NIELSEN THEORY AND REIDEMEISTER TORSION
BANACH CENTER PUBLICATIONS, VOLUME 49
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1999

ON ANALYTIC TORSION OVER C^*-ALGEBRAS

ALAN L. CAREY
Department of Pure Mathematics, University of Adelaide
Adelaide 5005, Australia
E-mail: acarey@maths.adelaide.edu.au

VARGHESE MATHAI
Department of Pure Mathematics, University of Adelaide
Adelaide 5005, Australia
E-mail: vmathai@maths.adelaide.edu.au

ALEXANDER S. MISHCHENKO
Department of Mathematics, Moscow State University
117 234 Moscow, Russia
E-mail: asmish@mech.math.msu.su

Abstract. In this paper, we present an analytic definition for the relative torsion for flat C^*-algebra bundles over a compact manifold. The advantage of such a relative torsion is that it is defined without any hypotheses on the flat C^*-algebra bundle. In the case where the flat C^*-algebra bundle is of determinant class, we relate it easily to the L^2 torsion as defined in [7], [5].

1. Introduction. We are interested in the problem of defining analogues of the torsion invariants found by Ray-Singer and Reidemeister-Franz for finite dimensional representations of the fundamental group of a compact manifold in the case where one considers infinite dimensional representations of the fundamental group. In the case where the fundamental group is represented in a finite von Neumann algebra and one has a cochain complex of free modules over this algebra for which the determinant of the associated Laplacian exists, a notion of von Neumann or L^2 torsion has been defined [2], [7], [5], [6], [1].

1991 *Mathematics Subject Classification*: 19, 57, 58.

Research of the third author partly supported by the Grant No.96-00-276 of Russian Foundation of Fundamental Sciences.

The paper is in final form and no version of it will be published elsewhere.

This paper is concerned with a generalisation of these notions to allow complexes which are free modules over the group C^*-algebra of the fundamental group. In order to describe our results we establish some notation and definitions. Our starting point is a non-simply connected Riemannian manifold M of dimension n with fundamental group $\pi = \pi_1(M)$ with $\mathcal{A}$ being the group C^*-algebra of the group π, $C^*[\pi] = \mathcal{A}$. Suppose in addition that the manifold M has a fixed simplicial structure. Consider the de Rham complex $\Omega^*(M;\mathcal{A}) = \oplus\Omega^i(M;\mathcal{A})$ and its completion with respect to the Sobolev norms $H\Omega^{*,s}(M;\mathcal{A}) = \oplus H^{s-i}\Omega^i(M;\mathcal{A})$ where s is chosen sufficiently large so that the exterior derivative is bounded on this complex. Introduce the simplicial chain complex $C^*(M;\mathcal{A}) = \oplus C^i(M;\mathcal{A})$ given by the simplicial structure on M and the de Rham homomorphism

$$\theta : \Omega^*(M;\mathcal{A}) \to C^*(M;\mathcal{A})$$

which is an epimorphism. Our aim is to define what we call a C^*-torsion or $\mathcal{A}$-torsion for the cone complex

$$\mathrm{Cyl}^k = H^0\Omega_k(M, C^*[\pi]) \oplus C^{k-1}(M, C^*[\pi])$$

of the de Rham map θ. The same methods work when we twist this cone complex by a flat $\mathcal{A}$-bundle E over M. The $\mathcal{A}$-torsion does not depend for its definition on any assumption about the large time asymptotics of the trace of the heat operator (that is, it does not assume anything about the so-called determinant type of M) [1].

There are a number of technical issues that need to be resolved for this to be well defined. We found it useful to introduce a subcomplex $U^*(M,\theta) = \mathrm{Ker}(\theta)$ which we show (Section 3) is acyclic in the algebraic sense and that there exists an orthogonal complement $V^*(M,\theta)$ such that the restriction

$$\theta|_{V^*(M,\theta)} : V^*(M,\theta) \to C^*(M;\mathcal{A})$$

is a chain homotopy equivalence. That is, the cone of this map, $\mathrm{Cone}(\theta|_{V^*(M,\theta)})$, is also an acyclic chain complex.

