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# LOCAL CHARACTERISTICS AND ABSOLUTE CONTINUITY CONDITIONS FOR *d*-DIMENSIONAL SEMI-MARTINGALES

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The purpose of what follows is to extend the results of [7] to the $d$-dimensional case. That paper [7] was devoted to the study of 1-dimensional semi-martingales. But, in the introduction, we asserted that extension to $d$-dimensional semi-martingales was obvious and straightforward. However, after reading Galtčuk [3] and Métivier–Pistone [9], we became aware that stochastic integrals with respect to $d$-dimensional martingales do not reduce to sums of $d$ stochastic integrals with respect to 1-dimensional martingales. Consequently, extension of [7] is not as easy as it was said.

This short paper is naturally divided into two parts: in the first one, which follows ideas of Galtčuk, Métivier and Pistone, stochastic integrals and the Girsanov theorem for $d$-dimensional martingales are described. We always give complete proofs, but only for the facts which are strictly necessary for the second part, which gives the extension of [7] to the $d$-dimensional case. This second part heavily relies upon [7], and we only describe in which way proofs are to be modified.

## 1. Stochastic integrals with respect to $d$-dimensional martingales

**1.1. Notations.** We consider a measurable space $(\Omega, \mathscr{F})$ equipped with an increasing and right-continuous family $(\mathscr{F}_t)_{t\geq 0}$ of sub-$\sigma$-algebras of $\mathscr{F}$. We denote by $\mathscr{P}$ the predictable $\sigma$-algebra of $\Omega\times[0, \infty[$ (for predictable processes and stopping times, we refer to Dellacherie [1]).

Let $P$ be a probability measure on $(\Omega, \mathscr{F})$. If $\mathscr{C}(P)$ is any class of processes, we denote by $\mathscr{C}_{\mathrm{loc}}(P)$ the class of those processes $X$ for which there exists a sequence $(T_n)$ of stopping times increasing $P$-a.s. to $\infty$, and such that $X^{T_n}$ belongs to $\mathscr{C}(P)$ for each $n$ (as usual, $X^T$ is the process $X$ "stopped at time $T$": $X_t^T = X_{T\wedge t}$). $\mathscr{V}^+(P)$ is the set of increasing, right-continuous processes $A = (A_t)$ such that $A_0 = 0$ $P$-a.s. and $E(A_\infty) < \infty$. Let $\mathscr{V}(P) = \mathscr{V}^+(P) - \mathscr{V}^+(P)$ be the set of differences of two processes of $\mathscr{V}^+(P)$.

When we speak of martingales, it is always with respect to $(\mathcal{F}_t)$; we consider only right-continuous and left-hand limited martingales, and we identify two martingales whose paths are $P$-a.s. the same. $\mathcal{M}(P)$ (resp. $\mathcal{M}^2(P)$) is the set of all uniformly integrable (resp. square-integrable) martingales. Let $\mathcal{M}^c(P) = \{M \in \mathcal{M}(P)$, with continuous paths$\}$. We have $\mathcal{M}^c_{\text{loc}}(P) \subset \mathcal{M}^2_{\text{loc}}(P)$. Each $M \in \mathcal{M}_{\text{loc}}(P)$ has a unique decomposition $M = M^c + M^d$ where $M^c \in \mathcal{M}^c_{\text{loc}}(P)$, $M^c_0 = 0$, and $M^d$ satisfies $M^d N \in \mathcal{M}_{\text{loc}}(P)$ for each $N \in \mathcal{M}^c_{\text{loc}}(P)$; $M^c$ is called the *continuous part* of $M$. The space $\mathcal{M}^2(P)$ is a Hilbert space under the norm $||M||_{M^2} = \sqrt{\sup_{(t)} E(M_t^2)}$.

For 1-dimensional stochastic integrals, we refer to Doléans-Dade and Meyer [2], or to Meyer [10]. We will be interested only in stochastic integrals with respect to elements of $\mathcal{M}^2(P)$ or $\mathcal{M}^2_{\text{loc}}(P)$: if $M, N \in \mathcal{M}^2_{\text{loc}}(P)$, we know the predictable element of $\mathcal{V}_{\text{loc}}(P)$, which is denoted by $\langle M, N\rangle$, and characterized by the fact that $MN - \langle M, N\rangle \in \mathcal{M}_{\text{loc}}(P)$. Let

$$L^2(M, P) = \{u \text{ predictable process, } u^2 \cdot \langle M, M\rangle \in \mathcal{V}^+(P)\}.(^1)$$

If $u \in L^2_{\text{loc}}(M, P)$, the stochastic integral $u \cdot M$ is the unique element of $\mathcal{M}^2_{\text{loc}}(P)$ such that $\langle u \cdot M, N\rangle = u \cdot \langle M, N\rangle$ for each $N \in \mathcal{M}^2_{\text{loc}}(P)$. If $u \in L^2(M, P)$, then $u \cdot M \in \mathcal{M}^2(P)$, even if $M \notin \mathcal{M}^2(P)$.

