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GENERALIZED LEFSCHETZ NUMBERS OF PUSHOUT MAPS DEFINED ON NON-CONNECTED SPACES

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Abstract. Let A, X_1 and X_2 be topological spaces and let $i_1 : A \to X_1$, $i_2 : A \to X_2$ be continuous maps. For all self-maps $f_A : A \to A$, $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ such that $f_1 i_1 = i_1 f_A$ and $f_2 i_2 = i_2 f_A$ there exists a *pushout map* f defined on the pushout space $X_1 \sqcup_A X_2$. In [F] we proved a formula relating the generalized Lefschetz numbers of f, f_A, f_1 and f_2. We had to assume all the spaces involved were connected because in the original definition of the generalized Lefschetz number given by Husseini in [H] the space was assumed to be connected. So, to extend the result of [F] to the not necessarily connected case, a definition of generalized Lefschetz number for a map defined on a not necessarily connected space is given; it reduces to the original one when the space is connected and it is still a trace-like quantity. It allows us to prove the pushout formula in this more general case and therefore to get a tool for computing Nielsen and generalized Lefschetz numbers in a wide class of spaces.

1. Introduction. As explained in the abstract, the aim of the paper is to give a proof of the pushout formula in the more general case where spaces are allowed to be non-connected. The main difference between the connected and the non-connected case is that if X is connected then so is the universal cover $\tilde{X}$ of X and the q-dimensional cellular chain group $C_q(\tilde{X})$ is a free finitely generated right $\mathbf{Z}\pi_1(X)$-module; on the other hand if X is disconnected then $\mathbf{Z}\pi_1(X;x)$ depends on the choice of the base point x and for all $x \in X$ the chain group $C_q(\tilde{X})$ is not a free $\mathbf{Z}\pi_1(X;x)$-module. Moreover we want to have a trace of the homomorphism $C_q(\tilde{f})$ where $\tilde{f} : \tilde{X} \to \tilde{X}$ and so a generalized Lefschetz number counting algebraically the number of fixed points of f. Hence we define the ring $\Lambda(X)$ which contains $\mathbf{Z}\pi_1(X;x_0)$ for every $x \in X$ and we prove that $C_q(\tilde{X})$ is a finitely generated projective $\Lambda(X)$-module. It is a free $\Lambda(X)$-module if and only if X is connected. In any case, it is possible to define traces following [S], [H] and the generalized Lefschetz numbers for non-connected spaces.

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We first have to extend the notion of Reidemeister classes to the case of a ring which could be not a group ring. This is done in section 2.1; in the same section we prove other propositions which will be used later. In the following sections we prove standard properties of traces and Lefschetz numbers in this algebraic setting. In section 3.1 we define the generalized Lefschetz number of a map defined on a finite CW-complex even when X is not connected and give the relation between this case and the connected one.

In section 3.2 we give standard definitions and examples of pushout construction and some preliminary facts. Finally in section 4 we give the statement and proof of the pushout formula.

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2. Algebraic preliminaries

2.1. The Reidemeister group of a ring homomorphism. In this section we will introduce a generalization of the classical Reidemeister set defined for group homomorphisms and will show some simple facts that will be needed later.

Let Λ be a ring (with unit element) and $f : \Lambda \to \Lambda$ be an endomorphism of Λ. Let $(\Lambda)_f$ denote the subgroup of Λ generated by all the elements $\lambda_1\lambda_2 - \lambda_2 f(\lambda_1)$ with $\lambda_1, \lambda_2 \in \Lambda$. We define the *Reidemeister group of f* as the additive group of Λ modulo $(\Lambda)_f$ and we will denote it with $\mathcal{R}(f)$. We will denote by $[\lambda]$ the obvious projection of λ in $\mathcal{R}(f)$.

Example 1. If $f = 1_\Lambda$ then $\mathcal{R}(f)$ is the group defined in [S], page 130.

Example 2. If $\Lambda = \mathbf{Z}G$ is the group ring of a group G over the ring of integers $\mathbf{Z}$ and $f = \mathbf{Z}\varphi$ is the linear extension of a group endomorphism $\varphi : G \to G$ then $\mathcal{R}(f)$ is the free abelian group generated by the set $R(\varphi)$ of orbits in G of the action of G over G defined by $g \cdot x := gx\varphi(g^{-1})$ $(\forall g, x \in G)$. In other words it is the classical Reidemeister set of a group homomorphism (see e.g. [B], [FH], [H], [J]).

Proof. Let $R(\varphi)$ denote the orbit set and $[g]$ denote the orbit of $g \in G$. The context will make clear whether $[g]$ is seen as an element of $\mathcal{R}(f)$ or of $R(\varphi)$. We want to prove that $\mathcal{R}(f) \equiv \mathbf{Z}R(\varphi)$. Let $p_0 : G \to \mathbf{Z}R(\varphi)$ be defined by $p_0(g) = [g]$ for each $g \in G$ and let p be the linear extension of p_0 to $\mathbf{Z}G$. Because p is onto, it suffices to prove that $\mathrm{Ker}(p) = (\Lambda)_f$. But $g_1g_2 = g_1g_2f(g_1)f(g_1^{-1})$ and hence $p(g_1g_2 - g_2f(g_1)) = 0$ $(\forall g_1g_2 \in G)$ therefore $\mathrm{Ker}(p) \supseteq (\Lambda)_f$. Now let $\lambda \in \mathrm{Ker}(p)$; this means that $\lambda = n_1g_1 + n_2g_2 + \ldots + n_kg_k$ with $n_i \in \mathbf{Z}$ and $g_i \in G$ for all $i = 1, \ldots, k$ and that $\sum_{i=1}^k n_i[g_i] = 0$. Up to rearranging indices, we can suppose that $[g_1] = [g_2] = \ldots = [g_{k_1}]$, $[g_{k_1+1}] = \ldots = [g_{k_2}]$, …, $[g_{k_l+1}] = \ldots = [g_k]$ with $1 \le k_1 \le k_2 \le \ldots \le k_l \le k$ suitable integers. In other words $\lambda = \sum_{j=0}^l \mu_j$ where $\mu_j = \sum_{i=k_j+1}^{k_{j+1}} n_{k_j+i}g_{k_j+i}$ and $k_0 = 0$, $k_{l+1} = k$. As $\mathbf{Z}R(\varphi)$ is free, $p(\lambda) = 0$ implies $p(\mu_j) = 0$ for all j. Therefore it is enough to prove that $\lambda \in (\Lambda)_f$ in the simple case $k_1 = k$ when $[g_1] = \ldots = [g_k]$. In this case there exist elements $\xi_2, \ldots, \xi_k \in G$ such that $g_i = \xi_i g_1 \varphi(\xi_i^{-1})$ for $i = 2, \ldots, k$ and $n_1 + \sum_{i=2}^k n_i = 0$. Hence

$$\lambda = n_1g_1 + n_2g_2 + \ldots + n_kg_k = -\sum_{i_2}^{k} n_i g_1 + \sum_{i=2}^{k} n_i \xi_i g_1 \varphi(\xi_i^{-1})$$

$$= -\sum_{i=2}^{k} n_i((\xi_i^{-1})(\xi_i g_1) - (\xi_i g_1)(\varphi(\xi_i^{-1})))$$

which is an element of $(\Lambda)_f$. ∎

Consider the rings Λ_1, Λ_2 and the following commutative diagram of ring homomorphisms:

$$\begin{tikzcd} \Lambda_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"'] & \Lambda_2 \arrow[d, "f_2"] \\ \Lambda_1 \arrow[r, "\varphi_2"] & \Lambda_2 \end{tikzcd}$$

If $\Lambda_1 = \sum_{i=1}^p \Lambda_1^i$ (direct sum) with $f_1(\Lambda_1^i) \subset \Lambda_1^i$ and $(\Lambda_1^i)_{f_1|_{\Lambda_1^i}} = (\Lambda_1)_{f_1} \cap \Lambda_1^i$ for each $i = 1, \dots, p$ we will say that Λ_1 is well-decomposed. This implies that $\mathcal{R}(f_1) = \sum_{i=1}^p \mathcal{R}(f_1|_{\Lambda_1^i})$.

Proposition 2.1. *If Λ_1 is well-decomposed into $\sum_{i=1}^p \Lambda_1^i$ and for every $i = 1, \dots, p$ there exists $\theta_i \in \Lambda_2$ such that for all $\lambda_i \in \Lambda_1^i$*

$$\theta_i \varphi_2(\lambda_i) = \varphi_1(\lambda_i)\theta_i$$

then there exists a well-defined group homomorphism $\theta_* : \mathcal{R}(f_1) \to \mathcal{R}(f_2)$ given by $\theta_*([\sum_{i=1}^p \lambda_i]) = \sum_{i=1}^p [\theta_i \varphi_2(\lambda_i)]$ with $\lambda_i \in \Lambda_1^i$ for each $i = 1, \dots, p$.

Proof. Let us prove that θ_* is well defined. We must show that $\theta_*([\eta]) = 0$ for all $\eta \in (\Lambda_1)_{f_1}$. Because for $i = 1, \dots, p$ we have by hypothesis that $(\Lambda_1^i)_{f_1|_{\Lambda_1^i}} = (\Lambda_1)_{f_1} \cap \Lambda_1^i$, then $\theta_*([\eta]) = 0$ for all $\eta \in (\Lambda_1)_{f_1}$ if and only if $\theta_*([\eta]) = 0$ for all $\eta \in (\Lambda_1^i)_{f_1|_{\Lambda_1^i}}$ for $i = 1, \dots, p$.