Our procedure is to then prove an analytic formula for the torsion for both acyclic complexes

$$\mathrm{Cone}(\theta|_{V^*(M,\theta)}) \tag{1}$$

and

$$U^*(M,\theta) = \mathrm{Ker}(\theta). \tag{2}$$

in the case where the Sobolev index $s = 0$. The technical results necessary for the definition of the latter are contained in Sections 5 and 6. Next in Section 7 we combine the torsion for (1) and (2) to define the $\mathcal{A}$-torsion for the cone complex Cyl^* of the de Rham map.

For the torsion of the complex (1) the procedure is straightforward as the complex $\mathrm{Cone}(\theta|_{V^*(M,\theta)})$ is a free finitely generated Hilbert $\mathcal{A}$-module. On the other hand, $U^*(M,\theta)$, though infinitely generated, is nevertheless a projective module (that is, the projection P_0 onto Ker θ is given by a matrix over $\mathcal{A}$.) It follows from this last fact that an analytic C^*-torsion may be defined for this complex. The result is an analogue of the formula for L^2-analytic torsion [7], [5].

Several technical results which are of independent interest are necessary to our analysis. For acyclicity of the cone complex we need a variation of a theorem of Dodziuk [3] (in Section 4) in which we prove that the de Rham map induces an isomorphism on the algebraic cohomology groups of the complexes $H^s\Omega^p(M;E_0)$ and $C^p(M;E_0)$. For the $\mathcal{A}$-torsion we need an extension of the analysis of [10] on the existence of complex powers (and the small time asymptotics of the heat operator) of the Laplacian of the complex Cyl^*.

Finally we turn to the question of how this $\mathcal{A}$-torsion relates to the earlier work on L^2 torsion. In these papers an essential assumption is made regarding the large time asymptotics of the trace of the heat operator on the de Rham complex which we refer to as the decay property for M. The $\mathcal{A}$-torsion does not depend for its definition on any assumption on the decay of M as we will show that the Laplacian is invertible on both of complexes (1) and (2). We know that the decay property for $H^s\Omega^p(M;E_0)$ and the chain complex $C^p(M;E_0)$ are the same so that for the purposes of determining decay it is sufficient to consider $\mathrm{Im}(1-P_0)$ which we show is homotopic to the chain complex $C^p(M;E_0)$. When the latter complex has the decay property the $\mathcal{A}$-torsion may be written as the ratio of the L^2-analytic and L^2-Reidemeister-Franz torsions. Hence by the theorem of [1] it is identically one. This of course raises the question of whether it is always one. We do not attempt to answer this here. We note however that if the projection onto the harmonic forms is given by a matrix over $\mathcal{A}$ then we may define, in the presence of decay, an $\mathcal{A}$-torsion for both the de Rham and simplicial complexes separately.

2. Further preliminaries. In view of the strategy outlined in the introduction the construction of the $\mathcal{A}$-torsion involves defining the torsion for an acyclic finitely generated differential complex of graded Hilbert $\mathcal{A}$-modules (1). There is a natural way to do this for any acyclic finitely generated differential complex of graded Hilbert $\mathcal{A}$-modules W^*. Let us consider the differential d_0 and its adjoint operator d_0^* defined by the Hilbert structure on the module W^*. Put

$$\Delta_0 = (d + d^*)^2.$$

Then the operator Δ_0 preserves the gradation and is invertible. Because the module W^* is free and finitely generated the operator Δ_0 can be represented as a matrix for some free basis in W^* with entries in the algebra $\mathcal{A}$. Thus given any trace tr on $\mathcal{A}$ one can define the function

$$f(t) = \operatorname{tr}\exp(-t\Delta_0). \tag{3}$$

Because the operator Δ_0 is bounded and invertible one has the inequality

$$\exp(-tC_1) < f(t) < \exp(-tC_2), \tag{4}$$

for some positive constant C_1, C_2. Therefore one can apply the zeta function definition of the determinant from which it is straightforward to define the torsion of the complex by analogy with [2].