Let $\mathcal{N} \subset \mathcal{M}^2_{\text{loc}}(P)$. Let

$$\mathcal{L}^0(\mathcal{N}) = \Big\{\sum_{i \leq n} u^i \cdot N^i:\ n \in N,\ N^i \in \mathcal{N},\ u^i \in L^2(N^i, P)\Big\}$$

and let $\mathcal{L}(\mathcal{N})$ be the closure of the space $\mathcal{L}^0(\mathcal{N})$ in $\mathcal{M}^2(P)$. $\mathcal{L}(\mathcal{N})$ is the *stable subspace* generated by $\mathcal{N}$. Remark that we do not impose $\mathcal{N} \subset \mathcal{M}^2(P)$, and that

$$\mathcal{L}^0_{\text{loc}}(\mathcal{N}) = \Big\{\sum_{i \leq n} u^i \cdot N^i:\ n \in N,\ N^i \in \mathcal{N},\ u^i \in L^2_{\text{loc}}(N^i, P)\Big\}.$$

**1.2. Stochastic integrals.** Now, let $\boldsymbol{M} = (M^1, \ldots, M^d)$ be a family of $d$ elements of $\mathcal{M}^2_{\text{loc}}(P)$. We want to construct stochastic integrals with respect to $\boldsymbol{M}$ in such a way that the set of all those stochastic integrals is exactly $\mathcal{L}(\boldsymbol{M})$. The difficulty arises from the fact that, except for $d = 1$, inclusion $\mathcal{L}^0(\boldsymbol{M}) \subset \mathcal{L}(\boldsymbol{M})$ may be strict.

We define a predictable element of $\mathcal{V}^+_{\text{loc}}(P)$ by

$$\langle \boldsymbol{M}, \boldsymbol{M}\rangle = \sum_{1 \leq i \leq d} \langle M^i, M^i\rangle.$$

The measures $d\langle M^i, M^j\rangle$ are absolutely continuous with respect to the measure $d\langle \boldsymbol{M}, \boldsymbol{M}\rangle$, so there exist predictable processes $m^{ij}$ such that

$$\langle M^i, M^j\rangle = m^{ij} \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle. \tag{1}$$

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(1) If $A$ is a process whose paths have bounded variation and if $z$ is any process, we define the process $z \cdot A$ by $z \cdot A_t(\omega) = \int_0^t z_s(\omega) dA_s(\omega)$ if this expression makes sense, and $z \cdot A_t(\omega) = \infty$ if it does not.



We may, and will, choose suitable versions of $m^{ij}$, such that for each $(\omega, t)$ the matrix $(m^{ij}(\omega, t))$ is symmetric, nonnegative.

Let $L^{2,0}(\boldsymbol{M}, P)$ denote the set of $\boldsymbol{R}^d$-valued processes $\boldsymbol{v} = (v^1, \ldots, v^d)$ such that $v^i \in L^2(M^i, P)$, for each $i$. If $\boldsymbol{v} \in L^{2,0}(\boldsymbol{M}, P)$, we put $\alpha(\boldsymbol{v}) = \sum_{1 \leq i \leq d} v^i \cdot M^i$, which defines an applications: $L^{2,0}(\boldsymbol{M}, P) \xrightarrow{\alpha} \mathcal{L}^0(\boldsymbol{M})$.

If $\boldsymbol{v} = (v^1, \ldots, v^d)$ is any $\boldsymbol{R}^d$-valued process, we define an increasing process $\boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle$ by

$$\boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle = \Big(\sum_{1 \leq i, j \leq d} v^i m^{ij} v^j\Big) \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle. \tag{2}$$