Therefore for all $\lambda, \mu \in \Lambda_1^i$ the following equalities hold:

$$\begin{aligned} \mathcal{R}(f_2) \ni [\theta_i \varphi_2(\lambda\mu - \mu f_1(\lambda))] &= [\theta_i \varphi_2(\lambda)\varphi_2(\mu) - \theta_i \varphi_2(\mu) f_2(\phi_1(\lambda))] \\ &= [\theta_i \varphi_2(\lambda)\varphi_2(\mu) - \varphi_1(\lambda)\theta_i \varphi_2(\mu)] \\ &= [\theta_i \varphi_2(\lambda)\varphi_2(\mu) - \theta_i \varphi_2(\lambda)\varphi_2(\mu)] = [0] \end{aligned}$$

and hence θ_* is well-defined on $\mathcal{R}(f_1)$. By trivial arguments it can be shown that θ_* is a group homomorphism. ∎

We note that if $\varphi_1 = \varphi_2$ we can always define $\varphi_* := \theta_*$ by setting $\theta = 1$ on the whole Λ_1. In this case $\varphi_*([\lambda_1]) = [\varphi_2(\lambda_1)]$ for all $\lambda_1 \in \Lambda_1$.

Here we prove some elementary properties of Reidemeister groups.

Commutativity: If $g : \Lambda_1 \to \Lambda_2$ and $f : \Lambda_2 \to \Lambda_1$ are ring homomorphisms, then $g_* : \mathcal{R}(fg) \to \mathcal{R}(gf)$ is an isomorphism. In fact $f_* : \mathcal{R}(gf) \to \mathcal{R}(fg)$ is its inverse: $f_* g_*([\lambda]) = [fg(\lambda)] = [\lambda]\ \forall \lambda \in \Lambda_1$ and $g_* f_*([\mu]) = [\mu]\ \forall \mu \in \Lambda_2$.

Conjugacy: If θ is a unit element in the ring Λ and $f : \Lambda \to \Lambda$ is a ring endomorphism we can define $\theta^{-1} f \theta$ as the endomorphism defined by $\theta^{-1} f \theta(\lambda) := \theta^{-1} f(\lambda)\theta$ for all $\lambda \in \Lambda$. Moreover if we define $\varphi_1(\lambda) := \lambda$ and $\varphi_2(\lambda) := \theta^{-1}\lambda\theta$ we get that $\varphi_2 f = \theta^{-1} f \theta \varphi_1$ and

$\theta\varphi_2(\lambda) = \varphi_1(\lambda)\theta \ \forall\lambda \in \Lambda$. Therefore $\varphi_* = \theta_*$ is a well-defined isomorphism $\varphi_* = \theta_* : \mathcal{R}(f) \to \mathcal{R}(\theta^{-1} f\theta)$.

2.2. The trace. In this section we introduce some basic facts about the trace of f-homomorphisms defined on projective modules, following the lines of [B], [H], [S].

Let Λ be a ring and M_1, M_2 be finitely generated right projective Λ-modules. An additive function $F : M_1 \to M_2$ is an f-homomorphism of M if f is an endomorphism of Λ and $F(x\lambda) = F(x)f(\lambda)$ for all $x \in M_1$ and $\lambda \in \Lambda$. If $M_1 = M_2$, F is an f-endomorphism. We want to define a trace function on the (additive) group of all such f-endomorphisms.

Let $\mathcal{M}_{p,q}(\Lambda)$ denote the group of all $p \times q$ matrices with entries in Λ. For any matrix F let F^f denote the matrix obtained by applying f to each of the entries of F.

Proposition 2.2. *For every integer p there exists a unique group homomorphism*

$$Tr_f : \mathcal{M}_{p,p} \to \mathcal{R}(f)$$

such that

$$Tr_f(FG) = Tr_f(GF^f)$$

for all $F \in \mathcal{M}_{q,p}$ and $G \in \mathcal{M}_{p,q}$. It is given by $Tr_f(F) = \sum_{i=1}^{p}[F_{ii}]$ where F_{ij} are the entries of F.

Proof. If $p = 1$, then $Tr_f : \Lambda \to \mathcal{R}(f)$ is just the projection $\lambda \to [\lambda]$. Let I_p be the $p \times p$ square matrix with the diagonal entries equal to $1 \in \Lambda$ and with the non-diagonal equal to 0. Let 0 be any matrix of zeros.

If $A \in \mathcal{M}_{p,p}$, $B \in \mathcal{M}_{q,q}$, $X \in \mathcal{M}_{p,q}$ and $Y \in \mathcal{M}_{q,p}$ then

$$Tr_f\begin{pmatrix} A & X \\ Y & B \end{pmatrix} = Tr_f\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + Tr_f\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} + Tr_f\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} + Tr_f\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

by additivity. But

$$Tr_f\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = Tr_f\begin{pmatrix} X \\ 0 \end{pmatrix}\begin{pmatrix} 0 & I_p \end{pmatrix} = Tr_f\begin{pmatrix} 0 & I_p \end{pmatrix}\begin{pmatrix} X^f \\ 0 \end{pmatrix} = 0$$

and similarly $Tr_f\left(\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right) = 0$. Moreover

$$\begin{aligned} Tr_f\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= Tr_f\begin{pmatrix} I_p \\ 0 \end{pmatrix}\begin{pmatrix} A & 0 \end{pmatrix} \\ &= Tr_f\begin{pmatrix} A & 0 \end{pmatrix}\begin{pmatrix} I_p^f \\ 0 \end{pmatrix} = Tr_f(AI_p^f) = Tr_f(I_pA) = Tr_f(A) \end{aligned}$$

and similarly $Tr_f\left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right) = Tr_f(B)$. Thus $Tr_f\left(\begin{smallmatrix} A & X \\ Y & B \end{smallmatrix}\right) = Tr_f(A) + Tr_f(B)$ and inductively $Tr_f(F) = \sum_{i=1}^{p}[F_{ii}]$ as required. It is clear that this function satisfies the hypotheses of the proposition. ∎

Now let us suppose that M is a free finitely generated Λ-module and $F : M \to M$ is an f-endomorphism. For any choice of a free Λ-basis of M there is a $p \times p$ matrix $\bar{F}$ with entries in Λ representing F. Let us define $Tr_f(F) := Tr_f(\bar{F}) = \sum_{i=1}^{p}[\bar{F}_{ii}]$. The definition is consistent: let $\{e_1, e_2, \ldots, e_p\}$ and $\{e'_1, e'_2, \ldots, e'_p\}$ be two bases. Let $\bar{F}_{ij}$ and $\bar{F}'_{ij}$ be the entries of the matrices $\bar{F}$ and $\bar{F}'$ representing F in these two bases. This means that $F(e_j) = \sum_{i=1}^{p} e_i \bar{F}_{ij}$ and $F(e'_j) = \sum_{i=1}^{p} e'_i \bar{F}'_{ij}$. Moreover, $e'_i = \sum_{h=1}^{p} e_h A_{hi}$ for a suitable invertible matrix A with entries $A_{ij} \in \Lambda$. Therefore $F(e'_j) = \sum_{i,h=1}^{p} e_h A_{hi} \bar{F}'_{ij}$ and $F(e'_j) = \sum_{i=1}^{p} F(e_i) f(A_{ij}) = \sum_{i,h=1}^{p} e_h \bar{F}_{hi} f(A_{ij})$ or equivalently the identity $A\bar{F}' = \bar{F}A^f$ holds true. Let A^{-1} denote the inverse of A. Then $\bar{F}' = A^{-1}\bar{F}A^f$ and hence $Tr_f(\bar{F}') = Tr_f(A^{-1}\bar{F}A^f) = Tr_f(AA^{-1}\bar{F}) = Tr_f(\bar{F})$ for the defining property of trace in proposition 2.2.

Let Λ be a ring and let M be a finitely generated projective right Λ-module. Let $F : M \to M$ be an f-endomorphism. Then there exists a Λ-module Q such that $M \oplus Q$ is a free finitely generated Λ-module and an f-endomorphism $F + 0 : M \oplus Q \to M \oplus Q$ defined by $(F + 0)(x + y) = F(x)$ for all $x \in M$ and $y \in Q$. It is an f-endomorphism of free finitely generated modules, hence there is a well-defined trace, and we define $Tr_f(F) := Tr_f(F + 0)$.

It does not depend on the choice of Q: if $M \oplus Q'$ is also free finitely generated, let us consider the free finitely generated Λ-module $M \oplus Q \oplus M \oplus Q'$ with the f-endomorphism $F + 0_Q + 0_M + 0_{Q'} : x + y + z + w \to F(x)$ for all $x, z \in M$, $y \in Q$ and $w \in Q'$. Using the same argument of the proof of proposition 2.2 it is easy to see that $Tr_f(F + 0_Q) = Tr_f(F + 0_Q + 0_M + 0_{Q'}) = Tr_f(F + 0_{Q'})$ thus it is well defined even in the case where M is a finitely generated projective right Λ-module.

Proposition 2.3 (Commutativity). *Let Λ be a ring and M_1, M_2 be two finitely generated projective right Λ-modules. Let $F : M_1 \to M_2$ and $G : M_2 \to M_1$ be respectively an f-endomorphism and a g-endomorphism, with f and g endomorphisms of Λ. Then*

$$f_* Tr_{gf}(GF) = Tr_{fg}(FG)$$

where $f_* : \mathcal{R}(gf) \to \mathcal{R}(fg)$ is the homomorphism defined in section 2.1.