However for the construction of an analytic torsion for an infinitely generated acyclic chain complex $U^*(M;\mathcal{A})$ there are difficulties. In this case we need to analyze the Laplacian as an unbounded operator. Actually, for the complex $H\Omega^{*,s}(M;\mathcal{A})$ with exterior differential d_s the operator

$$\Delta_s = (d_s + d_s^*)^2$$

is bounded, because

$$d_s : U^i = (U^* \cap H^{s-i}\Omega^i(M;\mathcal{A})) \to U^{i+1} = (U^* \cap H^{s-i-1}\Omega^{i+1}(M;\mathcal{A}))$$

is bounded. Therefore if one wants to regard the Laplacian as unbounded then one should consider the operator d_s as an unbounded operator d of degree 1:

$$\tilde{d} : H^0\Omega^i(M;\mathcal{A})) \to H^0\Omega^{i+1}(M;\mathcal{A}),$$

and its corresponding unbounded adjoint operator

$$\delta = \tilde{d}^* : H^0\Omega^{i+1}(M;\mathcal{A})) \to H^0\Omega^i(M;\mathcal{A}).$$

Then the Laplace operator

$$\Delta = (d+\delta)^2 : H^0\Omega^i(M;\mathcal{A})) \to H^0\Omega^i(M;\mathcal{A}))$$

is an unbounded operator. It is easy to show that $\Delta_s = \Delta(1+\Delta)^{-1}$. The main technicality associated with the definition of the torsion of (2) is to define the zeta function of the Laplacian and for this we need to determine the small time asymptotics of the trace of the heat operator for that complex. Then, to relate the torsions for (1) and (2) to that for the complex Cyl^*, we need the zeta function for the Laplacian of that complex as well. We defer a discussion of these matters to Section 6.

3. Adjoint to the de Rham homomorphism. In this section we deal with the technicalities required to prove the validity of the splitting of the complex Cyl^* into the complexes (1) and (2). Let the manifold M have a Riemannian metric which induces a scalar product on the vector bundles $\Lambda^p(M) = \Lambda^p(T^*M)$, denoted by

$$(\omega_1, \omega_2), \quad \omega_1, \omega_2 \in \Lambda^p_x(M), x \in M.$$

Put

$$\Gamma^\infty(\Lambda^p(M)) = \Omega^p(M) \tag{5}$$

and let $d\mu$ be the corresponding measure on M generated by the Riemannian metric. Consider the scalar product on the space $\Omega^p(M)$ which is determined by

$$\langle \omega_1, \omega_2 \rangle = \int_M (\omega_1, \omega_2) d\mu. \tag{6}$$

Let δ be the formal adjoint to d:

$$\delta : \Omega^p(M) \to \Omega^{p-1}(M), \tag{7}$$

$$\langle \delta\omega_1, \omega_2 \rangle = \langle \omega_1, d\omega_2 \rangle, \quad \omega_1 \in \Omega^p(M), \omega_2 \in \Omega^{p-1}(M). \tag{8}$$

3.1. The de Rham complex with values in a flat bundle. We extend the notation of the preceding discussion to the case where we have a flat bundle over M with fibre V being a Banach space which is an $\mathcal{A}$-module (free or projective) over a C^*-algebra $\mathcal{A}$. Let

$$\rho : \pi \to \mathrm{Aut}_A(V) \tag{9}$$

be a representation by isometries giving rise to the $\mathcal{A}$-module structure. Let E be the bundle over M associated with the representation ρ. Put

$$\Omega^p(M;E) = \Gamma^\infty(\Lambda^p(M) \otimes E). \tag{10}$$

The transition functions for E are locally constant and equal to $\rho(g) \in \mathrm{Aut}_{\mathcal{A}}(V)$ for some elements g depending on the choice of two charts from the atlas of charts. Then in $\Lambda^p(M) \otimes E$ there exists in each fibre a scalar product with values in $\mathcal{A}$:

$$(\omega_1 \otimes v_1, \omega_2 \otimes v_2) = (\omega_1, \omega_2)(v_1, v_2). \tag{11}$$

Because the transition functions are unitary (11) does not depend on the choice of a chart:

$$(\rho(g)v_1, \rho(g)v_2) = (v_1, v_2) \in \mathcal{A}, \quad v_1, v_2 \in V. \tag{12}$$

Then in $\Omega^p(M,E)$ we may define a scalar product with values in $\mathcal{A}$ and there exists an operator

$$d : \Omega^p(M;E) \to \Omega^{p+1}(M;E), \tag{13}$$

$$d(\omega \otimes v) = d\omega \otimes v, \quad \omega \in \Omega^p(M),\ v \in V\ (v \equiv \mathrm{Const}). \tag{14}$$

Let δ be defined by analogy with (7) and (9).