$L^2(\boldsymbol{M}, P)$ denotes the set of all predictable $\boldsymbol{R}^d$-valued processes $\boldsymbol{v}$ such that $\boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle \in \mathcal{V}^+(P)$, and formula $||\boldsymbol{v}||_{\boldsymbol{M}} = E(\boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle_\infty)^{1/2}$ defines a seminorm on that space. If $\boldsymbol{v} \in L^{2,0}(\boldsymbol{M}, P)$, trivial computations using (1) show that $\langle \alpha(\boldsymbol{v}), \alpha(\boldsymbol{v})\rangle = \boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle$. Therefore $\boldsymbol{v} \in L^2(\boldsymbol{M}, P)$ and $||\boldsymbol{v}||_{\boldsymbol{M}} = ||\alpha(\boldsymbol{v})||_{M^2}$. In other words, $L^{2,0}(\boldsymbol{M}, P) \subset L^2(\boldsymbol{M}, P)$ and $\alpha$ is a norm-preserving application. The next lemma is crucial for constructing stochastic integrals with respect to $\boldsymbol{M}$ (cf. [2], [9]).

LEMMA 1.1. *$L^{2,0}(\boldsymbol{M}, P)$ is a dense subspace of $L^2(\boldsymbol{M}, P)$ for the seminorm $||\cdot||_{\boldsymbol{M}}$.*

*Proof.* As $\langle \boldsymbol{M}, \boldsymbol{M}\rangle \in \mathcal{V}^+_{\text{loc}}(P)$ there exists a sequence $(T_n)$ of stopping times increasing $P$-a.s. to $\infty$, such that $E(\langle \boldsymbol{M}, \boldsymbol{M}\rangle_{T_n}) < \infty$ for each $n$. Let $\boldsymbol{v} \in L^2(\boldsymbol{M}, P)$. We put

$$A(n) = \{(\omega, t):\ t \leq T_n(\omega),\ |v^i(\omega, t)| \leq n,\ 1 \leq i \leq d\}.$$

We have $A(n) \in \mathcal{P}$ and if $\boldsymbol{v}_n$ is the process with coordinates $v^i_n = 1_{A(n)} v^i$, we clearly have $\boldsymbol{v}_n \in L^{2,0}(\boldsymbol{M}, P)$. Moreover, $||\boldsymbol{v} - \boldsymbol{v}_n||^2_{\boldsymbol{M}} = E[(1_{A(n)^c}\boldsymbol{v}^2) \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle_\infty]$, which tends to 0 as $n \uparrow \infty$ by the fact that the complement of $\bigcap_{(n)} A(n)$ in $\Omega \times [0, \infty[$ is $P$-evanescent, and by the Lebesgue convergence theorem.

Therefore $\alpha$ admits a unique extension as an application from $L^2(\boldsymbol{M}, P)$ in $\mathcal{M}^2(P)$. If $\boldsymbol{v} \in L^2(\mathcal{M}, P)$ we denote by $\boldsymbol{v} \cdot \boldsymbol{M}$ the image of $\boldsymbol{v}$ by that application: $\boldsymbol{v} \cdot \boldsymbol{M}$ is the *stochastic integral* of $\boldsymbol{v}$ with respect to $\boldsymbol{M}$. Moreover, this extension is again norm-preserving, and it is evident that $\langle \boldsymbol{v} \cdot \boldsymbol{M}, \boldsymbol{v} \cdot \boldsymbol{M}\rangle = \boldsymbol{v}^2 \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle$; by "polarisation" it follows that for $\boldsymbol{v} \in L^2_{\text{loc}}(\boldsymbol{M}, P)$ and $\boldsymbol{w} \in L^2_{\text{loc}}(\boldsymbol{M}, P)$ we have

$$\langle \boldsymbol{v} \cdot \boldsymbol{M}, \boldsymbol{w} \cdot \boldsymbol{M}\rangle = \Big(\sum_{1 \leq i, j \leq d} v^i m^{ij} w^j\Big) \cdot \langle \boldsymbol{M}, \boldsymbol{M}\rangle. \tag{3}$$

THEOREM 1.2. *$\mathcal{L}(\boldsymbol{M})$ is exactly the set of all $\boldsymbol{v} \cdot \boldsymbol{M}$, with $\boldsymbol{v} \in L^2(\boldsymbol{M}, P)$ (and, of course, $\mathcal{L}_{\text{loc}}(\boldsymbol{M}) = \{\boldsymbol{v} \cdot \boldsymbol{M},\ \boldsymbol{v} \in L^2_{\text{loc}}(\boldsymbol{M}, P)\}$).*

