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ON THE NIELSEN FIXED POINT THEORY FOR MULTIVALUED MAPPINGS

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Abstract. We present J. Jezierski's approach to the Nielsen fixed point theory for a broad class of multivalued mappings [Je1]. We also describe some generalizations and different techniques existing in the literature.

1. Notations and definitions. Let X, Y be metric spaces. By a multivalued mapping $\Phi : X \to Y$ we mean a transformation $\Phi : X \to 2^Y$ with nonempty compact values. Many notions known for singlevalued transformations can be generalized to multivalued mappings. For $A \subset X$ the *image* of A is the set $\Phi(A) = \bigcup_{x\in A} \Phi(x)$. The set

$$\Gamma_\Phi = \{(x, y) : y \in \Phi(x)\}$$

is called the *graph* of Φ.

There are several notions of continuity.

Definition 1. The mapping Φ is *lower semicontinuous (lsc)* (respectively *upper semicontinuous (usc))* if for every open subset $V \subset Y$ the set $\Phi^{-1}(V) = \{x \in X : \Phi(x) \cap V \neq \emptyset\}$ (respectively $\Phi_+^{-1}(V) = \{x \in X : \Phi(x) \subset V\}$) is an open subset of X. If Φ is both lsc and usc, then we say that Φ is *continuous*.

In the singlevalued case these three notions coincide. For basic properties and examples of usc (lsc) mappings we refer the reader to [AC] or [Gor].

In order to have a nontrivial fixed point theory we have to consider special classes of multivalued mappings.

Definition 2. A subset $A \subset X$ satisfies the $\star$*-property* if it is nonempty, connected and there exists an open neighbourhood U of A such that each loop in U is homotopic (with fixed ends) in X to a constant loop.

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Definition 3. A multivalued mapping $\Phi : Y \to X$ is called an *m-mapping* iff it is upper semicontinuous and the image of each point has the $\star$-property in X.

Let us denote by Δ^n the standard n-dimensional simplex and by $\partial\Delta^n$ its boundary.

Definition 4. A compact subset $A \subset X$ is ∞-*proximally connected* provided for every $\epsilon > 0$ there is $\delta > 0$ such that for any map $g : \partial\Delta^n \to O_\delta(A)$ there is a map $\tilde{g} : \Delta^n \to O_\epsilon(A)$ such that $g(x) = \tilde{g}(x)$ for $x \in \partial\Delta^n$ (for any natural n).

If X is an ANR, then the notion of ∞-proximally connected sets is equivalent to the notion of R_δ-sets (see e.g. [Gor]) . In particular, AR's are ∞-proximally connected.

Definition 5. An usc mapping $\Phi : X \to Y$ is called a *J-mapping* provided all values $\Phi(x)$ are ∞-proximally connected.

2. Reidemeister and Nielsen relations. In this section we assume that a space X is connected and admits a universal covering. Let us fix a universal covering $p : X' \to X$.

Definition 6. A mapping $F' : Y \to X'$ such that the following diagram commutes

$$\begin{tikzcd} & X' \arrow[d] \\ Y \arrow[ur, "F'"] \arrow[r, "F"] & X \end{tikzcd}$$

is called a *lift* of the mapping $F : Y \to X$.

The following natural property was proved in [Je1], 1.6.

Theorem 1. *Let $F : X \to X$ be an m-mapping and let $x_1, x_2 \in X'$ be two points such that $p(x_2) \in F(p(x_1))$. Then there exists a unique m-mapping $F' : X' \to X'$ for which $x_2 \in F'(x_1)$ and the following diagram commutes:*

$$\begin{tikzcd} X' \arrow[r, "F'"] \arrow[d, "p"] & X' \arrow[d, "p"] \\ X \arrow[r, "F"] & X \end{tikzcd}$$

Let us denote by $L(F)$ the set of all m-mappings $F' : X' \to X'$ making the above diagram commutative. We call the elements of $L(F)$ the *lifts* of the m-mapping F. Recall that all lifts of the identity mapping (called *deck transformations*) form a group isomorphic to the fundamental group of X. Denote this group by θ. If we fix one element $F' \in L(F)$, then each lift of F is of the form $\alpha F'$, where $\alpha \in \theta$.

