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THE TAIL STRUCTURE OF NONHOMOGENEOUS FINITE STATE MARKOV CHAINS: SURVEY

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1. Introduction and summary

Let S denote a finite set. Consider a (varying) probability distribution $p = (p(i))_{i\in S}$ on S and a (fixed) sequence of stochastic matrices $P_n = (p_n(i,j))_{i,j\in S}$, $n \geqslant 0$. These objects define a nonhomogeneous finite state Markov chain with state space S, initial probability distribution p and transition matrices $P_n, n \geqslant 0$. More precisely, as is well known, a probability space $(\Omega, \mathscr{K}, \mathsf{P}_p)$ can be set up and random variables $X(n), n \geqslant 0$, defined on it such that

$$\mathsf{P}_p(X(0) = i) = p(i), \quad i \in S,$$
$$\mathsf{P}_p(X(n+1) = j|X(u),\ 0 \leqslant u \leqslant n) = \mathsf{P}_p(X(n+1) = j|X(n)) = p_n(X(n), j), \quad n \geqslant 0, j \in S,$$

P_p-almost surely.

Let us denote by $\mathscr{K}^n \subset \mathscr{K}$ the σ-algebra generated by the random variables $X(n), X(n+1), \ldots, n \geqslant 0$, and put $\mathscr{T} = \bigcap_{n\geqslant 0} \mathscr{K}^n$. The σ-algebra $\mathscr{T}$ is called the *tail σ-algebra* of the Markov chain considered.

This paper is aimed at giving a self-contained account of recent investigations concerning the structure of the tail σ-algebra $\mathscr{T}$.

Citations in the text have been kept to a minimum. References and various remarks have been collected in the final section of the paper.

2. Preliminaries

2.1. Given a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ an event $A \in \mathscr{F}$ is said to be a P-*atom* if $\mathsf{P}(A) > 0$ and for any event $B \subset A$ either $\mathsf{P}(B) = 0$ or $\mathsf{P}(B) = \mathsf{P}(A)$. Next, an event $N \in \mathscr{F}$ is said to be P-*completely nonatomic* if for any positive number $c \leqslant \mathsf{P}(B)$ there exists an event $C \subset B$ such that $\mathsf{P}(C) = c$. It is well known (see, e.g. Loève [14], p. 100) that Ω can be partitioned as

$$\Omega = N \cup \Big(\bigcup_{i\in I} A_i\Big), \tag{1}$$

where the event N may be absent and the index set I may be empty, finite or denumerable; if present, N is a P-completely nonatomic event and, if I is not empty, the A_i, $i \in I$, are P-atoms. The decomposition (1) is unique modulo null P-probability events.

The σ-algebra $\mathcal{F}$ is said to be P-*finite* if $N = \emptyset$ and the set I is finite. In particular, it is said to be P-*trivial* if there is just a P-atom (thus coinciding with Ω).

2.2. In the case of the tail σ-algebra $\mathcal{T}$ of a Markov chain (even with a denumerable state space) the representation (1) can be constructed as follows.

There exists a sequence $(S(n), S_r(n), r \in I)_{n\geqslant 0}$ *of partitions of the state space* S *such that*

$$N = \lim_{n\to\infty}\{X(n) \in S(n)\}, \quad A_r = \lim_{n\to\infty}\{X(n) \in S_r(n)\}, \quad r \in I.$$

To prove this assertion let us set

$$S_r(n) = [i:\ \mathsf{P}_p(A_r|X(n) = i) > 1/2], \qquad S(n) = S - \bigcup_{r\in I} S_r(n), \qquad r \in I,\ n \geqslant 0.$$

For any $n \geqslant 0$, the sets $S_r(n)$, $r \in I$, are clearly pairwise disjoint. Therefore $(S(n), S_r(n), r \in I)$ is a partition of the state space S for any $n \geqslant 0$. Next, by the Markov property and the martingale convergence theorem (see, e.g., Loève [14], p. 409)

$$\lim_{n\to\infty}\mathsf{P}_p(A_r|X(n)) = \lim_{n\to\infty}\mathsf{P}_p(A_r|X(n), \ldots, X(0)) = \chi_{A_r}, \quad r \in I, \tag{2}$$