*Proof.* It is clear that $\boldsymbol{v} \cdot \boldsymbol{M} \in \mathcal{L}(\boldsymbol{M})$ whenever $\boldsymbol{v} \in L^2(\boldsymbol{M}, P)$. By definition $\mathcal{L}(\boldsymbol{M})$ is the closure of $\mathcal{L}^0(\boldsymbol{M})$, which is the image of $L^{2,0}(\boldsymbol{M}, P)$ for $\alpha$. Therefore we only have to prove that if $\boldsymbol{v}_n \cdot \boldsymbol{M}$ tends to $N$ in $\mathcal{M}^2(P)$, with $\boldsymbol{v}_n \in L^{2,0}(\boldsymbol{M}, P)$, then $N = \boldsymbol{v} \cdot \boldsymbol{M}$ for some $\boldsymbol{v} \in L^2(\boldsymbol{M}, P)$. But if we put $\boldsymbol{v}^2_n = \sum v^i_n m^{ij} v^j_n$, the sequence $(\boldsymbol{v}^2_n)$ is a Cauchy sequence in the space $L^1(\Omega \times [0, \infty[, \mu)$, where $\mu$ is the positive measure $\mu(d\omega, dt) = P(d\omega) \times \langle \boldsymbol{M}, \boldsymbol{M}\rangle(\omega, dt)$. Therefore $\boldsymbol{v}^2_n$ tends to a limit $w$ in that space and we may find a subsequence $(v_{n_k}^2)$ which tends to $w$ $\mu$-almost everywhere. If $A$ is the set of convergence of that subsequence, we put $w' = 1_A w$. The existence of a predictable $\mathbf{R}^d$-valued process $\mathbf{v}$ such that $w' = \sum v^i m^{ij} v^j$ easily follows from the definition of $v_n^2$ and the predictability of $A$. Finally, the convergence of $\mathbf{v}_n^2$ towards $w'$ in $L^1(\Omega\times[0,\infty[,\mu)$ implies that $\mathbf{v}\in L^2(\mathbf{M},P)$ and that $\|\mathbf{v}_n\cdot\mathbf{M}-\mathbf{v}\cdot\mathbf{M}\|_{M^2} = \|\mathbf{v}_n-\mathbf{v}\|_M \to 0$. Therefore $N = \mathbf{v}\cdot\mathbf{M}$.

**1.3. The Girsanov theorem for continuous $d$-dimensional martingales.** In this section, $\mathbf{M} = (M^1,\dots,M^d)$ is a family of $d$ elements of $\mathscr{M}^c_{\mathrm{loc}}(P)$, and $m^{ij}$ is again defined by (1). Suppose $P'$ is another probability measure on $(\Omega,\mathscr{F})$ such that $P' \ll P$. We want to study which properties $\mathbf{M}$ has with respect to $P'$. We may take advantage of the following lemma, which is a version of the Girsanov theorem and is well known (see, for example, Van Schuppen and Wong [11]).

LEMMA 1.3. *Let $Z$ be the martingale $Z_t = E\left(\frac{dP'}{dP}|\mathscr{F}_t\right)$. If $N\in\mathscr{M}^c_{\mathrm{loc}}(P)$, the process $N' = N-N_0-\frac{1}{Z_-}\cdot\langle N, Z^c\rangle$ belongs to $\mathscr{M}^c_{\mathrm{loc}}(P')$ and $\langle N,N\rangle$ is a version of the bracket associated with $N'$ for $P'$.*

Of course this lemma applies to each coordinate $M^i$ separately. But we may also obtain a "global" version as follows:

THEOREM 1.4. *There exists a predictable $\mathbf{R}^d$-valued process $\mathbf{v}$ such that $M'^i = M^i - M^i_0 - \left(\sum_{1\leqslant j\leqslant d} v^j m^{ji}\right)\cdot\langle \mathbf{M},\mathbf{M}\rangle \in \mathscr{M}^c_{\mathrm{loc}}(P')$, and $\langle M^i, M^j\rangle$ is a version of the bracket associated with $M'^i$ and $M'^j$ for $P'$.*