Proof. Let Q_1 and Q_2 be Λ-modules such that $M_1 \oplus Q_1$ and $M_2 \oplus Q_2$ are free finitely generated. With the same notation as above, it is easily seen that

$$Tr_f((G + 0_{Q_2})(F + 0_{Q_1})) = Tr_f(GF),$$
$$Tr_f((F + 0_{Q_1})(G + 0_{Q_2})) = Tr_f(FG),$$

hence substituting M_1 with $M_1 \oplus Q_1$ and M_2 with $M_2 \oplus Q_2$ we can suppose M_1 and M_2 to be free. Let $e_1, \ldots, e_p$ be a free basis of M_1 and $e'_1, \ldots, e'_q$ be a free basis of M_2. Then $F(e_j) = \sum_{i=1}^{q} e'_i F_{ij}$ and $G(e'_j) = \sum_{i=1}^{p} e_i G_{ij}$. Hence

$$GF(e_j) = \sum_{i=1}^{q} G(e'_i) g(F_{ij}) = \sum_{i=1}^{q} \sum_{h=1}^{p} e_h G_h i g(F_{ij})$$

and similarly

$$FG(e'_j) = \sum_{i=1}^{p} \sum_{h=1}^{q} e'_h F_{hi} f(F_{ij})$$

Moreover

$$f_*(Tr_{gf}(GF)) = \sum_{i=1}^{q} \sum_{h=1}^{p} [f(G_h i) fg(F_{ih})] = \sum_{i=1}^{q} \sum_{h=1}^{p} [F_{ih} f(G_h i)] = Tr_{fg}(FG)$$

hence the assertion. ∎

2.3. The Lefschetz number of an f-endomorphism of a Λ-complex. Let Λ be a ring and let $\mathcal{C} = \{C_n \to \ldots \to C_0\}$ be a finite projective Λ-complex, i.e. a finite-dimensional chain complex of finitely generated projective right Λ-modules. Let $f : \Lambda \to \Lambda$ be a given ring endomorphism; an f-endomorphism $F : \mathcal{C} \to \mathcal{C}$ of the Λ-complex $\mathcal{C}$ is any set of f-endomorphisms $F_n : C_i \to C_i$ for $i = 0, \ldots, n$ which commute with the boundary homomorphisms. The traces $Tr_f(F_i)$ are well-defined. The *Lefschetz number* of F is defined to be

$$\mathcal{L}(F) = \sum_{q \geq 0} (-1)^q Tr_f(F_q)$$

and it is an element of $\mathcal{R}(f)$.

PROPOSITION 2.4 (Homotopy). *If $F, G : C \to C$ are chain-homotopic f-endomorphisms then $\mathcal{L}(F) = \mathcal{L}(G)$.*

PROOF. Let $d_i : C_i \to C_{i+1}$ and $\partial_i : C_{i+1} \to C_i$ for $i = 1, \ldots, n$ be the chain homotopy between F and G and the boundary homomorphisms; let us recall that d_i simply is a Λ-homomorphism of Λ-modules. By additivity $\mathcal{L}(F) - \mathcal{L}(G) = \mathcal{L}(F - G)$ and

$$\begin{aligned} \mathcal{L}(F - G) &= \sum_{q \geq 0} (-1)^q (Tr_f(\partial_{i+1} d_i) + Tr_f(d_{i-1} \partial_i)) \\ &= \sum_{q \geq 0} (-1)^q (Tr_f(\partial_{i+1} d_i) - Tr_f(d_i \partial_{i+1})) = 0 \end{aligned}$$

by commutativity of Tr_f as in proposition 2.3. Hence $\mathcal{L}(F) = \mathcal{L}(g)$. ∎

PROPOSITION 2.5 (Commutativity). *Let Λ be a ring and C, C' be two chain complexes of finitely generated projective right Λ-modules. Let $F : C \to C'$ and $G : C' \to C$ be respectively an f-endomorphism and a g-endomorphism, with f and g endomorphisms of Λ. Then*

$$f_* \mathcal{L}(GF) = \mathcal{L}(FG)$$

where $f_* : \mathcal{R}(gf) \to \mathcal{R}(fg)$ is the homomorphism defined in section 2.1.

PROOF. It is a trivial corollary of 2.3. ∎

3. Topological preliminaries

3.1. The generalized Lefschetz number of a continuous self-map on a finite CW-complex. Let X be a finite CW-complex. We are not requiring it to be connected. Let $X^1, X^2, \ldots, X^p$ be its connected components and let $x^i \in X^i$ be a base point of X^i for each $i = 1, \ldots, p$. Let $\Lambda(X)$ denote the free abelian group generated by the elements of the fundamental groups of these components, i.e.

$$\Lambda(X) := \mathbf{Z}\pi_1(X^1, x^1) \oplus \mathbf{Z}\pi_1(X^2, x^2) \oplus \ldots \oplus \mathbf{Z}\pi_1(X^p, x^p)$$

and let the product in $\Lambda(X)$ be defined by the linear extension of

$$gh = \begin{cases} gh & \text{if } g, h \in \pi_1(X^i, x^i) \text{ for some } i, \\ 0 & \text{if } g \in \pi_1(X^i, x^i) \text{ and } h \in \pi_1(X^j, x^j) \text{ with } i \neq j. \end{cases}$$

If X is connected then $\Lambda(X) = \mathbf{Z}\pi_1(X)$ is simply the group ring of the fundamental group of X.

Let 1_i denote the constant loop in $\pi_1(X^i, x^i)$. Then $1 := \sum_{i=1}^{p} 1_i$ is the unit element of Λ.

Let $f : X \to X$ be a self-map. Let $J := \mathbf{Z}_p \times I$ be the cartesian product of the set of the first p integers $\mathbf{Z}_p = \{1, 2, \ldots, p\}$ with discrete topology and the unit interval $I = [0, 1]$. A continuous map $w : J \to X$ is called a *base multipath* if for all $j = 1, \ldots, p$ there exist j' such that

$$w(j, 0) = x_{j'}, \quad w(j, 1) = f(x_j).$$

Let us note that j' is uniquely determined once we have the second identity; it is because $X^{j'}$ is connected. We say that the self-map f is *multipath-based* if a base multipath w has been chosen and we denote it by (f, w). Up to rearranging indices it is always possible to assume that $f(x^i) \in X^i$ for $i = 1, \ldots, p_0$ and $f(x^i) \in X^{i'}$ with $i' \neq i$ for $i = p_0 + 1, \ldots, p$.

For any multipath-based self-map $(f, w) : X \to X$ there is an induced endomorphism $f_\Lambda : \Lambda(X) \to \Lambda(X)$ defined as the linear extension of

$$f_\Lambda(g_i) = 1_i f_\pi(g_i)$$

if $g_i \in \pi_1(X^i, x^i)$ and $f_\pi : \pi_1(X^i, x^i) \to \pi_(X^{i'}, x^{i'})$ is defined by

$$f_\pi(\alpha) = w(i, -) f(\alpha) w(i, -)^{-1}$$

where $\alpha : (I, \partial I) \to (X^i, x^i)$ is a loop in X^i and $w(i, -) : I \to \{i\} \times I \xrightarrow{w} X^{i'}$ is the path in $X^{i'}$ from $x^{i'}$ to $f(x^i)$ we have previously chosen. In other words $f_\Lambda(g_i) = f_\pi(g_i)$ if $1 \leq i \leq p_0$ and $f_\Lambda(g_i) = 0$ if $p_0 + 1 \leq i \leq p$.

Let $\tilde{X}$ be the universal covering space of X. It is the disjoint union of the universal covering spaces of $X^1, \ldots, X^p$. If the set of paths $PX := \{\lambda : (I, \{0\}) \to (X, \{x^1, \ldots, x^p\})\}$ is endowed with the compact-open topology, then $\tilde{X}$ is the quotient space of PX under the relation of homotopy equivalence relative to endpoints. Therefore we can view a point in $\tilde{X}$ as a homotopy class of paths $[\lambda]$. For any $g \in \pi_1(X_i, x_i)$ let

$$[\lambda]g = \begin{cases} [g^{-1}\lambda] & \text{if } \lambda(0) = x_i, \\ [\lambda] & \text{if } \lambda(0) \neq x_i, \end{cases}$$

be defined as above. The map $[\lambda] \to [\lambda]g$ is the cellular homeomorphism of $\tilde{X}$ induced by g.

For every integer $q \geq 0$ let $C_q(\tilde{X})$ denote the q-th cellular chain group $C_q(\tilde{X}) = H_q(\tilde{X}^{(q)}, \tilde{X}^{(q-1)}; \mathbf{Z})$ where $\tilde{X}^{(q)}$ is the q-dimensional skeleton of $\tilde{X}$ for all positive integers q. We know that $C_q(\tilde{X}) = C_q(\tilde{X}^1) \oplus \ldots \oplus C_q(\tilde{X}^p)$.

Let $\Lambda(X)$ act on $C_q(\tilde{X})$ on the right by extending linearly the function defined for each $x \in C_q(\tilde{X}^i)$ and $g \in \pi_1(X^j, x^j)$ by

$$xg = \begin{cases} C_q(g)(x) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $C_q(g) : C_q(\tilde{X}^i) \to C_q(\tilde{X}^i)$ is the homomorphism induced by the map $[\lambda] \in \tilde{X}^1 \to [g^{-1}\lambda] \in \tilde{X}^1$. Thus $C_q(\tilde{X})$ is a right Λ-module. If X is connected, it is free and finitely generated. In our general setting a weaker proposition holds.