P_p-almost surely. Here χ_A denotes the indicator of the event A, i.e.

$$\chi_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Consequently, modulo null P-probability events,

$$A_r = \{\omega:\ \lim_{n\to\infty}\mathsf{P}_p(A_r|X(n)) > 1/2\} = \lim_{n\to\infty}\{X(n) \in S_r(n)\}, \quad r \in I,$$

and it follows that

$$N = (\bigcup_{r\in I} A_r)^c = \lim_{n\to\infty}\{X(n) \in \bigcup_{r\in I} S_r(n)\}^c = \lim_{n\to\infty}\{X(n) \in S(n)\}.$$

Although tautological [$S_r(n)$ is defined in terms of A_r], the above representation of the events N and A_r, $r \in I$, is instrumental in proving the main theorem about $\mathcal{T}$.

3. The main theorem

3.1. We are now able to prove

THEOREM 1. *The tail σ-algebra $\mathcal{T}$ of a (nonhomogeneous) finite state Markov chain is P_p-finite whatever the initial probability distribution p and the number of its atoms does not exceed the number of states of the chain.*

Proof. Assume for a contradiction that the chain has s states and the index set I in (1) contains more than s elements. Then there would exist $r_1, \ldots, r_{s+1} \in I$ such that for a sufficiently large n one has $\mathsf{P}_p(X(n) \in S_{r_k}(n)) > 0$, $1 \leqslant k \leqslant s+1$. But these inequalities contradict the fact the chain has s states. This contradiction shows that the number of the atoms of $\mathcal{T}$ does not exceed the number of states of the chain.

It remains to prove that $\mathsf{P}_p(N) = 0$. Assuming again for a contradiction that $\mathsf{P}_p(N) > 0$, there would exist $s+1$ pairwise disjoint events $N_1, \ldots, N_{s+1} \in \mathcal{T}$ contained in N such that $\mathsf{P}_p(N_k) > 0$, $1 \leqslant k \leqslant s+1$. Defining the subset $B_k(n)$ of the state space by $B_k(n) = [i:\ \mathsf{P}_p(N_k|X(n) = i) > 1/2]$, we deduce as in Section 2.2 that $N_k = \lim_{n\to\infty}\{X(n) \in B_k(n)\}$, $1 \leqslant k \leqslant s+1$. These relations contradict again the fact the chain has s states. Therefore one should have $\mathsf{P}_p(N) = 0$ and the proof is complete.

Of course, it would be desirable to describe an effective procedure to construct the partitions $(S(n), S_r(n), r \in I)$, $n \geqslant 0$. As we shall see in the next section this can be done in the homogeneous case (i.e. the case where $P_n = P = (p(i,j))_{i,j\in S}$ for any $n \geqslant 0$).

3.2. We begin by considering the case of a homogeneous finite state Markov chain without transient states.

THEOREM 2. *Consider a homogeneous finite state Markov chain with no transient states. If its (closed) classes of states are C^l with cyclical moving subclasses $C_0^l, \ldots, C_{d_l-1}^l$, $1 \leqslant l \leqslant r$, then the P_p-atoms of the tail σ-algebra $\mathcal{T}$ are the events $A_s^l = \{X(0) \in C_s^l\}$, $0 \leqslant s \leqslant d_l - 1$, $1 \leqslant l \leqslant r$, for which $\sum_{i\in C_s^l} p(i) > 0$.*

Proof. For the sake of clarity we shall treat only the case where $\sum_{i\in C_s^l} p(i) > 0$ for any $0 \leqslant s \leqslant d_l - 1$, $1 \leqslant l \leqslant r$. The reader will be able to modify the proof in order to make it applicable to cases where some of the above sums are null.