3.2. The simplicial complex. Let $C^p(M;E)$ be the simplicial cochain space with values in E. This means that each simplex σ lies in some chart U_α and since the restriction $E|_{U_\alpha} \approx U_\alpha \times V$ then the value of the cochain $x \in C^p(M;E)$ can be determined as an element $x(\sigma)_\alpha \in V$. If the simplex σ lies in the intersection of two charts $U_{\alpha\beta} = U_\alpha \cap U_\beta \supset \sigma$, then x has two values $x(\sigma)_\alpha$ and $x(\sigma)_\beta$ which are connected by the relation

$$\varphi_{\alpha\beta}(x(\sigma)_\alpha) = x(\sigma)_\beta, \tag{15}$$

($\varphi_{\alpha\beta}$ being a **constant** transition function). The de Rham homomorphism

$$\theta : \Omega^p(M;E) \to C^p(M;E) \tag{16}$$

is defined as follows. If $\sigma \subset U_\alpha$, $\omega \otimes v \in \Omega^p(M;E)|_{U_\alpha}$, then

$$\theta(\omega \otimes v) = \Big(\int_\sigma \omega\Big) v \in E. \tag{17}$$

The definition (3.2) does not depend on the choice of a chart. Then

$$\theta d = \partial\theta, \tag{18}$$

where $\partial : C^p(M;E) \to C^{(p+1)}(M;E)$ is the coboundary homomorphism.

3.3. Existence of the adjoint to the de Rham map. The first fact is an immediate consequence of the definition.

LEMMA 1. *The modules $C^p(M;E)$ are free $\mathcal{A}$-modules.*

Now let us define a scalar product

$$\langle \omega_1, \omega_2 \rangle_s = \langle \omega_1, (1+\Delta)^s \omega_2 \rangle \in \mathcal{A}, \tag{19}$$

where $\Delta = (d+\delta)^2$. Denote by $H^s\Omega^p(M;E)$ the completion of Ω^p with respect to the norm (19).

LEMMA 2. *If $s \gg 1$ then θ can be extended to an epimorphism*

$$\theta : H^s\Omega^p(M;E) \to C^p(M;E). \tag{20}$$

THEOREM 1. *The homomorphism (20) has an adjoint in the sense of Hilbert C^*-modules.*

PROOF. It is sufficient to check the property on restriction to one simplex σ, with $\dim\sigma = p, \sigma \subset U_\alpha$, that is, for the map

$$\theta_\sigma(\omega) = \int_\sigma \omega. \tag{21}$$

Let φ be a smooth function such that $\varphi|_\sigma \equiv 1$, $\operatorname{Supp}\varphi \subset U_\alpha$. If $\omega \in \Omega^p(M;E)$ then

$$\theta_\sigma(\omega) = \theta_\sigma(\varphi\omega). \tag{22}$$

We are looking for an element $\xi \in H^s\Omega^p(M;E)$ such that

$$\theta_\sigma(\omega) = \langle \xi, \omega\rangle_s. \tag{23}$$

Consider a neighborhood $O(\sigma)$ such that $O(\sigma) \subset \overline{O(\sigma)} = Y$, $\operatorname{Supp}\varphi \subset O(\sigma)$. Let $H = H(Y) \subset H^s\Omega^p(M;E)$ be the subspace generated by the forms $\omega \in \Omega^p(M;E)$ for which one has $\operatorname{Supp}\omega \subset Y$. Then H is a Hilbert $\mathcal{A}$-module. We show now how the result follows from the next three lemmas (whose proofs we defer to the end of the section).