Proposition 3.1. *The q-th cellular chain group $C_q(\tilde{X})$ is a finitely generated projective right $\Lambda(X)$-module.*

Proof. We have to prove that each $C_q(\tilde{X}^i)$ is a finitely generated projective $\Lambda(X)$-module. We already know that for each $i = 1, \ldots, p$, $C_q(\tilde{X}^i)$ is a free finitely generated $\Lambda(X^i)$-module. Let $\{e_1, \ldots, e_k\}$ be a free basis. Just by taking the projection $pr_i : \Lambda(\tilde{X}) = \Lambda(\tilde{X}^1) \oplus \ldots \oplus \Lambda(\tilde{X}^p) \to \Lambda(\tilde{X}^i)$ we can define a right action of $\Lambda(X)$ on $C_q(\tilde{X}^i)$ by $x\,\lambda := x\, pr_i(\lambda)$ for each $x \in C_q(\tilde{X}^i)$ and each $\lambda \in \Lambda(X)$. Hence $C_q(\tilde{X}^i)$ is a right finitely generated $\Lambda(X)$-module. Let $\Lambda^i := \sum_{j \neq i} \Lambda(\tilde{X}^j)$ be the complement of $\Lambda(\tilde{X}^i)$ in $\Lambda(X)$. Let Q_i be the direct sum of k copies of Λ^i. Let $\Lambda(X)$ act on Q_i by the usual ring product in $\Lambda(X)$ and distributive law. Therefore

$$C_q(\tilde{X}^i) \oplus Q_i \cong \bigoplus_{u=1}^{k} (\Lambda(\tilde{X}^u) \oplus \Lambda^u)$$

and hence it is a free finitely generated right $\Lambda(X)$-module. ∎

If (f, w) is a multipath-based cellular self-map of X then there is a canonical cellular lifting of (f, w), namely $\tilde{f} : \tilde{X} \to \tilde{X}$, defined by $\tilde{f}([\lambda]) = [w(i, -)f(\lambda)]$ for each path $\lambda : (I, 0) \to (X^i, x^i)$. It induces an endomorphism $C_q(\tilde{f}) : C_q(\tilde{X}) \to C_q(\tilde{X})$ at the cellular chain group level. Let $P : C_q(\tilde{X}) = \bigoplus_{i=1}^{p} C_q(\tilde{X}^i) \to C_q(\tilde{X})$ be the homomorphism defined by

$$P(x) := \begin{cases} x & \text{if } x \in C_q(\tilde{X}^i) \text{ with } i \leq p_0, \\ 0 & \text{if } x \in C_q(\tilde{X}^i) \text{ with } i \geq p_0 + 1. \end{cases}$$

It will be called the projection homomorphism for $C_q(\tilde{X})$.

It is easy to see that the composition $C_q(\tilde{f})P$ is an f_Λ-endomorphism, where $f_\Lambda : \Lambda(X) \to \Lambda(X)$ is defined as above. Therefore we can define the generalized Lefschetz number of the multipath-based cellular self-map (f, w) as the Lefschetz number of $C_q(\tilde{f})P$

$$\mathcal{L}(f, w) = \sum_{q \geq 0} (-1)^q Tr_{f_\Lambda}[C_q(\tilde{f})P]$$

which is an element of $\mathcal{R}(f_\Lambda)$(see [H], [FH]). Let us note that when $p = 1$ this is the generalized Lefschetz number as defined in [H].

It is expected that $\mathcal{L}(f, w)$ is independent of the base multipath w and depends only on the homotopy class of f. This is truly the case: for $i = 1, \ldots, p$, let $x'^i \in X^i$ be another base point and $w' : J \to X$ another corresponding base multipath. The paths $w(i, -)$ and $w'(i, -)$ will be denoted simply with w_i and w'_i. Let $\tilde{X}'$ denote the universal covering space pointed at $x'^1, \ldots, x'^p$ and $\tilde{f}' : \tilde{X}' \to \tilde{X}'$ the canonical lifting of f at $\tilde{X}'$. The rings

$$\Lambda(X) := \mathbf{Z}\pi_1(X^1, x^1) \oplus \mathbf{Z}\pi_1(X^2, x^2) \oplus \ldots \oplus \mathbf{Z}\pi_1(X^p, x^p),$$
$$\Lambda'(X) := \mathbf{Z}\pi_1(X^1, x'^1) \oplus \mathbf{Z}\pi_1(X^2, x'^2) \oplus \ldots \oplus \mathbf{Z}\pi_1(X^p, x'^p)$$

are given. For each $i = 1, \ldots, p$, let $\gamma_i : (I, 0, 1) \to (X^i, x^i, x'^i)$ be a continuous path from x^i to x'^i. Let $\varphi_1, \varphi_2 : \Lambda(X) \to \Lambda'(X)$ be defined by extending linearly $\varphi_1(g) := \gamma_i^{-1} g \gamma_i$ and $\varphi_2(g) := w'_i f(\gamma_i^{-1}) w_i^{-1} g w_i f(\gamma_i) w_i'^{-1}$ if $g \in \pi_1(X^i, x^i) \subseteq \Lambda(X)$ and $f(x^i) \in X^i$. Otherwise $\varphi_1(g) := \varphi_2(g) := \gamma_i^{-1} g \gamma_i$. It is easy to see that $\varphi_2 f_\Lambda = f_{\Lambda'} \varphi_1$.

Let us note that $\Lambda(X)$ and $\Lambda'(X)$ are well-decomposed (see section 2.2) into

$$\bigoplus_{i=1}^{p} \Lambda(X^i) \quad \text{and} \quad \bigoplus_{i=1}^{p} \Lambda'(X^i)$$

respectively, and that if we set

$$\theta_i := \gamma_i^{-1} w_i f(\gamma_i) {w'_i}^{-1}$$

if $f(x_i) \in X_i$ and otherwise $\theta_i := 1$, then the identity $\theta_i \varphi_2(\lambda_i) = \varphi_1(\lambda_i)\theta_i$ holds true for all $\lambda_i \in \Lambda(X_i)$. Therefore there exists a well-defined group homomorphism $\theta_* : \mathcal{R}(f_\Lambda) \to \mathcal{R}(f_{\Lambda'})$ defined as in section 2.1.

Let $\Phi_1, \Phi_2 : \tilde{X} \to \tilde{X}'$ be the homeomorphisms defined by $\Phi_1([\lambda]) := [\gamma_i^{-1}\lambda]$ if $\lambda(0) = x^i$ and $\Phi_2([\lambda]) := [w'_i f(\gamma_i^{-1}) w_i^{-1}\lambda]$ if $\lambda(0) = x^i$ and $f(x^i) \in X^i$; otherwise $\Phi_2([\lambda]) := \Phi_1([\lambda])$. We can see that at the cellular complex level $C_q(\Phi_2)C_q(\tilde{f})P = C_q(\tilde{f}')P'C_q(\Phi_1)$ where $P : C_q(\tilde{X}) \to C_q(\tilde{X})$ and $P' : C_q(\tilde{X}') \to C_q(\tilde{X}')$ are defined as above. Moreover $C_q(\Phi_1)$ and $C_q(\Phi_2)$ are a φ_1 and φ_2-homomorphism respectively which satisfy the identity

$$C_q(\Phi_2)(x) = C_q(\Phi_1)(x) \cdot \theta_i$$

for all $x \in C_q(\tilde{X}_i)$. We have

$$C_q(\tilde{f}')P' = C_q(\phi_2)C_q(\tilde{f})PC_q(\Phi_1^{-1})$$

and hence by commutativity

$$\mathcal{L}(f, w') = \mathcal{L}(C_q(\tilde{f}')P') = \varphi_{1*}\mathcal{L}(C_q(\Phi_1^{-1})C_q(\Phi_2)C_q(\tilde{f})P).$$

But $C_q(\Phi_2)(x) = C_q(\phi_1)(x) \cdot \theta_i$ for each $x \in C_q(\tilde{X}_i)$; therefore

$$C_q(\Phi_1^{-1})C_q(\phi_2)(x) = x \cdot \varphi_1^{-1}(\theta_i)$$

for all $x \in C_q(\tilde{X}_i)$. Hence

$$\varphi_{1*}\mathcal{L}(C_q(\Phi_1^{-1})C_q(\Phi_2)C_q(\tilde{f})P) = \theta_*\mathcal{L}(C_q(\tilde{f})P) = \theta_*\mathcal{L}(f, w)$$

where θ_* is the isomorphism defined in section 2.1.

If $H : f \sim f'$ is a cellular homotopy then it can be shown that H induces an isomorphism $H_* : \mathcal{R}(f_\Lambda) \to \mathcal{R}(f'_\Lambda)$ such that $\mathcal{L}(f', w') = H_*\mathcal{L}(f, w)$ for suitable base multipaths w and w'. H_* can be defined by considering the chain homotopy at the chain complex level, in the same way as in [H]. We prefer to give a slightly different proof which follows the lines of [F]. Let $\bar{H} : X \times I \to X \times I$ be a cellular approximation of the fat homotopy (cf. [J], [B]) such that $\bar{H}(-, 0) = f$ and $\bar{H}(-, 1) = f'$. Let w, w' be base multipaths for f and f' respectively with the same base points $x^1, \dots, x^p$. It is easy to see that, if $i_0, i_1 : X \to X \times I$ are defined by $i_0(x) := (x, 0)$ and $i_1(x) := (x, 1)$ for all $x \in X$, then $i_{0*}(\mathcal{L}(f, w)) = \mathcal{L}(\bar{H}, i_0(w))$ and $i_{1*}(\mathcal{L}(f', w')) = \mathcal{L}(\bar{H}, i_1(w'))$. Let $\gamma_i : (I, 0, 1) \to \big(X \times I, (x^i, 0), (x^i, 1)\big)$ be the vertical path from $(x^i, 0)$ to $(x^i, 1)$. Then for the previous arguments there exists an isomorphism θ_* such that $\theta_*(\mathcal{L}(\bar{H}, i_0(w))) = \mathcal{L}(\bar{H}, i_1(w'))$. We can therefore define

$$H_* := i_{1*}^{-1}\theta_* i_{0*} : \mathcal{R}(f_\Lambda) \to \mathcal{R}(f'_\Lambda)$$

which coincides with the one defined in [F] if X is connected.

