviour of the control. But it concerns chains with a countable state space. In this case additional hypotheses on the transition probabilities of the system are required to ensure the stability of its basic parameters. In stating these hypotheses, we begin with a Liapounov type condition for the existence of an optimal stationary policy introduced in monograph [3]. The methods of this monograph are used in Section 2. In the remaining sections, where general non-anticipative controls are investigated, the validity of several Liapounov type conditions is assumed.

We consider a system S, which is observed at times $0, 1, 2, \ldots$, and has countably many states labelled by numbers $1, 2, 3, \ldots$ We write $I = \{1, 2, 3, \ldots\}$. Let X_n be the state of S at time n. We assume the following law of motion: For arbitrary $i \in I$, whenever S is in state i, the probability distribution of the next state is

$$\{p(i, 1; z), p(i, 2; z), \ldots\}, \quad z \in \mathscr{Z}(i). \tag{1}$$

z is a control parameter ranging in a compact metric space $\mathscr{Z}(i)$. The probabilities p are supposed to be continuous in z. They are transition probabilities of S.

Under a stationary control policy the control parameter value is a function of the actual state of S only. This function is a vector $\sigma \in \mathscr{Z}(1)\times\mathscr{Z}(2)\times \ldots = \mathscr{Z}^\infty$. The random sequence $\{X_n, n = 0, 1, \ldots\}$ is then a homogenous Markov chain with transition probability matrix

$$P_\sigma = \|p(i, j; (\sigma)_i)\|_{i,j\in I}.$$

$(\sigma)_i$ denotes the ith component of the (column) vector σ. Space $\mathscr{Z}^\infty$ is called the *set of stationary policies*. Under policy σ the control parameter value at time n equals

$$Z_n = (\sigma)_{X_n}, \quad n = 0, 1, \ldots \tag{2}$$

The random sequence $\{Z_n, n = 0, 1, \ldots\}$ is called the *control*.