LEMMA 3. *The subspace $H \subset H^s\Omega^p(M;E)$ is the image of a selfadjoint projector*

$$P : H^s\Omega^p(M;E) \to H^s\Omega^p(M;E).$$

In this case one has

$$\theta_\sigma(\omega) = \theta_\sigma(\varphi\omega) = \theta_\sigma(P\varphi\omega). \tag{24}$$

LEMMA 4. *The restriction $\theta_\sigma|_H : H \to \mathcal{A}$ has an adjoint operator.*

This means that

$$I_\sigma|_H(h) = \langle \xi_H, h\rangle_s, \tag{25}$$

where $\xi_H \in H$, $h \in H$. Hence

$$\theta_\sigma(\omega) = \theta_\sigma(P\varphi\omega) = \langle \xi_H, P\varphi\omega\rangle_s = \langle P^*\xi_H, \varphi\omega\rangle_s. \tag{26}$$

LEMMA 5. *The operator $\hat\varphi : H^s\Omega^p(M;E) \to H^s\Omega^p(M;E)$*

$$\hat\varphi : \omega \mapsto \varphi\omega \tag{27}$$

has an adjoint.

Then

$$\theta_\sigma(\omega) = \langle P^*\xi_H, \varphi\omega\rangle_s = \langle P^*\xi_H, \hat\varphi(\omega)\rangle_s = \langle \hat\varphi^* P^*\xi_H, \omega\rangle_s, \tag{28}$$

which is the desired conclusion. Therefore it remains to prove Lemmas 3–5.

PROOF OF LEMMA 5. We have

$$\begin{aligned}\langle \xi, \hat\varphi\omega\rangle_s = \langle \xi, (1+\Delta)^s\hat\varphi\omega\rangle &= \langle \hat\varphi(1+\Delta)^s\xi, \omega\rangle \\ &= \langle (1+\Delta)^{-s}\hat\varphi(1+\Delta)^s\xi, (1+\Delta)^s\omega\rangle \\ &= \langle (1+\Delta)^{-s}\hat\varphi(1+\Delta)^s\xi, \omega\rangle_s.\end{aligned} \tag{29}$$

The operator $(1+\Delta)^{-s}\hat\varphi(1+\Delta)^s$ is bounded from $H^s\Omega^p(M;E)$ to itself. ∎

Proof of Lemma 3. Consider the bundle $\xi = \Lambda^p(M;E)$. Because M is compact, there exists an $\mathcal{A}$-bundle η such that $\xi \oplus \eta$ is trivial,

$$\zeta = \xi \oplus \eta = M \times (\mathcal{A} \oplus \ldots \oplus \mathcal{A}) = M \times \mathcal{A}^N. \tag{30}$$

This means that there exists a continuous family of $\mathcal{A}$-projectors,

$$Q_x : \mathcal{A}^N \to \mathcal{A}^N, \quad x \in M$$

such that $\operatorname{Im} Q_x = \xi$. All Q_x are selfadjoint. Therefore the inclusion

$$\psi : \Gamma^\infty(\xi) \to \Gamma^\infty(\zeta)$$

is an isometry. Therefore the inclusion ψ extendes to a continuous operator

$$\psi : H^s\Gamma^\infty(\xi) \to H^s\Gamma^\infty(\zeta) \tag{31}$$

and $H^s\Gamma^\infty(\xi)$ is the image of the projector Q generated by the family $\{Q_x\}$. ∎

Let $H_1 = H_1(Y) \subset H^s\Gamma^\infty(\zeta)$ be generated by sections with support in Y. We need the following fact.

Lemma 6. *There exists a selfadjoint projector P_1, $P_1 = P_1^*$, $H_1 = \operatorname{Im} P_1$.*

We defer the proof until the end of the proof of Lemma 3. Now we have

$$H = H^s\Gamma^\infty(\xi) \cap H_1. \tag{32}$$

Indeed, if $\omega \in H$, $\omega = \lim \omega_n$, $\omega_n \in \Gamma^\infty$, $\operatorname{Supp} \omega_n \subset Y$, then $\omega_n \in H_1$ implies that $\omega \in H_1$. Connversely, if $\omega \in H^s\Gamma^\infty(\xi) \cap H_1$ then $\omega = \lim \omega_n$, $\operatorname{Supp} \omega_n \subset Y$, $Q\omega = \omega$.