Let us remark that such an isomorphism exists even if f is not cellular; in this case $\mathcal{L}(f, w)$ is not yet defined, but at the $\mathcal{R}(f_\Lambda)$-level everything works. So if (f, w) is not cellular we can define $\mathcal{L}(f, w) := H_*\mathcal{L}(f', w')$ where $f' : X \to X$ is any cellular approximation of f and H is the homotopy between f and f' and w' is a base multipath for f'; it turns out that $\mathcal{L}(f, w)$ does not depend on the choice of f'.

Proposition 3.2. *Let $X^1, \ldots, X^p$ be the connected components of X, with base points $x^i \in X^i$. Let us suppose that $f(x^i) \in X^i$ for $i = 1, \ldots, p_0$ and $f(x^i) \in X^j$ with $j \neq i$ for $i = p_0, \ldots, p$. Let $w : J \to X$ be a base multipath for f, and $w_i := w(i, -)$. Let $f_i : X^i \to X^i$ be the restriction of f to X^i for $i = 1, \ldots, p_0$. Then*

$$\mathcal{R}(f_\Lambda) = \bigoplus_{i=1}^{p_0} \mathcal{R}(f_{i\Lambda}(X^i)) \quad \textit{and} \quad \mathcal{L}(f, w) = \sum_{i=1}^{p_0} \mathcal{L}(f_i, w_i)$$

where $\mathcal{L}(f_i, w_i)$ is the generalized Lefschetz number of $f_i : X^i \to X^i$.

Proof. It is trivial to check that $\Lambda(X)$ is well-decomposed into $\Lambda(X_1), \ldots, \Lambda(X_p)$. Therefore

$$\mathcal{R}(f_\Lambda) = \bigoplus_{i=1}^{p} \mathcal{R}(f|_{\Lambda(X_i)})$$

but for $i = 1 \cdot p_0$, $\mathcal{R}(f|_{\Lambda(X_i)}) = \mathcal{R}(f_{i\Lambda(X_i)})$ and for $i = p_0, \ldots, p$, $\mathcal{R}(f|_{\Lambda(X_i)}) = 0$. Hence the first identity.

Now let $e_1, \ldots, e_k$ be a free $\Lambda(X^1)$-basis for $C_q(\tilde{X}^1)$ which is a finitely generated free right $\Lambda(X^1)$-module, because X^1 is connected; let us remark that $\Lambda(X^1) = \mathbf{Z}\pi_1(X^1, x^1)$. Then $C_q(\tilde{f}_1)(e_j) = \sum_{h=1}^{k} e_h F^1_{hj}$ for suitable $F^1_{hj} \in \Lambda(X^1)$. Now we can take a $\Lambda(X)$-module Q_1 such that $C_q(\tilde{X}^1) \oplus Q_1$ is a free $\Lambda(X)$-module with $e_1, \ldots, e_k$ as a free $\Lambda(X)$-basis, as done in proposition 3.1. This argument can be applied to each $j = 1, \ldots, p_0$. Therefore, if $i_j : \mathcal{R}(f_{j\Lambda(X_j)}) \to \mathcal{R}(f_{\Lambda(X)})$ is the obvious inclusion, we have that

$$Tr(C_q(\tilde{f})P) = \sum_{j=1}^{p_0} i_j Tr(C_q(\tilde{f}_j))$$

and taking alternating sums,

$$\mathcal{L}(C_q(\tilde{f})P) = \sum_{j=1}^{p_0} \mathcal{L}(C_q(\tilde{f}_j))$$

and so the conclusion follows. ∎

For each $i = 1, \ldots, p_0$ there exists a coordinate function $\mathrm{cd}_i : \mathrm{Fix}(f_i) \to \mathcal{R}(f_{i\Lambda(X^i)})$ defined by $\mathrm{cd}_i(y) := \lambda f_i(\lambda^{-1}) w_i^{-1}$ for all $y \in \mathrm{Fix}(f_i)$ with a path $\lambda : (I, 0, 1) \to (X^i, x^i, y)$ (see e.g. [B], [J]). The main theorem of [H] states that

$$\mathcal{L}(f_i, w_i) = \sum_{x \in \mathrm{Fix}(f_i)} \mathrm{Ind}(f_i, x) \dot{\mathrm{cd}}_i(x)$$

where $\mathrm{Ind}(f_i, x)$ is the index of the fixed point x, and $\mathrm{Fix}(f_i) := \{y \in X_i \mid f_i(y) = y\}$ is the fixed point set for f_i. We can always assume $\mathrm{Fix}(f)$ to be a finite subset of X. The same formula holds for $f : X \to X$; if $\mathrm{cd} : \mathrm{Fix}(f) \to \mathcal{R}(f)$ is defined by $\mathrm{cd}(y) := \mathrm{cd}_i(y)$ for every $y \in X_i \cap \mathrm{Fix}(f)$ then we have the identity

$$\mathcal{L}(f, w) = \sum_{x \in \mathrm{Fix}(f)} \mathrm{Ind}(f, x) \cdot \mathrm{cd}(x).$$

Let us recall that the number of nontrivial distinct free generators of $\mathcal{R}(f_i, w_i)$ which have to be used in writing $\mathcal{L}(f_i, w_i)$ is the Nielsen number $N(f_i)$ of the map f_i as defined in

[B], [J]. The same is true for $f : X \to X$ in the sense that we can define the Nielsen number of f, $N(f)$, to be the number of nontrivial distinct free generators of $\mathcal{R}(f)$ which have to be used in writing $\mathcal{L}(f, w)$. It is the sum $N(f) = \sum_{i=1}^{p_0} N(f_i)$ of the Nielsen numbers of the restrictions $f_i : X_i \to X_i$. In the same way the inequality $N(f) \le \#\mathrm{Fix}(f)$ holds. The Nielsen number naturally continues to be a lower bound of the number of fixed points of the self-map f.

3.2. Pushout maps. Let A, X_1 and X_2 be finite, not necessarily connected CW-complexes. Let $i_1 : A \to X_1$ and $i_2 : A \to X_2$ be cellular continuous maps. Then the pushout space $X := X_1 \sqcup_A X_2$ of X_1 and X_2 via i_1 and i_2, or the pushout space of i_1 and i_2 for short, is the set of all equivalence classes of the topological sum $X_1 \sqcup X_2$ under the equivalence relation generated by $x_1 \sim x_2 \iff (\exists a \in A) x_1 = i_1(a), x_2 = i_2(a)$. It can be shown that X is a finite CW-complex. Let $q : X_1 \sqcup X_2 \to X_1 \sqcup_A X_2$ be the identification function and define $j_1 : X_1 \to X$ and $j_2 : X_2 \to X$ as the compositions of q with the inclusions of X_1 and X_2 in $X_1 \sqcup X_2$. For more details see [P]. The main property of a pushout space is the universal property: given two maps with the same codomain $h_1 : X_1 \to Z$, $h_2 : X_2 \to Z$ such that $h_1 i_1 = h_2 i_2$, there exists a unique $l : X \to Z$ such that $l j_1 = h_1$ and $l j_2 = h_2$.

Here is a list of very common pushout-type constructions.

Example 3. *Union spaces.* If $X = X_1 \cup X_2$ is the union of two subcomplexes X_1 and X_2, then $X = X_1 \sqcup_A X_2$ where $A = X_1 \cap X_2$ and $i_1 : A \to X_1$, $i_2 : A \to X_2$ are the inclusions. For any cellular self-maps f_1 and f_2 of X_1 and X_2 that coincide on the common intersection A, there exists the extended map $f : X \to X$ which is the pushout map of f_1 and f_2 via f_A.

Example 4. *Quotient spaces.* Let (X, A) be a pair of finite CW-complexes. Then the quotient space X/A is the pushout space of $i_1 : A \to X$ and $i_2 : A \to \{*\}$ where i_1 is the inclusion and i_2 the constant map.

Example 5. *One-point unions.* The one-point union of two spaces X_1 and X_2 is simply the pushout space of $i_1 : \{*\} \to X_1$ and $i_2 : \{*\} \to X_2$.

Example 6. *Connected sums.* Let M_1 and M_2 be two compact triangulated n-manifolds. Let $X_1 := M_1 - D^n$ and $X_2 := M_2 - D^n$ be the manifolds minus an open ball D^n, and $A := \partial \bar{D}^n$. Then the connected sum $M_1 \# M_2$ is the pushout space of $i_1 : A \to X_1$ and $i_2 : A \to X_2$ if i_1 and i_2 are the natural inclusions of $\partial \bar{D}^n$ in M_1 and M_2.

Example 7. *Mapping cylinder.* Let $i_2 : A \to X_2$ be any cellular map. The pushout space of $i_0 : A \to A \times I$ and i_2, where $i_0(a) := (a, 0) (\forall a \in A)$, is called the mapping cylinder $M(i_2)$ of i_2 and is useful in the proof of the pushout formula of this paper.

Example 8. *Mapping torus.* Let Y be a finite CW-complex and $f : Y \to Y$ be a self-map. Let $A := Y \times \partial I$, $X_1 := Y \times I$ and $X_2 := Y$. If i_1 is the inclusion $A \to X_1$ and i_2 is defined by $i_2(y, 0) = y$ and $i_2(y, 1) = f(y)$ for all $y \in Y$, then the pushout space is the mapping torus T_f of f, as defined in [J], [J1].

Now let us consider cellular self-maps $f_A : A \to A$, $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ such that $i_1 f_A = f_1 i_1$ and $i_2 f_A = f_2 i_2$. There exists a unique cellular self-map $f : X \to X$

defined on the pushout space X such that the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[rrr, "i_2"] \arrow[ddd, "i_1"'] & & & X_2 \arrow[ddd, "j_2"] \\
& A \arrow[ul, "f_A"] \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \arrow[ur, "f_2"'] \arrow[d, "j_2"] & \\
& X_1 \arrow[dl, "f_1"'] \arrow[r, "j_1"] & X \arrow[dr, "f"] & \\
X_1 \arrow[rrr, "j_1"] & & & X
\end{tikzcd}$$

The map f is called the *pushout map* of f_1 and f_2 via f_A and can be denoted by $f_1 \sqcup_{f_A} f_2$ in analogy with topological spaces.