Let us denote by d the least common multiple of the periods d_l, $1 \leqslant l \leqslant r$. Then for n large enough ($\geqslant n_0$) we shall have $\mathsf{P}_p(X(nd) = i) > 0$ for any $i \in S$ (see, e.g., Loève [14], p. 35). This fact allows us to write

$$\begin{aligned}\mathsf{P}_p(A|X(nd) = i) &= \sum_{j\in S}\mathsf{P}_p(A \cap \{X((m+n)d) = j\}|X(nd) = i) \\ &= \sum_{j\in S} p^{(md)}(i,j)\mathsf{P}_p(A|X((m+n)d) = j),\end{aligned} \tag{3}$$

for any $n \geqslant n_0$, $m \geqslant 0$, $A \in \mathcal{T}$ and $i \in S$. Next, consider the sequences $\big(\mathsf{P}_p(A|X((m+n)d) = j)\big)_{m\geqslant 0}$, $j \in S$. Since they are bounded, it is possible to find an increasing sequence $(n_s)_{s\geqslant 1}$ of natural numbers such that the limits

$$\lim_{s\to\infty}\mathsf{P}_p(A|X(n_s d) = j), \quad j \in S,$$

do exist. Let us now remark the matrix P^d can be considered as the transition matrix of a homogeneous finite state Markov chain with $d_1+\ldots+d_r$ aperiodic classes and with no transient states. Consequently, on account of a well-known result the matrix P^{nd} will approach as $n\to\infty$ a matrix of the form

$$\begin{bmatrix} \Pi_{1,0} & & & & & 0 \\ & \ddots & & & & \\ & & \Pi_{1,d_1-1} & & & \\ & & & \ddots & & \\ & & & & \Pi_{r,0} & \\ 0 & & & & & \ddots \\ & & & & & \Pi_{r,d_r-1} \end{bmatrix},$$

where the matrices on the main diagonal are positive stochastic stable (i.e. with identical rows) matrices.

If we replace m by n_s-n in (3) and let $s\to\infty$, we conclude that the conditional probabilities $\mathsf{P}_p(A|X(nd)=i)$ are independent of $n\geqslant n_0$ and the vector $\big(\mathsf{P}_p(A|X(nd)=i)\big)_{i\in S}$ is a linear combination with coefficients depending on A and P of $d_1+\ldots+d_r$ linearly independent vectors with constant components. Finally, if A is a P_p-atom of $\mathscr{T}$, then there will exist an $i\in S$ such that $\mathsf{P}_p(A|X(nd)=i)>0$ for $n\geqslant n_0$ [otherwise it would follow that $\mathsf{P}_p(A)=\sum_{i\in S}\mathsf{P}_p(X(nd)=i)\mathsf{P}_p(A|X(nd)=i)=0$] and (2) leads us to the conclusion that $\mathsf{P}_p(A|X(nd)=i)=1$, $n\geqslant n_0$.

We are now able to prove that if $A_1,\ldots,A_q$ are distinct P_p-atoms of $\mathscr{T}$, then the vectors $v_m=\big(\mathsf{P}_p(A_m|X(n_0d)=i)\big)_{i\in S}$, $1\leqslant m\leqslant q$, are linearly independent. Indeed, if $\sum_{m=1}^{q}c_mv_m=0$, on account of the fact that $\mathsf{P}_p(A_1|X(n_0d)=i)=1$ for a certain $i\in S$ [implying $\mathsf{P}_p(A_2|X(n_0d)=i)=\ldots=\mathsf{P}_p(A_2|X(n_0d)=i)=0$], we conclude that $c_1=0$, and in an analogous manner, $c_2=\ldots=c_q=0$.

It follows, therefore, that there can be at most $d_1+\ldots+d_r$ atoms. Since the $d_1+\ldots+d_r$ pairwise disjoint events A_s^l, $0\leqslant s\leqslant d_l-1$, $1\leqslant l\leqslant r$, belong to $\mathscr{T}$ [we have $A_s^l=\{X(nd_l)\in C_s^l \text{ for any } n\geqslant 0\}=\lim_{n\to\infty}\{X(nd_l)\in C_s^l\}$] they are, clearly, the P_p-atoms of $\mathscr{T}$.