Now we can define an equivalence relation R on the set $L(F)$:

$$F' \, R \, F'' \text{ iff } F' = \gamma F'' \gamma^{-1} \text{ for some } \gamma \in \theta.$$

This is the so called *Reidemeister relation.* Let us denote the set of equivalence classes of the Reidemeister relation by $\nabla(F)$. The cardinality of $\nabla(F)$ is independent of the choice of a universal covering of X.

Definition 7. The number of elements in ∇F is called the *Reidemeister number* of the mapping F.

This number is a homotopy invariant ([Je1], 1.12):

THEOREM 2. *If two m-mappings are homotopic by an m-homotopy, then their Reidemeister classes are in one to one correspondence.*

Now we can define the Nielsen relation in two equivalent ways (comp. [Jia]). Consider the fixed point set of the m-mapping $F : X \to X$

$$\operatorname{Fix} F = \{x \in X : x \in F(x)\}.$$

DEFINITION 8. Let $x, x' \in \operatorname{Fix} F$. We say that x and x' are *Nielsen equivalent* iff there exists a lift $F' \in L(F)$ such that $x, x' \in p(\operatorname{Fix} F')$. We write $x \sim_N x'$, and denote the quotient set by $\Phi'(F)$.

Similarly to the singlevalued case one proves

THEOREM 3 (see [Je1]). *Let $F : X \to X$ be an m-mapping. Then*

(i) $\operatorname{Fix} F = \bigcup_{F' \in L(F)} p(\operatorname{Fix} F')$;
(ii) *the sets $p(\operatorname{Fix} F'), p(\operatorname{Fix} F'')$ are either equal or disjoint for any $F', F'' \in L(F)$;*
(iii) $p(\operatorname{Fix} F') = p(\operatorname{Fix} F'') \neq \emptyset$ *implies that $F' \sim_R F''$.*

Consequently there is an injective mapping from $\Phi'(F)$ to $\nabla(F)$.

Now we turn to the more popular definition of the Nielsen relation. If $f : X \to X$ is a singlevalued mapping then two points $x, x' \in \operatorname{Fix} f$ are equivalent iff there is a path $\omega : I \to X$ joining them such that ω and $f\omega$ are fixed end homotopic. We can reformulate it in the language of the fundamental grupoid. Recall that the *fundamental grupoid* $\Pi(X)$ is a category. Objects of this category are points of the space X and morphisms from x to x' are the fixed end homotopy classes of paths joining these two points. We denote the set of all morphisms between x and x' by $\Pi(X; x, x')$. Every continuous singlevalued mapping $f : X \to Y$ induces a functor $\Pi(f) : \Pi(X) \to \Pi(Y)$ by

$$\Pi(f)(x) = f(x); \quad \Pi(f)[\omega] = [f\omega].$$

PROPOSITION 4. *Let $f : X \to X$ be a continuous singlevalued mapping. Then two points $x, x' \in \operatorname{Fix} f$ are equivalent if and only if the mapping*

$$\Pi(f) : \Pi(X; x, x') \to \Pi(X; x, x')$$

has a fixed point.

Now let X be a connected, locally pathwise connected, semilocally simply-connected topological space (i.e. admitting a universal covering). Let A_0, A_1 be two subsets of X satisfying the $\star$-property. Then the sets $\Pi(X; a_0, a_1), \Pi(X; a'_0, a'_1)$ are identified as follows: let U_i be a pathwise connected neighbourhood of A_i as in the $\star$-property (definition 2) for $i = 0, 1$. Let $a_i, a'_i \in A_i$ and let ω_i be a path in U_i joining the points a_i and a'_i. We identify $[\alpha] \in \Pi(X; , a_0, a_1)$ with $[\omega_0^{-1} * \alpha * \omega_1] \in \Pi(X; , a'_0, a'_1)$ and define the quotient set

$$\widehat{\Pi}(X; A_0, A_1) = \bigcup_{(a_0,a_1) \in A_0 \times A_1} \Pi(X; a_0, a_1)/\sim .$$

For each $a_0 \in A_0, a_1 \in A_1$ we denote by

$$i_{a_0,a_1} : \Pi(X; a_0, a_1) \to \widehat{\Pi}(X; A_0, A_1)$$

the natural bijection.