Let w_A, w_1, w_2 and w be base multipaths for f_A, f_1, f_2 and f. We wish to show that there exist well-defined homomorphisms $i_{*1} : \mathcal{R}(f_A, w_A) \to \mathcal{R}(f_1, w_1)$, $i_{2*} : \mathcal{R}(f_A, w_A) \to \mathcal{R}(f_2, w_2)$, $j_{1*} : \mathcal{R}(f_1, w_1) \to \mathcal{R}(f, w)$ and $j_{2*} : \mathcal{R}(f_2, w_2) \to \mathcal{R}(f, w)$ such that $j_{2*}i_{2*} = j_{1*}i_{1*}$. Let us consider one of the squares of the previous diagram, e.g.

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "f_A"'] & X_1 \arrow[d, "f_1"'] \\
A \arrow[r, "i_1"'] & X_1
\end{tikzcd}$$

where $A^1, \dots, A^p$ are the connected components of A and $X_1^1, \dots, X_1^q$ those of X_1. Let $a^1 \in A^1, \dots a^p \in A^p, x_1^1 \in X_1^1, \dots, x_1^q \in X_1^q$ be the base points. For each $i = 1, \dots, p$, let us choose a path $\gamma_i : (I, 0, 1) \to (X_1, i_1(a^i), x_1^i)$. The diagram

$$\begin{tikzcd}
\Lambda(A) \arrow[r, "\varphi_1"] \arrow[d, "f_{A\Lambda}"] & \Lambda(X_1) \arrow[d, "f_{1\Lambda}"'] \\
\Lambda(A) \arrow[r, "\varphi_2"'] & \Lambda(X_1)
\end{tikzcd}$$

commutes, if φ_1 and φ_2 are defined by extending linearly

$$\varphi_1(g) := \gamma_i^{-1} i_1(g) \gamma_i, \qquad \varphi_2(g) := w_1 f_1(\gamma_i^{-1}) i_1(w_A^{-1} g w_A) f_1(\gamma_i) w_1^{-1}$$

if $g \in \pi_1(A^i, a^i)$ with $f_A(a^i) \in A^i$, otherwise $\varphi_1(g) = \varphi_2(g) = 0$ if $f_A(a^i) \in A^j$ with $j \neq i$. Let θ_i be defined by

$$\theta_i := \gamma_i^{-1} i_1(w_A) f_1(\gamma_i) w_1^{-1}$$

for each i such that $f_A(a^i) \in A^i$ and otherwise $\theta_i := 1_i$. Because $\Lambda(A)$ is well-decomposed into $\bigoplus_{i=1}^p \Lambda(A^i)$ and for every i we have

$$\theta_i \varphi_2(\lambda_i) = \varphi_1(\lambda_i)\theta_i$$

for each $\lambda_i \in \Lambda(A^i)$, according to proposition 2.1, there exists a well-defined group homomorphism $\theta_* : \mathcal{R}(f_A, w_A) \to \mathcal{R}(f_1, w_1)$. We will denote it by i_{1*}. It turns out that

$$i_{1*}([g]) = [\gamma_i^{-1} i_1(g) i_1(w_A) f_1(\gamma_i) w_1^{-1}]$$

if $g \in \pi_1(A^i, a^i)$ with $f_A(a^i) \in A^i$. It is easy to see that it does not depend on the choice of the paths γ_i in the sense that if other paths δ_i are chosen then the corresponding induced homomorphism is the same. We could do the same thing for i_2, j_1 and j_2, and it can be easily shown that $j_{2*} i_{2*} = j_{1*} i_{1*}$.

In other words the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{R}(f_A, w_A) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"] & \mathcal{R}(f_2, w_2) \arrow[d, "j_{2*}"'] \\
\mathcal{R}(f_1, w_1) \arrow[r, "j_{1*}"'] & \mathcal{R}(f, w)
\end{tikzcd}$$

4. The pushout formula. We are now in a position to state the main theorem of this paper. If all the spaces involved are connected then the statement is the same as that of [F].

THEOREM 4.1 (Pushout formula). *Let $i_1 : A \to X_1$, $i_2 : A \to X_2$, $f_A : A \to A$, $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ be cellular maps such that $f_1 i_1 = i_1 f_A$ and $f_2 i_2 = i_2 f_A$. Let $f := f_1 \sqcup_{f_A} f_2$ be the pushout map of f_1 and f_2 via f_A. If i_1 is an inclusion, then*

$$\mathcal{L}(f, w) = j_{1*}\mathcal{L}(f_1, w_1) + j_{2*}\mathcal{L}(f_2, w_2) - j_{1*} i_{1*}\mathcal{L}(f_A, w_A).$$

PROOF. Let $M(i_2)$ be the mapping cylinder of i_2 as defined in example 3.2. Let $ii_2 : A \to M(i_2)$ be defined by $ii_2(a) := \bar{\imath}_2(a, 1)$ where $\bar{\imath}_2 : A \times I \to M(i_2)$ is the map of the pushout construction, and let $p : M(i_2) \to X_2$ be defined by $p\bar{\imath}_2(a, t) = i_2(a)$ for every $(a, t) \in A \times I$ and $p\bar{\imath}_0 = 1_{X_2}$. Well-known facts are that ii_2 is a cellular inclusion (hence a cofibration) and that p is a homotopy equivalence whose inverse is $\bar{\imath}_0$. For more details see e.g. [P].

Let $f_{A\times I} : A \times I \to A \times I$ be defined by $f_{A\times I}(a, t) = (f_A(a), t)$ for all $(a, t) \in A \times I$. Then $f_{A\times I} i_0 = i_0 f_A$ and $f_2 i_2 = i_2 f_A$, hence the pushout map $f_{A\times I} \sqcup_{f_A} f_2 : M(i_2) \to M(i_2)$ is defined. Denote it by f_2'. It is a cellular self-map of $M(i_2)$ and $f_2' ii_2 = ii_2 f_A$. Therefore the pushout map $f_1 \sqcup_{f_A} f_2'$ can be defined on the pushout space $X_1 \sqcup_A M(i_2)$ of $i_1 : A \to X_1$ and $ii_2 : A \to M(i_2)$. Let $\bar{p} : X_1 \sqcup_A M(i_2) \to X_1 \sqcup AX_2$ be the cellular map such that $\bar{p}\bar{ii}_2 = j_1$ and $\bar{p}\bar{\imath}_1 = j_2 p$.

As we did in the previous section, induced homomorphisms are defined such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{R}(f_{A\Lambda}) \arrow[r, "ii_{2*}"] \arrow[d, "i_{1*}"'] & \mathcal{R}(f'_{2\Lambda}) \arrow[r, "p_*"] \arrow[d, "\bar{\imath}_{1*}"'] & \mathcal{R}(f_{2\Lambda}) \arrow[d, "j_{2*}"] \\
\mathcal{R}(f_{1\Lambda}) \arrow[r, "ii_{2*}"] & \mathcal{R}(f_1 \sqcup_{f_A} f'_{2\Lambda}) \arrow[r, "\bar{p}_*"] & \mathcal{R}(f_\Lambda)
\end{tikzcd}$$

Let us note that base multipaths are omitted for the sake of simplicity.

Lemma 4.2. *We have the identities*

$$p_*(\mathcal{L}(f'_2)) = \mathcal{L}(f_2), \quad \bar{p}_*(\mathcal{L}(f_1 \sqcup_{f_A} f'_2)) = \mathcal{L}(f).$$

Proof. If all the spaces involved are connected this is exactly the statement of lemma 4.2 and lemma 4.3 of [F]. Otherwise X_2 or X may be disconnected. But in this case, as p and $\bar{p}$ are homotopy equivalences (see e.g. [P]), they induce bijections at the 0-homotopy set level $\pi_0(M(i_2)) = \pi_0(X_2)$ and $\pi_0(X_1 \sqcup_A M(i_2)) = \pi_0(X)$. Moreover, according to proposition 3.2,

$$\mathcal{R}(f'_{2\Lambda}) = \bigoplus_{i=1}^{p'_0} \mathcal{R}({f'_2}^i{}_\Lambda(M(i_2)^i)), \quad \mathcal{L}(f'_2) = \sum_{i=1}^{p'_0} \mathcal{L}({f'_2}^i),$$

$$\mathcal{R}(f_{2\Lambda}) = \bigoplus_{i=1}^{p_0} \mathcal{R}(f^i_{2\Lambda}(X^i_2)), \quad \mathcal{L}(f_2) = \sum_{i=1}^{p_0} \mathcal{L}(f^i_2),$$

where $X^1_2, \ldots, X^{p_0}_2$ are the connected components of X_2 such that $f_2(X^i_2) \subset X^i_2$ and the same holds for $M(i_2)$. It can be seen that $p_0 = p'_0$ and that, as proved in [F],

$$p_* \mathcal{L}({f'_2}^i) = \mathcal{L}(f^i_2)$$

because f^i_2 and $f^{i'}_2$ are self-maps of connected spaces. Therefore, because of the additivity of p_*,

$$p_*(\mathcal{L}(f'_2)) = \mathcal{L}(f_2)$$

and hence the first part of the lemma. The second one can be proved in the same way. ■

Because $\bar{p}_* \overline{ii}_{2*} = j_{1*}$ and $p_* ii_{2*} = i_{2*}$, we have

$$j_{1*}\mathcal{L}(f_1) + j_{2*}\mathcal{L}(f_2) - j_{1*}i_{1*}\mathcal{L}(f_A) = \bar{p}_*\overline{ii}_{2*}\mathcal{L}(f_1) + \bar{p}_*\bar{\imath}_{1*}\mathcal{L}(f'_2) - \bar{p}_*\overline{ii}_{2*}i_{1*}\mathcal{L}(f_A)$$

and hence the pushout formula holds if and only if

$$\mathcal{L}(f_1 \sqcup_{f_A} f'_2) = \overline{ii}_{2*}\mathcal{L}(f_1) + \bar{\imath}_{1*}\mathcal{L}(f'_2) - \overline{ii}_{2*}i_{1*}\mathcal{L}(f_A).$$

Let us remark that both i_1 and ii_2 are supposed to be inclusions. This means that it suffices to prove the theorem in case both i_1 and i_2 are cellular inclusions.