*Proof.* Let $Z$ as in Lemma 1.3. Owing to Theorem 1.2, we see that $Z^c$ may be written as $Z^c = \mathbf{u}\cdot\mathbf{M}+Z^{\prime c}$, where $\mathbf{u}\in L^2_{\mathrm{loc}}(\mathbf{M},P)$ and $Z^{\prime c}$ satisfies $\langle Z^{\prime c}, N\rangle = 0$ for each $N\in\mathscr{L}_{\mathrm{loc}}(\mathbf{M})$. Moreover, $Z$ is a nonnegative martingale, thus $Z_t = Z_t^c = \mathbf{u}\cdot\mathbf{M}_t = 0$ if $t\geqslant \inf(s\colon Z^c_{s-}=0)$: therefore if $\mathbf{v} = \mathbf{u}(1/Z_-)$ on the set $\{Z_->0\}$ and $\mathbf{v}=0$ elsewhere, we have $Z^c = (Z_-\mathbf{v})\cdot\mathbf{M}+Z^{\prime c}$ and $Z_-\mathbf{v}\in L^2_{\mathrm{loc}}(\mathbf{M},P)$. Now, by using the fact that $M^i = \mathbf{w}_i\cdot\mathbf{M}$ with $w_i^j = 0$ or $=1$ if $i\neq j$ or $i=j$, respectively, and by using (3), Lemma 1.3 yields $M'^i\in \mathbf{M}^c_{\mathrm{loc}}(P)$. Finally, the statement concerning $\langle M^i, M^j\rangle$ again follows from Lemma 1.3, directly if $i=j$ and by polarisation if $i\neq j$. ∎

As for the 1-dimensional case, this theorem presents a partial converse. Namely, let $\mathbf{v}\in\mathscr{L}^c_{\mathrm{loc}}(\mathbf{M},P)$ and $N = \mathbf{v}\cdot\mathbf{M}$. We define the exponential of $N$ by

$$(4)\qquad Z_t = \exp(N_t - \tfrac{1}{2}\langle N,N\rangle_t),$$

which is a nonnegative local martingale. Then

THEOREM 1.5. *If $E(Z_\infty) = 1$, then*

$$M'^i = M^i - M^i_0 - \Big(\sum_{1\leqslant j\leqslant d} v^j m^{ij}\Big)\cdot\langle\mathbf{M},\mathbf{M}\rangle$$

*belongs to $\mathscr{M}^c_{\mathrm{loc}}(P')$, where $P'$ is the probability measure defined by $P' = Z_\infty\cdot P$.*



*Proof.* It is sufficient to notice that $Z_t = E\left(\frac{dP'}{dP}|\mathscr{F}_t\right)$ and that (4) implies $Z = 1+Z_-\cdot N$, thus $Z^c = (Z_-\mathbf{v})\cdot\mathbf{M}$, and compare with the proof of 1.5. ∎

## 2. Application to $d$-dimensional semi-martingales

**2.1. Local characteristics.** Now we proceed to the extension of [7], without restricting ourselves to using the results of that paper.

$(\Omega,\mathscr{F})$ and the family $(\mathscr{F}_t)$ are as in Section 1. We put $E = \mathbf{R}^d\setminus\{0\}$, with its Borel $\sigma$-algebra $\mathscr{E}$, and $\tilde{\Omega} = \Omega\times[0,\infty[\times E$, $\tilde{\mathscr{P}} = \mathscr{P}\otimes\mathscr{E}$. If $\eta(\omega; dt, dx)$ is a "random measure" on $[0,\infty[\times E$, and if $W$ is a function on $\tilde{\Omega}$, we define the process $W*\eta$ by

$$W*\eta_t(\omega) = \int_{[0,t]\times E} W(\omega,s,x)\eta(\omega; ds, dx)$$

if this expression makes sense, and

$$W*\eta_t(\omega) = \infty$$

if it does not. A random measure $\eta$ is said to be *predictable* if $W*\eta$ is a predictable process for any $\tilde{\mathscr{P}}$-measurable function $W$.

$\mathbf{X} = (X^1,\dots,X^d)$ is an adapted $\mathbf{R}^d$-valued process whose paths are right-continuous and left-hand limited. We consider the random measure $\mu$ on $[0,\infty[\times E$ which is associated with the jumps of $\mathbf{X}$ by $\mu(\omega;[0,t]\times A) = \sum_{s\leqslant t} 1_A(\Delta X_s)$, where $A\in\mathscr{E}$ ($\Delta X_s$ is the size of the jump of $\mathbf{X}$ at time $s$). We associate with $\mathbf{X}$ another $\mathbf{R}^d$-valued process $\mathbf{Y}$ by

$$Y^i_t = X^i_t - X^i_0 - \sum_{s\leqslant t}\Delta X^i_s 1_{\{|\Delta X^i_s|>1\}}.$$

Let $P$ be a probability measure on $(\Omega,\mathscr{F})$ for which $\mathbf{X}$ is a ($d$-dimensional) *semi-martingale*, i.e. for each $i$, $X^i = A^i+N^i$ where $N^i\in\mathscr{M}_{\mathrm{loc}}(P)$ and $A^i$ is a process whose paths have bounded variation over each finite interval.