Next we consider $\omega_n' = Q\omega_n' \in \Gamma^{i\infty}(\xi)$. Then $\operatorname{Supp} \omega_n' \subset \operatorname{Supp} \omega_n \subset Y$, that is, the relation $\omega_n' \in H$ implies

$$\lim \omega_n' = Q \lim \omega_n = Q\omega = \omega \in H.$$

Moreover

$$P_1 Q = Q P_1. \tag{33}$$

Indeed, let $\omega \in H^s\Gamma^\infty(\zeta)$. Then $P_1\omega = \lim \omega_n$, $\operatorname{Supp} \omega_n \subset Y$ and so $P_1\omega_n = \omega_n$ and $P_1 Q\omega_n = Q\omega_n$. Therefore $(QP_1 - P_1Q)\omega_n = 0$ that is, $(QP_1 - P_1Q)\omega = 0$, which means that $QP_1 = P_1QP_1$ and hence $P_1Q = QP_1$.

Let $P_2 = P_1QP_1$ and calculate:

$$P_2P_2 = P_1QP_1P_1QP_1 = P_1QP_1QP_1 = P_1QQP_1 = P_1QP_1 = P_2 \tag{34}$$

so that P_2 is a projector. Next we show that $\operatorname{Im} P_2 = H$. If $P_2\omega = \omega$ then

$$\omega = QP_1\omega = P_1QP_1\omega = P_1\omega.$$

Hence $\omega = \lim \omega_n$, $\operatorname{Supp} \omega_n \subset Y$, which implies that $\operatorname{Supp} Q\omega_n \subset Y$, that is, $\lim Q\omega_n \in H$. But $\lim Q\omega_n = Q\omega = \omega \in H$.

Conversely, let $\omega \in H$, $H \subset H^s\Gamma^\infty(\xi)$. Then

$$Q\omega = \omega, \quad \omega = \lim \omega_n, \quad Q\omega_n = \omega_n, \quad \operatorname{Supp} \omega_n \subset Y,$$

which implies that $P_1\omega_n = \omega_n$ and hence $P_2\omega_n = \omega_n \in \operatorname{Im} P_2$. Therefore if $\omega_1, \omega_2 \in$

$H^s\Gamma^\infty(\xi)$ then

$$\langle P\omega_1, \omega_2\rangle_s = \langle PQ\omega_1, \omega_2\rangle_s$$

showing that the restriction of P_2 on $H^s\Gamma^\infty(\xi)$ coincides with P.

If $\omega \in H^s\Gamma^\infty(\xi)$ then

$$Q\omega = \omega, \quad P_2\omega = \lim \omega_n, \quad \operatorname{Supp} \omega_n \subset Y.$$

Hence $\operatorname{Supp} Q\omega_n \subset Y$, that is, $\lim Q\omega_n = QP_2\omega = P_2Q\omega = P_2\omega$ and so $P_2\omega \in \operatorname{Im} P$. If $\omega \in \operatorname{Im} P$ then $\omega = \lim \omega_n$, $Q\omega = \omega$, $Q\omega_n = \omega_n$, $\operatorname{Supp} \omega_n \subset Y$. Hence $\omega \in \operatorname{Im} P_1$, that is, $P_1\omega = P_1$ and so $P_2\omega = \omega$.

Therefore

$$\langle P\omega_1, \omega_2\rangle_s = \langle P_2\omega_1, \omega_2\rangle_s = \langle \omega_1, P_2\omega_2\rangle_s = \langle \omega_1, P\omega_2\rangle_s. \ \blacksquare$$

Now we return to the discussion of Lemma 6.