Hence let us suppose that A is a subcomplex of X_1 and X_2 and that $X = X_1 \cup X_2$, $A = X_1 \cap X_2$. Clearly $i_1 : A \to X_1$, $i_2 : A \to X_2$, $j_1 : X_1 \to X$ and $j_2 : X_2 \to X$ are all inclusions. The maps f_A, f_1 and f_2 are simply the restrictions of f to the subcomplexes A, X_1 and X_2. Let $w_A = w^1_A, w^2_A, \ldots, w^1_1, w^2_1, \ldots, w^1_2, w^2_2, \ldots$ and $w^1, w^2, \ldots$ be the base multipaths. Let $A^1, A^2, \ldots, A^{p_A}$ be the connected components of A with base points $a^1, a^2, \ldots, a^{p_A}$; let $X_1^1, X_1^2, \ldots, X_1^{p_1}$ be those of X_1 with base points $x_1^1, x_1^2, \ldots, x_1^{p_1}$; $X_2^1, X_2^2, \ldots, X_2^{p_2}$, those of X_2 with base points $x_2^1, x_2^2, \ldots, x_2^{p_2}$ and $X^1, X^2, \ldots, X^p$ those of X with base points $x^1, x^2, \ldots, x^p$. Let $\tilde{A}$, $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}$ be the universal covering spaces of A, X_1, X_2 and X, and $C_q(\tilde{A})$, $C_q(\tilde{X}_1)$, $C_q(\tilde{X}_2)$ and $C_q(\tilde{X})$ their q-dimensional cellular chain groups. Fix an integer $q \geq 0$. Let $P_A : C_q(\tilde{A}) \to C_q(\tilde{A})$, $P_1 : C_q(\tilde{X}_1) \to C_q(\tilde{X}_1)$, $P_2 : C_q(\tilde{X}_2) \to C_q(\tilde{X}_2)$ and $P : C_q(\tilde{X}) \to C_q(\tilde{X})$ be the projection homomorphisms as defined in section 3.1. Now let us consider the canonical liftings of (f_A, w_A), (f_1, w_1), (f_2, w_2) and (f, w) and the corresponding chain homomorphisms

$$C_q(\tilde{f}_A),\ C_q(\tilde{f}_1),\ C_q(\tilde{f}_2),\ C_q(\tilde{f})$$

at the q-dimensional chain group level. Then the theorem follows easily from the following lemma.

LEMMA 4.3. *For any $q \geq 0$ we have the identity*

$$j_{1*}(Tr_{f_{1\Lambda}}(C_q(\tilde{f}_1)P_1)) + j_{2*}(Tr_{f_{2\Lambda}}(C_q(\tilde{f}_2)P_2)) \\ -j_{1*}i_{1*}(Tr_{f_{A\Lambda}}(C_q(\tilde{f}_A)P_A)) = Tr_{f_\Lambda}(C_q(\tilde{f})P).$$

PROOF. Let us start by considering the square diagram

$$\begin{tikzcd}
C_q(\tilde{A}) \arrow[d, "C_q(\tilde{i}_1')P_A"'] & C_q(\tilde{A}) \arrow[l, "C_q(\tilde{f}_A)P_A"'] \arrow[d, "C_q(\tilde{i}_1)P_A"] \\
C_q(\tilde{X}_1) & C_q(\tilde{X}_1) \arrow[l, "C_q(\tilde{f}_1)P_1"']
\end{tikzcd}$$

where $C_q(\tilde{f}_A)$, $C_q(\tilde{f}_1)$ and P_A are defined as above, and $\tilde{i}_1$ and $\tilde{i}_1'$ are defined as follows: for each $j = 1, \ldots, p_A$ let γ_j be a continuous path $\gamma_j : (I, 0, 1) \to (X_1, a^j, x_1^{j'})$ where a^j is the base point of A^j and $x_1^{j'}$ is the base point of the component $X_1^{j'}$ of X_1 which contains A^j; let us remark that points of $\tilde{A}$ are homotopy classes rel. endpoints of paths $\lambda : (I, 0) \to (A, \{a^1, a^2, \ldots, a^{p_A}\})$ and points of $\tilde{X}_1$ are homotopy classes of paths $\lambda : (I, 0) \to (X_1, \{x_1^1, x_1^2, \ldots, x_1^{p_1}\})$; we set

$$\tilde{i}_1([\lambda]) := [\gamma_j^{-1}\lambda]$$

for each $\lambda : (I, 0) \to (A^j, a^j)$ and

$$\tilde{i}_1'([\lambda]) := \begin{cases} [w_1^j f_1(\gamma_j^{-1})(w_A^j)^{-1}\lambda] & \text{if } f_A(a^j) \in A^j, \\ [\gamma_j{}^{-1}\lambda] & \text{otherwise,} \end{cases}$$

for each $\lambda : (I, 0) \to (A^j, a^j)$.

It is not difficult to see that the diagram is commutative. Moreover, if we recall the definition of i_{1*} of section 3.2 by taking $\varphi_1, \varphi_1' : \Lambda(A) \to \Lambda(X_1)$ as the unique ring homomorphisms such that

$$\varphi_1(g) := \begin{cases} \gamma_j^{-1} g \gamma_j & (\forall g \in \pi_1(A^j, a^j) \text{ s.t. } f(a^j) \in A^j), \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_1'(g) := \begin{cases} w_1^j f_1(\gamma_j^{-1})(w_A^j)^{-1} g w_A^j f_1(\gamma_j) w_1^{j\,-1} & (\forall g \in \pi_1(A^j, a^j), \\ & \text{s.t. } f(a^j) \in A^j), \\ 0 & \text{otherwise,} \end{cases}$$

we get the commutative diagram

$$\begin{tikzcd} \Lambda(A) \arrow[d, "\varphi_1'"'] & \Lambda(A) \arrow[l, "f_{A\Lambda}"'] \arrow[d, "\varphi_1"] \\ \Lambda(X_1) & \Lambda(X_1) \arrow[l, "f_{1\Lambda}"'] \end{tikzcd}$$

of ring homomorphisms. The main point is that $C_q(\tilde{i}_1)P_A$ is a φ_1-homomorphism and $C_q(\tilde{i}_1')P_A$ is a φ_1'-homomorphism. Moreover, as shown in section 3.2, $i_{1*} : \mathcal{R}(f_A, w_A) \to \mathcal{R}(f_1, w_1)$ is defined by

$$i_{1*}([\lambda]) := [\theta_j \varphi_1'(\lambda)] \ \big(\forall \lambda \in \Lambda(A^j) \ \text{ s.t.} f(a^j \in A^j)\big)$$

and is zero otherwise, where θ_j is a suitable element of $\Lambda(X_1)$ such that $\theta_j \varphi_1'(\lambda) = \varphi_1(\lambda)\theta_j$ for each $\lambda \in \Lambda(A^j)$.

The same arguments apply to i_2, j_1 and j_2; therefore we get the following commutative diagram of ring homomorphisms:

$$\begin{tikzcd} \Lambda(A) \arrow[rrr, "\varphi_2'"] \arrow[ddd, "\varphi_1'"'] & & & \Lambda(X_2) \arrow[ddd, "\psi_2'"] \\ & \Lambda(A) \arrow[ul, "f_{A\Lambda}"] \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \Lambda(X_2) \arrow[ur, "f_{2\Lambda}"'] \arrow[d, "\psi_2"] & \\ & \Lambda(X_1) \arrow[dl, "f_{1\Lambda}"'] \arrow[r, "\psi_1"] & \Lambda(X) \arrow[dr, "f_\Lambda"] & \\ \Lambda(X_1) \arrow[rrr, "\psi_1'"] & & & \Lambda(X) \end{tikzcd}$$

which induces, at the Reidemeister group level, the diagram of section 3.2.

Now, let Q_A be a $\Lambda(A)$-module such that $C_q(\tilde{A}) \oplus Q_A$ is a free finitely generated $\Lambda(A)$-module. As shown in the proof of proposition 3.1 and in [H], [FH] we can suppose that a free basis of $C_q(\tilde{A}) \oplus Q_A$ can be given by taking liftings of the q-dimensional cells of A $\{e_1, e_2, \dots, e_k\}$ to $\tilde{A}$ (and hence they can be thought of as elements of $C_q(\tilde{A})$). If we consider the $f_{A\Lambda}$-endomorphism $C_q(f_A)P + 0_{Q_A}$ of $C_q(\tilde{A}) \oplus Q_A$ we get exactly the $f_{A\Lambda}$-endomorphism whose trace is the trace of $C_q(\tilde{f}_A)P_A$. The same argument applies to X_1, X_2 and X, and therefore we have the $f_{1\Lambda}$-endomorphism $C_q(\tilde{f}_1)P_1 + 0_{Q_1}$ of the free finitely generated $\Lambda(X_1)$-module $C_q(\tilde{X}_1) \oplus Q_1$, the $f_{2\Lambda}$-endomorphism $C_q(\tilde{f}_2)P_2 + 0_{Q_2}$ of the free finitely generated $\Lambda(X_2)$-module $C_q(\tilde{X}_2) \oplus Q_2$ and the f_Λ-endomorphism $C_q(\tilde{f})P + 0_Q$ of the free finitely generated $\Lambda(X)$-module $C_q(\tilde{X}) \oplus Q$. Their traces are by definition exactly the traces of the corresponding endomorphisms at the chain group level. Let us call them F_A, F_1, F_2 and F.