DEFINITION 9. The *generalized fundamental grupoid* of the space X is the category in which objects are subsets of X satisfying the $\star$-property and $\widehat{\Pi}(X; A_0, A_1)$ is the set of all morphisms between the objects A_0 and A_1. We will denote this category by $\widehat{\Pi}(X)$.

PROPOSITION 5. *Let X be a connected space admitting a universal covering and let Y be a topological space. Then each m-mapping $F : Y \to X$ induces a functor*

$$\widehat{\Pi}(F) : \Pi(Y) \to \widehat{\Pi}(X)$$

which coincides with $\Pi(F)$ when F is a singlevalued mapping.

PROOF. We define $\widehat{\Pi}(F)(y) = F(y)$ for each $y \in Y$. Let $[\omega] \in \Pi(Y; y_0, y_1)$. Let us fix a universal covering $p : X' \to X$ and points $x_0 \in F(y_0), x_1 \in F(y_1), x'_0 \in p^{-1}(x_0)$. Consider the diagram

$$\begin{tikzcd} & & X' \arrow[d, "p"] \\ I \arrow[r, "\omega"] & Y \arrow[r, "F"] & X \end{tikzcd}$$

It admits a unique lift $\widetilde{(F\omega)} : I \to X'$ such that $x'_0 \in \widetilde{(F\omega)}(0)$. Let $\{x'_1\} = \widetilde{(F\omega)}(1) \cap p^{-1}(x_1)$ and let τ be a path in X' joining x'_0 to x'_1. We define

$$\widehat{\Pi}(F)[\omega] = i_{x_0,x_1}[p\tau] \in \widehat{\Pi}(X; F(y_0), F(y_1)).$$

One checks that the definition is independent of the choice of the points x_0, x_1, x'_1 and the path τ. ∎

DEFINITION 10. Two fixed points x_1, x_2 of the m-mapping $F : X \to X$ are *in $\sim_{N'}$ relation* iff the maps

$$\widehat{\Pi}(F), i_{x_1,x_2} : \Pi(X; x_1, x_2) \to \widehat{\Pi}(X; F(x_1), F(x_2))$$

have a coincidence point.

THEOREM 6 ([Je1], 3.12). *The above relation $\sim_{N'}$ is equal to $\sim_N$.*

Equivalence classes of the above relation are called *Nielsen classes* of the mapping F.

3. The Nielsen number.

In order to define the Nielsen number one has to define the notion of an essential class. The best way to do it is to use a fixed point index. Therefore we have to consider those classes of m-mappings which admit an index theory.

Let X be a metric ANR.

DEFINITION 11. An m-mapping $F : X \to X$ is called *a Nielsen m-mapping* if the image of each point is a Q-acyclic continuum and the image $F(X)$ is relatively compact in X.

For the above class of mappings there exists a fixed point index theory satisfying all the standard properties (see e.g. [Dz1]).

Let $F : X \to X$ be a Nielsen m-mapping and let A be one of its Nielsen classes. Choose an open set $U \subset X$ such that $U \cap \operatorname{Fix} F = A$.

Definition 12. The class A is *essential* provided $i(X, F, U) \neq 0$. The number of all essential classes is called the *Nielsen number* of F and is denoted by $N(F)$.

The proofs of two basic properties of the Nielsen number are standard.

Theorem 7 ([Dz1], 9.15). *If $H : X \times I \to X$ is a homotopy in the class of Nielsen m-mappings, then $N(H_0) = N(H_1)$.*

Theorem 8 ([Dz1], 9.13). *A Nielsen m-mapping $F : X \to X$ has at least $N(F)$ fixed points.*

4. Approximation approach. Observe that the property from Definition 4 is stronger than the $\star$ property if X is an ANR. Therefore our definition of the Nielsen relation works for the class of J-mappings. Roughly speaking, mappings of this class can be approximated on the graph by singlevalued transformations, so the Nielsen number of a J-mapping is just the Nielsen number of its sufficiently close approximation. This approach has been used in [KrM] for J-mappings and also for their finite compositions.