Proof of Lemma 6. It suffices to prove this, not for η, but for the 1-dimensional bundle $M \times \mathcal{A}$. From $\Gamma^\infty(M\times\mathcal{A}) \supset C^\infty(M)\otimes\mathcal{A}$ we have $H^sC^\infty(M)\otimes\mathcal{A} \subset H^s\Gamma^\infty(M\times\mathcal{A})$. Now consider

$$H_2 \subset H^sC^\infty \tag{35}$$

generated by those $f \in C^\infty(M)$ with $\operatorname{Supp} f \subset Y$. Let

$$P_3 : H^sC^\infty(M) \to H^sC^\infty(M)$$

be the selfadjoint operator onto H_2. Then

$$P_3 \otimes 1 : H^sC^\infty(M) \otimes \mathcal{A} \to H^sC^\infty(M) \otimes \mathcal{A}$$

is a bounded operator and has a unique selfadjoint extension

$$P_4 : H^s\Gamma^\infty(M \times \mathcal{A}) \to H^s\Gamma^\infty(M \times \mathcal{A})$$

and $\operatorname{Im} P_4$ is generated by functions $f \in \Gamma^\infty(M \times \mathcal{A})$, $\operatorname{Supp} f \subset Y$. $\blacksquare$

Proof of Lemma 4. Let $H_4 \subset H^s\Omega^p(M)$ be the completion of the subspace $\{\omega \mid \operatorname{Supp}\omega \subset Y\}$. Then there exists a natural inclusion

$$H_4 \otimes \mathcal{A} \to H \subset H^s\Omega^p(M).$$

The restriction $\theta_\sigma|_H$ has the following form on the element $\omega \otimes b \in H_4 \otimes \mathcal{A}$:

$$\theta_\sigma|_H(\omega \otimes b) = \theta_\sigma(\omega) \otimes b.$$

This means that $\theta_\sigma|_H = \theta_\sigma \otimes \mathrm{id} : H_4 \otimes \mathcal{A} \to C \otimes \mathcal{A} = \mathcal{A}$. As $\theta_\sigma : H_4 \to C$ has an adjoint operator, so $\theta_\sigma \otimes \mathrm{id}$ has an adjoint as well. $\blacksquare$

3.4. The subcomplex $\operatorname{Ker}\theta$. Let $H_0 = l_2(\pi)$ and let $\mathcal{A}$ act on H_0 via the left regular representation. If we let E_0 be the corresponding bundle then

$$C^*(M; E_0) = C^*(M; E)_{\mathcal{A}} \otimes H_0.$$

Hence

$$\Omega^p(M; E_0) \supset \Omega^p(M; E)_{\mathcal{A}} \otimes H_0$$

and

$$H^s\Omega^p(M;E_0) \supset H^s\Omega^p(M;E)_{\mathcal{A}} \otimes H_0,$$

the last being a dense subspace. The de Rham map (20) can be extended to a homomorphism

$$\theta_0 = \theta \otimes \mathrm{id} : H^s\Omega^p(M;E_0) \to C^p(M;E_0) \tag{36}$$

which coincides with the homomorphism $\int$ of lemma 3.2 of [3]. Since (20) has an adjoint, $\operatorname{Ker}\theta$ can be split with a selfadjoint projector

$$P : H^s\Omega^p(M;E) \to H^s\Omega^p(M;E)$$

so that

$$\theta_1 = \theta|_{\mathrm{Im}(1-P)} : \mathrm{Im}(1-P) \to C^p(M;E) \tag{37}$$

is an isomorphism. Hence the map θ_1 is a chain homotopy equivalence.

Consider the projector

$$P_0 = P \otimes 1 : P : H^s\Omega^p(M;E_0) \to H^s\Omega^p(M;E_0).$$

It is clear from the fact that (37) is a chain homotopy equivalence that the homomorphism

$$\theta_1 \otimes 1 : \mathrm{Im}(1-P_0) \to C^p(M;E_0) \tag{38}$$

is also a chain homotopy equivalence. We will show later that the Laplacian is invertible on the range of P_0.

4. The Dodziuk theorem. In [3] it was proved that the L^2-cohomology of a nonsimply connected compact closed manifold for the simplicial and de Rham complexes are isomorphic. There are two ways to define the L^2-cohomology for these complexes. The first, considered in [3], is to define each cohomology group as the quotient of the kernel of the coboundary by the closure of the image of the coboundary. The second is to consider the algebraic quotient group of the kernel by the image (without taking the closure). In [3] the question is posed as to whether a similar theorem about the isomorphism of the simplicial and the de Rham L^2-cohomology holds for the second version of L^2-cohomology.