Finally we define the homomorphisms corresponding to i_1, i_2, j_1 and j_2. As an example, let $\Phi_1 := C_q(\tilde{i_1})P_A + 0_{Q_A} : C_q(\tilde{A}) \oplus Q_A \to C_q(\tilde{A}) \oplus Q_A$ be defined in the obvious way. The others are defined in the same way. We arrange them as in the following diagram.

$$\begin{tikzcd}
C_q(\tilde{A}) \oplus Q_A \arrow[rrr, "\Phi'_2"] \arrow[ddd, "\Phi'_1"'] & & & C_q(\tilde{X}_2) \oplus Q_2 \arrow[ddd, "\Psi'_2"] \\
& C_q(\tilde{A}) \oplus Q_A \arrow[ul, "F_A"'] \arrow[r, "\Phi_2"] \arrow[d, "\Phi_1"'] & C_q(\tilde{X}_2) \oplus Q_2 \arrow[ur, "F_2"] \arrow[d, "\Psi_2"] & \\
& C_q(\tilde{X}_1) \oplus 0_{Q_1} \arrow[r, "\Psi_1"] \arrow[dl, "F_1"'] & C_q(\tilde{X}) \oplus Q \arrow[dr, "F"] & \\
C_q(\tilde{X}_1) \oplus Q_1 \arrow[rrr, "\Psi'_1"] & & & C_q(\tilde{X}) \oplus Q
\end{tikzcd}$$

Let us note that

$$(\forall \lambda \in \Lambda(A^j))\ \Phi'_1(\lambda) = \Phi_1(\lambda) \cdot \theta_j.$$

Similarly $(\forall j = 1, \ldots, p_A)$ there exist elements $\eta_j \in \Lambda(X_2)$ such that

$$(\forall \lambda \in \Lambda(A^j))\ \Phi'_2(\lambda) = \Phi_2(\lambda) \cdot \eta_j$$

and $(\forall j = 1, \ldots, p_1, \forall i = 1, \ldots, p_2)$ there exist $\zeta_j, \epsilon_i \in \Lambda(X)$ such that

$$(\forall \lambda \in \Lambda(X_1^j))\ \Phi'_1(\lambda) = \Phi_1(\lambda) \cdot \zeta_j$$

and

$$(\forall \lambda \in \Lambda(X_2^i))\ \Phi'_2(\lambda) = \Phi_2(\lambda) \cdot \epsilon_i$$

for each $j = 1, \ldots, p_1$ and $i = 1, \ldots, p_2$. Moreover as before $i_{1*} = \theta_*$, $i_{2*} = \eta_*$, $j_{1*} = \epsilon_*$ and $j_{2*} = \zeta_*$. Let us recall that these homomorphisms do not depend on the choice of the paths that occur in the definition of θ_j, η_j, ζ_j and ϵ_j.

As shown in [H], [F] we can take a free $\Lambda(X_1)$-basis $b_1, b_2, \ldots, b_{k+s}$ of $C_q(\tilde{X}_1) \oplus Q_1$ such that $b_j = \Phi_1(e_j)$ for each $j = 1, \ldots, k$ such that e_j and $f_A(e_j)$ belong to the same connected component of A. In the same way let $c_1, c_2, \ldots, c_{k+t}$ be a free $\Lambda(X_2)$-basis of $C_q(\tilde{X}_2) \oplus Q_2$ such that $c_j = \Phi_2(e_j)$ for each $j = 1, \ldots, k$ such that e_j and $f_A(e_j)$ belong to the same connected component of A. Therefore we can take a free $\Lambda(X)$-basis $d_1, d_2, \ldots, d_{k+s+t}$ of $C_q(\tilde{X}) \oplus Q$ such that $d_j = \Psi_1(b_j)$ for all $j = 1, \ldots, k+s$ such that

$f_1(b_j)$ and b_j are in the same connected component of X_1, and $d_j = \Psi_2(c_{j-s})$ for all $j = k+s+1, \ldots, k+s+t$ such that $f_2(c_j)$ and c_j are in the same connected component of X_2.

Let E_{vu}, B_{vu}, C_{vu} and D_{vu} be the entries of the matrices representing F_A, F_1, F_2 and F respectively. In other words $\{E_{vu}\} \in \mathcal{M}_{k,k}(\Lambda(A))$ and

$$F_A(e_u) = \sum_{v=1}^{k} e_v E_{vu}$$

for all $u = 1, \ldots, k$; $\{B_{vu}\} \in \mathcal{M}_{k+s,k+s}(\Lambda(X_1))$ and

$$F_1(b_u) = \sum_{v=1}^{k+s} b_v B_{vu}$$

for all $u = 1, \ldots, k+s$; $\{C_{vu}\} \in \mathcal{M}_{k+t,k+t}(\Lambda(X_2))$ and

$$F_2(c_u) = \sum_{v=1}^{k+t} c_v C_{vu}$$

for all $u = 1, \ldots, k+t$; finally $\{D_{vu}\} \in \mathcal{M}_{k+s+t,k+s+t}(\Lambda(X))$ and

$$F(d_u) = \sum_{v=1}^{k+s+t} d_v D_{vu}$$

for all $u = 1, \ldots, k+s+t$.

We know that

$$Tr_{f_\Lambda}(C_q(\tilde{f})P) = Tr_{f_\Lambda}(F) = \sum_{u=1}^{k+s+t} [D_{uu}]$$

and that similar formulae hold for $C_q(\tilde{f}_A)P_A$, $C_q(\tilde{f}_1)P_1$ and $C_q(\tilde{f}_2)P_2$. But for $u = 1, \ldots, k+s$

$$F(d_u) = F(\Psi_1(b_u)) = \Psi_1' F_1(b_u)$$

and hence

$$F(d_u) = \sum_{u=1}^{k+s} \Psi_1'(b_v B_{vu})$$

for each $u = 1, \ldots, k+s$. Because Ψ_1' is a ψ_1'-homomorphism, $\Psi_1'(b_v B_{vu}) = \Psi_1'(b_v)\psi_1'(B_{vu})$ for all $u, v = 1, \ldots, k+s$. But for a suitable j depending on v, $\Psi_1'(b_v) = \Psi_1(b_v)\zeta_{j(v)}$ and hence $\Psi_1'(b_v) = d_v \zeta_{j(v)}$. This implies that

$$F(d_u) = \sum_{u=1}^{k+s} d_v \left(\zeta_{j(v)} \psi_1'(B_{vu})\right)$$

for all $u = 1, \ldots, k+s$ and hence that

$$D_u^u = \zeta_{j(u)} \psi_1'(B_{uu})$$

for $u = 1, \ldots, k+s$. In a similar way we can show that

$$D_{uu} = \epsilon_{j'(u-s)} \psi_2'(C_{u-s,u-s})$$

for all $u = k+s+1, \ldots, k+s+t$ and suitable indices j' depending on u; moreover for other i's depending on u

$$B_{uu} = \theta_{i(u)} \phi_1'(E_{uu}), \qquad C_{uu} = \eta_{i(u)} \phi_2'(E_{uu})$$

for all $u = 1, \ldots, k$.

Now by definition of i_{1*}, i_{2*}, j_{1*} and j_{2*}

$$j_{1*}(Tr_{f_{1\Lambda}}(C_q(\tilde{f}_1)P_1)) = \sum_{u=1}^{k+s}[\zeta_{j(u)}\psi_1'(B_{uu})]$$

$$j_{2*}(Tr_{f_{2\Lambda}}(C_q(\tilde{f}_2)P_2)) = \sum_{u=1}^{k+t}[\epsilon_{j'(u)}\psi_2'(C_{uu})]$$

$$j_{1*}i_{1*}(Tr_{f_{A\Lambda}}(C_q(\tilde{f}_A)P_A)) = \sum_{u=1}^{k}[\epsilon_{j'(u)}\psi_2'(\eta_{i(u)}\phi_2'(E_{uu}))]$$

and hence

$$\begin{aligned} j_{1*}(Tr_{f_{1\Lambda}}(C_q(\tilde{f}_1)P_1)) + j_{2*}(Tr_{f_{2\Lambda}}(C_q(\tilde{f}_2)P_2)) \\ &= Tr_{f_\Lambda}(C_q(\tilde{f})P) + \sum_{u=1}^{k}[\epsilon_{j'(u)}\psi_2'(C_{uu})] \\ &= Tr_{f_\Lambda}(C_q(\tilde{f})P) + j_{1*}i_{1*}(Tr_{f_{A\Lambda}}(C_q(\tilde{f}_A)P_A)) \end{aligned}$$

concluding the proof. ∎

With this lemma the proof of the theorem is complete. Let us note that the hypothesis that at least i_1 is an inclusion cannot be omitted. Let $A = S^1$ be a circle and let $X_1 = X_2 = \{*\}$ be a single point. Then $i_1 : A \to X_1$ and $i_2 : A \to X_2$ are constant maps, and the pushout space $X = X_1 \sqcup_A X_2$ is a single point $\{*\}$ too. Therefore if $f_A : S^1 \to S^1$ is a map of degree d and f_1 and f_2 are constant maps, then the pushout map f is the obvious constant map, and $\mathcal{L}(f) = [1] \neq \mathcal{L}(f_1) + \mathcal{L}(f_2) - \mathcal{L}(f_A)$ because the right hand side of this equation is equal to $[1] + [1] - (1-d)[1] = (d+1)[1] \neq [1]$. ∎

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