COROLLARY. *The tail σ-algebra of a regular Markov chain (i.e. a homogeneous finite state Markov chain having just an aperiodic class and no transient states) is P_p-trivial whatever the initial probability distribution p.*

We are now able to consider the general case of a homogeneous finite state Markov chain with both recurrent and transient states.

THEOREM 3. *Consider a homogeneous finite state Markov chain having r classes C^l with cyclical moving subclasses $C_0^l,\ldots,C_{d_l-1}^l$, $1\leqslant l\leqslant r$, and an arbitrary number of transient states. Then the P_p-atoms of the tail σ-algebra $\mathscr{T}$ are the events $A_s^l=\lim_{n\to\infty}\{X(nd_l)\in C_s^l\}$ for which*

$$\sum_{i\in S}\sum_{j\in C_s^l}p(i)f_{d_l}(i,j)>0,$$

$0\leqslant s\leqslant d_l-1$, $1\leqslant l\leqslant r$, where

$$f_d(i,j)=\sum_{n\geqslant 1}\mathsf{P}_p(X(nd)=j,\ X(u)\neq j,\ 0<u<nd|X(0)=i)$$

for any $d>0$ and $i,j\in S$.

Proof. Denote by ν the time to absorption in the set of the recurrent states, i.e.

$$\nu=\min(n\colon X(n)\in C^1\cup\ldots\cup C^r).$$

Clearly, ν is a Markov time of the chain considered. On account of the strong Markov property the sequence $\big(X(n+\nu)\big)_{n\geqslant 0}$ is a homogeneous Markov chain with state space $C^1\cup\ldots\cup C^r$ and whose transition probabilities coincide with the corresponding ones of the initial chain $\big(X(n)\big)_{n\geqslant 0}$. It is easily seen that the P_p-atoms of the latter are precisely the $\mathsf{P}_{\bar p}$-atoms of the former, where $\bar p$ is the probability distribution on $C^1\cup\ldots\cup C^r$ with components

$$\bar p(i)=\mathsf{P}_{\bar p}(X(\nu)=i),\qquad i\in C^1\cup\ldots\cup C^r.$$

Thus, Theorem 3 obtains from Theorem 2.

COROLLARY. *The tail σ-algebra of an indecomposable Markov chain (i.e. a homogeneous finite state Markov chain having just an aperiodic class and an arbitrary number of transient states) is P_p-trivial whatever the initial probability distribution p.*

4. An application to reverse probabilities

The tail structure established by Theorem 1 can be connected with investigations by Kolmogorov [13] and Blackwell [1] concerning the asymptotic properties of the *reverse* transition probabilities

$$\hat p_n^{(n-m)}(i,j)=\mathsf{P}_{\bar p}(X(m)=j|X(n)=i),\quad m\leqslant n,$$

defined for all $i,j\in S$ [for the sake of clarity we shall assume that $\mathsf{P}_{\bar p}(X(n)=i)>0$ for any $n\geqslant 0$, $i\in S$]. It is easily seen that putting $\hat P_n^{(n-m)}=(\hat p_n^{(n-m)}(i,j))_{i,j\in S}$, one has

$$\hat P_l^{(l-n)}\hat P_n^{(n-m)}=\hat P_l^{(l-m)},\qquad m\leqslant n\leqslant l. \tag{4}$$

Following Kolmogorov let us show that there exist probability distributions (row vectors) p_n on S, $n\geqslant 0$, such that

$$p_n\hat P_n^{(n-m)}=p_m,\qquad 0\leqslant m\leqslant n,\ n\geqslant 0. \tag{5}$$