Let X, Y be two metric spaces. For $\Gamma \subset X \times Y$ we will denote by $O_\epsilon(\Gamma)$ the ϵ-neighbourhood of Γ in $X \times Y$ with the "maximum" metric.

Definition 13. A continuous mapping $f : X \to Y$ is an *ϵ-approximation* of a multivalued mapping $F : X \to Y$ provided

$$\Gamma_f \subset O_\epsilon(\Gamma_F).$$

Theorem 9. *Let X be a compact ANR and $F : X \to Y$ a J-mapping. Then for any $\epsilon > 0$ there exists an ϵ-approximation of F. Moreover, for each $\delta > 0$ there exists an ϵ such that any two ϵ-approximations of F are homotopic by a homotopy H such that $H_t = H(\cdot, t)$ is a δ-approximation of F for every $t \in I$.*

For the proof we refer the reader to [Gor], [GGK]. We now apply Theorem 9 to define a fixed point index (a Nielsen number) of F to be the index (the Nielsen number) of a sufficiently fine approximation of F. This definition does not depend on the choice of the approximation ([KrM], 6.5).

Remark 1. There are examples in [KrM] showing that if one defines the Nielsen number in an analogous way for compositions of J-mappings dropping out the assumption of the type $\star$ then the numbers obtained may depend on the way of composition.

5. Other classes of mappings. There were some attempts to consider other classes of mappings for Nielsen theory (see [Dz1-2], [Mas], [S1-4]). If we do not assume that images of points are connected, then we have to consider only continuous (i.e. both usc and lsc) mappings, as the following simple example shows.

Example 1. The map $F : [-1,1] \to [-1,1]$ given by

$$F(x) = \begin{cases} \{x+1, 1\} & \text{if } x < 0, \\ \{-1, 1\} & \text{if } x = 0, \\ \{-1, x-1\} & \text{if } x > 0, \end{cases}$$

is usc and has no fixed points.

One can use the following observation of S. Banach.

Proposition 10. *Let $F : X \to Y$ be a continuous mapping such that for each point $x \in X$ the image $F(x)$ consists of exactly n components. If X is path connected and simply connected then the graph Γ_F has exactly n components and consequently F splits into n disjoint usc mappings.*

This is true, in particular, for n-valued continuous mappings. If we assume that the components of images of points are Q-acyclic continua satisfying the $\star$-property, then we are able to prove a version of Thm. 1 (see [Dz1], 9.4). Therefore one can repeat the definition of Nielsen relation and Nielsen number after Section 2 and 3 (see [Dz1] for details).

In general it is hard to expect minimum theorems for multivalued mappings. But for n-valued continuous mappings the following theorem is true.

Theorem 11 (H. Schirmer [S2]). *If X is a compact, connected manifold of dimension at least 3 and $F : X \to X$ is an n-valued continuous mapping, then there exists an n-valued mapping G homotopic to F such that the number of fixed points of G is equal to $N(F)$.*

One could try to extend the theory to finite-valued continuous mappings. Unfortunately even the Lefschetz fixed point theorem is not true then. In fact in [Je2] an example was given of a continuous mapping with values which are either 1- or 2- or 3-point sets from the two-dimensional disc onto itself without fixed points. However, if we restrict our attention to simpler combinations (e.g. if the images of points have either one or n components) then the index theory works (see [Dz1]). Such mappings determine continuous mappings into symmetric products. This approach has been used to build a Nielsen theory in [Mas], [Dz2], [S3-4]. But the following observation was made in [Mik]. Recall that the symmetric product X_n is the orbit space of the action of the symmetric group $S(n)$ on the n-fold Cartesian product X^n by permutation of coordinates.

Proposition 12. *If X is a pathwise connected space, then every symmetric product mapping $f : X \to X_n$ has at most one fixed point class.*

This is true also in a multivalued setting (see [Dz2]). Using this fact the following minimum theorem has been proved by H. Schirmer [S4].

Theorem 13. *Let X be a compact, connected manifold of dimension at least 3, and $F : X \to X$ a continuous mapping with images of points being either singletons or two-point sets. Then there exists a mapping G of the same type, which is homotopic to F with only one fixed point if the Lefschetz number of F is non-zero, and fixed point free otherwise.*

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