DEFINITION 2.1. The *set of $P$-local characteristics* of $\mathbf{X}$ is the triplet $\mathscr{C} = (\nu, \underline{\alpha}, \underline{\beta})$ defined by

$\nu$ is the unique predictable random measure such that the process $(\mu([0,t]\times A)-\nu([0,t]\times A))_{t\geqslant 0}$ belongs to $\mathscr{M}_{\mathrm{loc}}(P)$ for each $A\in\mathscr{E}$ lying at a positive distance of 0.

$\underline{\alpha}$ is the unique predictable $\mathbf{R}^d$-valued process such that $\alpha^i\in\mathscr{V}_{\mathrm{loc}}(P)$ and $Y^i - \alpha^i\in\mathscr{M}_{\mathrm{loc}}(P)$ for each $i$.

$\underline{\beta} = (\beta^{ij} = \langle M^i, M^i\rangle)_{1\leqslant i,j\leqslant d}$, where $M^i$ is the continuous part of the local martingale $Y^i-\alpha^i$.

Existence and uniqueness for $\nu$ were proved in [6], while existence and uniqueness for each $\alpha^i$ were proved in [7] or [10]. The concept of local characteristics has been introduced by Grigelionis [4], [5]. $\nu$ is also called the "Lévy system" of $\mathbf{X}$.

The usefulness of this concept relies upon its relationship with stochastic differential equations: quite frequently, solutions of those equations are semi-martingales whose local characteristics are related in a simple way to the "coefficients" of the equation. Several examples (including processes with jumps) are displayed in [7] and [8], as well as in several papers by Grigelionis. But here, in order to illustrate the notion, we content ourselves with the example of processes with independent increments: namely, let us suppose that for $P$, $X$ is a process with independent increments (say, homogeneous), with Lévy measure $F$, diffusion coefficients $b^{ij}$ and drift coefficients $a^i$. Then $X$ is a $P$-semi-martingale whose local characteristics are

$$\nu(\omega; dt, dx) = dt \times F(dx), \quad \alpha^i(\omega, t) = a^i t, \quad \beta^{ij}(\omega, t) = b^{ij} t.$$

(Of course, we must choose a suitable version for the drift coefficients.)

**2.2. Changes of measures.** We will state the main results of [7], namely theorems (3.3), (4.1), (4.2), (4.3), and (4.5), for the $d$-dimensional case, leaving the extension of other parts of [7] to the reader (this time, this is really obvious and straightforward!). Theorem (4.4) of [7] has already been extended to a much more general case in [8].

At first let $P$ be a probability measure for which $X$ is a semi-martingale whose local characteristics are $\mathscr{C} = (\nu, \underline{\alpha}, \underline{\underline{\beta}})$. We put $U^i(\omega, t, x) = x^i_{\{|x^i| \leqslant 1\}}$ (where $x = (x^1, \ldots, x^d) \in E$). Let $P'$ be another probability measure such that $P' \ll P$.

THEOREM 2.2. (a) *$X$ is a $P'$-semi-martingale.*

(b) *There exist a $\tilde{\mathscr{P}}$-measurable nonnegative function $Y$ on $\Omega$ and a predictable $\boldsymbol{R}^d$-valued process $\boldsymbol{v}$ such that the formulae*

$$(5) \qquad \begin{aligned} &\nu'(\omega; dt, dx) = Y(\omega, t, x)\nu(\omega; dt, dx), \\ &\alpha'^i = \alpha^i + \Big(\sum_{1 \leqslant j \leqslant d} v^j m^{ij}\Big) \cdot \beta + [U^i(Y-1)] * \nu, \end{aligned}$$

*where* $\beta = \sum_{1 \leqslant i \leqslant d} \beta^{ii}$ *and* $m^{ij}$ *are predictable processes with* $\beta^{ij} = m^{ij} \cdot \beta$,

$$\beta'^{ij} = \beta^{ij}$$

*define a triplet* $\mathscr{C}' = (\nu', \underline{\alpha}', \underline{\underline{\beta}}')$ *which is the set of $P'$-local characteristics* of $X$.