To this end choose an increasing sequence $(s_k)_{k\geqslant 1}$ of natural numbers such that the limits

$$\lim_{k\to\infty}\hat P_{s_k}^{(s_k-m)}=Q(m) \tag{6}$$

do exist for all $m \geqslant 0$. Letting $l \to \infty$ through the sequence $(s_k)_{k \geqslant 1}$ in (4), we obtain

$$Q(n)\hat{P}_n^{(n-m)} = Q(m), \quad 0 \leqslant m \leqslant n,\ n \geqslant 0.$$

In other words, the rows of the same rank of the matrices $Q(n)$, $n \geqslant 0$, provide us with probability distributions p_n on S, $n \geqslant 0$, satisfying (5). Next, if q_n, $n \geqslant 0$, are probability distributions on S such that $q_n \hat{P}_n^{(n-m)} = q_m$, $0 \leqslant m \leqslant n$, $n \geqslant 0$, then denoting by $(t_{k'})_{k' \geqslant 1}$ a subsequence of $(s_k)_{k \geqslant 1}$ such that the limit $\lim_{k' \to \infty} q_{t_{k'}} = q$ does exist and letting $k' \to \infty$ in the equation $q_{t_{k'}} \hat{P}_{t_{k'}}^{(t_{k'}-m)} = q_m$, we deduce that $qQ(m) = q_m$, $m \geqslant 0$. It follows that q_m should be a convex linear combination of the rows of Kolmogorov's matrix $Q(m)$ for any $m \geqslant 0$. Consequently, the number of linearly independent solutions $(p_n)_{n \geqslant 0}$ of (5) does not exceed the number of the states of the chain considered.

Now, let us remark that the Markov property being time-reversible, i.e.

$$\mathsf{P}_p\big(X(m) = i | X(n)\big) = \mathsf{P}_p\big(X(m) = i | X(n), X(n+1), \ldots\big)$$

P_p-almost surely for any $0 \leqslant m \leqslant n$, $n \geqslant 0$, $i \in S$, on account of the martingale convergence theorem we have already made use of in Section 2.2, we deduce that

$$\lim_{n \to \infty} \mathsf{P}_p\big(X(m) = j | X(n)\big) = \mathsf{P}_p\big(X(m) = j | \mathscr{T}\big), \quad m \geqslant 0,\ j \in S,$$

P_p-almost surely. In particular

$$\lim_{n \to \infty} \mathsf{P}_p\big(X(m) = j | X(n)\big) = \mathsf{P}_p\big(X(m) = j | A_r\big), \quad m \geqslant 0,\ j \in S,$$

on any P_p-atom A_r of $\mathscr{T}$.

When compared with (6), the last equation shows that the elements of a basis of the set of solutions $(p_n)_{n \geqslant 0}$ of (5) are precisely given by $p_n^{(r)} = \big(\mathsf{P}_p(X(n) = j | A_r)\big)_{j \in S}$, their number being thus equal to the number of P_p-atoms of $\mathscr{T}$. (The linear independence of the $(p_m^{(r)})_{m \geqslant 0}$ is easily established on account of the representation of the A_r given in Section 2.2.) In particular, there exists only one solution if and only if $\mathscr{T}$ is P_p-trivial.

5. References and comments

The finiteness of $\mathscr{T}$ was first explicitly proved by Cohn [4]. A clearer treatment including also the continuous parameter case was given by Iosifescu [10]. The proofs given in these two papers depend on estimating certain random quantities related to mixing coefficients. Subsequently, Senčenko [15] and Cohn [5] (unaware of Senčenko's paper) proved the fact that the number of atoms does not exceed the number of states of the chain. Cohn [6], to whom the present proof of Theorem 1 is essentially due, showed that theorem is in fact implicit in Blackwell [1]. It is worth noting that Blackwell was the first to make use of martingales in the study of nonhomogeneous Markov chains. The representation of atoms in Section 2.2 is his. Further, Kingman [12] devised a geometric approach to nonhomogeneous (both discrete and continuous parameter) finite state Markov processes, Theorem 1 resulting as a simple corollary of it. The idea of Kingman's treatment is to regard the matrix P_n, $n \geqslant 0$, as a linear operator $x \to P_n x$ acting on the space of column vectors $x = \big(x(i)\big)_{i \in S}$ and to study the compact convex images of the unit cube $[x\colon x = x(i)_{i \in S},\ 0 \leqslant x(i) \leqslant 1,\ i \in S]$ under composition of such linear operators associated with $P_n, P_{n+1}, \ldots, P_{n+m}$, $m \geqslant 0$. (Somewhat similar approaches are to be found in Caubet [3] and Senčenko [15].)

Theorem 2 is a special instance of a more general result by Blackwell and Freedman [2] for homogeneous denumerable state space recurrent Markov chains [result extended by Jamison and Orey [11] to homogeneous general state space Markov chains recurrent in Harris' sense]. The finiteness of the state space allowed the elementary given proof that is motivated by Senčenko's treatment. Theorem 3 seems to be new (at least as to its explicit statement). Of course, it does not extend to homogeneous denumerable state Markov chains having an infinite set of transient states.

It is worth while to note the recent work by Griffeath [9], who makes use of coupling techniques [that originate with the so-called Doeblin [8] "two particle method" for proving the basic ergodic theorem for homogeneous finite state Markov chains] to derive results on the structure of discrete or continuous-parameter Markov processes on a typically uncountable state space.

Since in the homogeneous case the tail structure is nothing but classification of states, it is natural to try to use Theorem 1 to set up a theory of classification of states for nonhomogeneous finite state Markov chains. (This assertion is to be associated with the remark made at the end of Section 3.1.) In this respect one should consult Cohn [7].

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LOCAL CHARACTERISTICS AND ABSOLUTE CONTINUITY CONDITIONS FOR *d*-DIMENSIONAL SEMI-MARTINGALES

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The purpose of what follows is to extend the results of [7] to the *d*-dimensional case. That paper [7] was devoted to the study of 1-dimensional semi-martingales. But, in the introduction, we asserted that extension to *d*-dimensional semi-martingales was obvious and straightforward. However, after reading Galtčuk [3] and Métivier–Pistone [9], we became aware that stochastic integrals with respect to *d*-dimensional martingales do not reduce to sums of *d* stochastic integrals with respect to 1-dimensional martingales. Consequently, extension of [7] is not as easy as it was said.

This short paper is naturally divided into two parts: in the first one, which follows ideas of Galtčuk, Métivier and Pistone, stochastic integrals and the Girsanov theorem for *d*-dimensional martingales are described. We always give complete proofs, but only for the facts which are strictly necessary for the second part, which gives the extension of [7] to the *d*-dimensional case. This second part heavily relies upon [7], and we only describe in which way proofs are to be modified.

1. Stochastic integrals with respect to *d*-dimensional martingales

1.1. Notations. We consider a measurable space $(\Omega, \mathscr{F})$ equipped with an increasing and right-continuous family $(\mathscr{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathscr{F}$. We denote by $\mathscr{P}$ the predictable σ-algebra of $\Omega\times[0, \infty[$ (for predictable processes and stopping times, we refer to Dellacherie [1]).

Let P be a probability measure on $(\Omega, \mathscr{F})$. If $\mathscr{C}(P)$ is any class of processes, we denote by $\mathscr{C}_{\text{loc}}(P)$ the class of those processes X for which there exists a sequence (T_n) of stopping times increasing P-a.s. to ∞, and such that X^{T_n} belongs to $\mathscr{C}(P)$ for each n (as usual, X^T is the process X "stopped at time T": $X_t^T = X_{T\wedge t}$). $\mathscr{V}^+(P)$ is the set of increasing, right-continuous processes $A = (A_t)$ such that $A_0 = 0$ P-a.s. and $E(A_\infty) < \infty$. Let $\mathscr{V}(P) = \mathscr{V}^+(P) - \mathscr{V}^+(P)$ be the set of differences of two processes of $\mathscr{V}^+(P)$.