*Proof.* We can prove (a) for each component $X^i$, and this is done in [7]. The formula giving $\nu'$ is proved in [6]. Let $M^i$ be the continuous part of the local martingale $Y^i - \alpha^i$ and $N^i = Y^i - \alpha^i - M^i$. Theorem 1.4 implies that there exists a predictable $\boldsymbol{R}^d$-valued process $\boldsymbol{v}$ such that

$$M'^i = M^i - \Big(\sum_{1 \leqslant j \leqslant d} v^j m^{ij}\Big) \cdot \beta \in \mathscr{M}^c_{\mathrm{loc}}(P')$$

and that $\beta^{ij}$ is a version of the bracket associated with $M'^i$ and $M'^j$ for $P'$. Finally, trivial modifications in the proofs of [7] show that $N'^i = N^i - [U^i(Y-1)] * \nu \in \mathscr{M}_{\mathrm{loc}}(P')$ and that the continuous part $N'^i$ for $P'$ is 0, thus yielding the result. ∎

From now on, we suppose that $P$ and $P'$ are two probability measures for which $X$ is a semi-martingale. We denote by $\mathscr{C}$ (resp. $\mathscr{C}'$) the set of $P$- (resp. $P'$-) local characteristics of $X$. As we are interested in conditions for having $P' \ll P$, it is only natural to assume that *$\mathscr{C}'$ is obtained from $\mathscr{C}$ by formulae* (5), for some $\tilde{\mathscr{P}}$-measurable function $Y$ and some predictable process $\boldsymbol{v}$. We will also assume that $X$ is *quasi-left continuous for $P$* (this is equivalent to assuming $\nu(\omega; \{t\} \times E) = 0$ identically and, in view of (5), $X$ is also quasi-left continuous for $P'$).



We have also to consider an "initial condition", which is described by a sub-$\sigma$-algebra $\mathscr{F}^0_0$ of $\mathscr{F}_0$ (usually $\mathscr{F}^0_0 = \sigma(x_0)$) and the restriction $Q$ and $Q'$ of $P$ and $P'$ to $\mathscr{F}^0_0$.

If $T$ is any stopping time, we say that "$T$-uniqueness" holds for $(\mathscr{C}, Q)$ if, $\hat{P}$ being another probability measure whose restriction to $\mathscr{F}^0_0$ is $Q$ and such that the stopped process $X^T$ is a $\hat{P}$-semi-martingale with local characteristics $\mathscr{C}^T = (\nu(\omega; dt, dx) 1_{\{t \leqslant T(\omega)\}}, \underline{\alpha}^T, \underline{\underline{\beta}}^T)$, the restrictions of $P$ and $\hat{P}$ to $\mathscr{F}_{T-}$ coincide. In [7] conditions for having $T$-uniqueness for each predictable stopping time are examined, and everything goes through the $d$-dimensional case. Similarly, we define $T$-uniqueness for $(\mathscr{C}', Q')$. Let us define:

$$A = \Big(\sum_{1 \leqslant i, j \leqslant d} v^i m^{ij} v^j\Big) \cdot \beta + (Y-1) 1_{\{Y > 2\}} * \nu + (Y-1)^2 1_{\{Y \leqslant 2\}} * \nu,$$

$$S_n = \inf(t\colon A_t \geqslant n), \quad S = \inf(t\colon A_t = \infty) = \lim_{(n)} S_n,$$

$$G = (A_\infty < \infty), \quad \hat{G} = \{1_{\{Y=0\}} * \nu_\infty = 0\}.$$

THEOREM 2.3. (a) *If $P' \ll P$, we have $Q' \ll Q$ and $P'(G) = 1$.* (b) *If $P \ll P'$, we have $Q \ll Q'$ and $P(G \cap \hat{G}) = 1$.*

THEOREM 2.4. *Assume that $\mathscr{F} = \bigvee_{(t)} \mathscr{F}_t$. Assume that for each $n$, $S_n$-uniqueness holds for $P$ (resp. $P'$). Then*

(a) *If $P'(G \cap \hat{G}) = 1$ (resp. if $P'(G) = 1$) and $Q' \ll Q$, we have $P' \ll P$.*

(b) *If $P(G \cap G) = P'(\hat{G}) = 1$ (resp. if $P(G \cap \hat{G}) = 1$) and $Q \ll Q'$, we have $P \ll P'$.*

These theorems are exact reproductions of the statements of [7]. The only difficulty with respect to the 1-dimensional case was to find the correct first term to be put in the definition of $A$. We will not reproduce the proofs here: they would be much too long. Let us just give some hints. Let $M^i$ be the continuous part of $Y^i - \alpha^i \in \mathscr{M}_{\mathrm{loc}}(P)$, and $\boldsymbol{M} = (M^1, \ldots, M^d)$. Put $N(n) = (1_{[0,S_n]} \boldsymbol{v}) \cdot \boldsymbol{M} + (1_{[0,S_n]}(Y-1)) * (\mu - \nu)$ (owing to the definition of $S_n$, the first stochastic integral makes sense; for the second one, we refer to [7]), and

$$(6) \qquad Z_t = \begin{cases} e^{N_t(n) - \frac{1}{2}\langle N(n)^c, N(n)^c \rangle_t} \prod_{s \leqslant t} \big(1 + \Delta N_s(n)\big) e^{-\Delta N_s(n)} & \text{if } t \leqslant S_n, \\ \liminf_{(n)} Z_{S_n} & \text{if } t \geqslant S. \end{cases}$$

The proofs in [7] were based upon the properties of $N(n)$ and $Z$. The continuous part $N(n)^c$ and the purely discontinuous part $N(n)^d$ were constantly separated and treated independently. For the discontinuous part nothing has changed. For the continuous part (which is the easiest one) we only have to use Theorems 1.4 and 1.5: this explains the first term in $A$, which on $[0, S_n]$ reduces to $\langle N(n)^c, N(n)^c \rangle$. The details are left to the reader.

Finally, let us reproduce Theorem 4.5 of [7], which shows why the process $Z$ defined by (6) intervenes.

THEOREM 2.5. (a) *Let $q$ be a nonnegative $\mathcal{F}_0^0$-measurable random variable such that $E(q) = 1$. Assume that $E(qZ_\infty) = 1$ and that $qZ_S = 0$ $P$-a.s. on the set $\bigcup_{(n)} \{S_n = S < \infty\}$. Then $X$ admits $\mathscr{C}'$ for $\hat{P}$-local characteristics if $\hat{P} = (qZ_\infty)\cdot P$.*

(b) *Assume $P' \ll P$ and let $q = \frac{dQ'}{dQ}$. If $S_n$-uniqueness holds for $(\mathscr{C}', Q')$ for each $n$, we have $E(qZ_\infty) = 1$, $qZ_S = 0$ on the set $\bigcup_{(n)} \{S_n = S < \infty\}$ and $qZ$ is a version of the martingale $E\left(\frac{dP'}{dP} | \mathcal{F}_t\right)$.*

(For statement (b) above, we recall that $P'$ is supposed to be given a priori, with $P'$-local characteristics $\mathscr{C}'$ for $X$; statement (b) remains true of we replace $S$-uniqueness by the "property of representation for martingales" with respect to $X$, for $P$; cf. [7], [8].)

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# SEMI-STABLE MEASURES

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## 1

We shall deal with the theory of infinitely divisible measures. One of the most important and most interesting problems in this theory is to describe some natural subclasses of the class of all infinitely divisible distributions. By a natural class we mean here a class of measures which coincides with the set of all possible limit laws for some more or less standard array of random variables. Some natural subclasses of infinitely divisible measures are well known and have been examined in detail, for example: stable measures, self-decomposable measures, all infinitely divisible measures.

The class of stable measures will play some role in the sequel, so we recall now that it can be defined as the class of all limit laws for normed sums of random variables. Namely, for a sequence of independent, identically distributed random variables $\xi_1, \xi_2, \ldots$ we consider normed sums of the form $\eta_n = A_n \sum_{k=1}^{n} \xi_k + B_n$, where $A_n > 0$ and $B_n$ are arbitrary real numbers. The class of stable measures consists of all limit laws for the sums $\eta_n$. Of course, we can consider random variables $\xi_k$ taking their values in the linear vector space $Y$. Then $B_n$ are vectors from $Y$ and we obtain the definition of stable measures in $Y$. In this case the sums may be normed by linear operators, that is, the numbers $A_n$ may be replaced by linear operators acting in $Y$. In particular, if we consider $\mathbf{R}^N$-valued random variables and $A_n$ are non-singular linear operators acting in the $N$-dimensional Euclidean space, then we obtain the class of operator-stable measures. That interesting class has been introduced and examined by M. Sharpe [12]. More precisely, M. Sharpe described the class of full operator-stable measures. Recall that a measure in $\mathbf{R}^N$ is said to be *full* if its support is not contained in any $(N-1)$-dimensional hyperplane.

We shall now quote the theorem of Sharpe [12].

THEOREM 1. *A full measure $\mu$ in $\mathbf{R}^N$ is operator-stable if and only if it is infinitely divisible and there is a non-singular linear operator $B$ in $\mathbf{R}^N$ such that $\mu^t = t^B\mu * \delta_{b(t)}$, $t > 0$, for some $b(t) \in \mathbf{R}^N$. We put here by definition $t^B = \exp\{\lg t \cdot B\}$.*