Here we give a positive answer to this question not for L^2-cohomology but for the cohomology over the group C^*-algebra $C^*[\pi]$ of the fundamental group π.

Let M be as usual a nonsimply connected closed compact manifold, $\pi = \pi_1(M)$, with a fixed smooth simplicial structure and let

$$\rho : \pi \to C^*[\pi]$$

be the regular representation. Denote by $C^k(M, C^*[\pi]) = C^k$ the space of k-dimensional cochains with values in the local system of coefficients generated by the representation ρ. Let ∂ be the coboundary operator $\partial = \partial_k : C^k \to C^{k+1}$. Let E be a locally flat vector bundle with fibre $\mathcal{A}$ and transition functions generated by the representation ρ. Let $\Omega_k^\infty(M, C^*[\pi]) = \Omega_k^\infty$ be the space of k-dimensional smooth differential forms with values in the fibres of the vector bundle E. One can think of the space Ω_k^∞ as the space of sections for the vector bundle

$$\Lambda_k = \Lambda_k(M, C^*[\pi]) = \Lambda_k(M) \otimes E, \qquad \Omega_k^\infty = \Gamma^\infty(\Lambda_k).$$

Let $d : \Omega_k^\infty \to \Omega_{k+1}^\infty$ be the exterior differential with adjoint δ and as usual let θ be the (surjective) de Rham homomorphism $\theta : \Omega_k^\infty \to C^k$. Let $\Delta = (d+\delta)^2$ and recall the Sobolev norm (19) on Ω_k^∞ with completion $H^s\Omega_k$. Then for sufficiently large s, θ can be extended to a bounded operator $\theta : H^s\Omega_k \to C^k$ and hence one has the commutative diagram

$$\begin{array}{ccc} H^s\Omega_k & \xrightarrow{d} & H^{s-1}\Omega_{k+1} \\ \downarrow \theta & & \downarrow \theta \\ C^k & \xrightarrow{\partial} & C^{k+1} \end{array} \tag{39}$$

The problem is to prove that θ induces an isomorphism of the homology groups in the following morphism of complexes:

$$\begin{array}{ccccccc} H^s\Omega_0 & \xrightarrow{d} & H^{s-1}\Omega_1 & \xrightarrow{d} & \dots & \xrightarrow{d} & H^{s-n}\Omega_n \\ \downarrow \theta & & \downarrow \theta & & & & \downarrow \theta \\ C^0 & \xrightarrow{\partial} & C^1 & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C^n \end{array} \tag{40}$$

where the homology groups should be understood in the algebraic sense:

$$H^k(\Omega) = \operatorname{Ker} d / \operatorname{Im} d, \qquad H^k(C) = \operatorname{Ker} \partial / \operatorname{Im} \partial.$$

It will be sufficient to restrict to smooth version sections. Consider the diagram

$$\begin{array}{ccccccc} \Omega_0^\infty & \xrightarrow{d} & \Omega_1^\infty & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega_n^\infty \\ \downarrow \theta & & \downarrow \theta & & & & \downarrow \theta \\ C^0 & \xrightarrow{\partial} & C^1 & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C^n \end{array} \tag{41}$$

We now prove that in (41) the homomorphism θ induces an isomorphism in homology. Denote by $C^{p,q}$ the space of p-dimensional cochains with values in q-dimensional differential forms Ω_q^∞:

$$C^{p,q} = C^p(\Omega_q^\infty).$$

Then one has a bicomplex:

$$\begin{array}{cccccc} C^{0,0} & \xrightarrow{d} & C^{0,1} & \xrightarrow{d} & \dots & C^{0,n} \\ \downarrow \partial & & \downarrow \partial & & & \downarrow \partial \\ C^{1,0} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & \dots & C^{1,n} \\ \downarrow \partial & & \downarrow \partial & & & \downarrow \partial \\ \vdots & & \vdots & & & \vdots \\ \downarrow \partial & & \downarrow \partial & & & \downarrow \partial \\ C^{n,0} & \xrightarrow{d} & C^{n,1} & \xrightarrow{d} & \dots & C^{n,n} \end{array} \tag{42}$$

The horizontal complexes in the diagram (42) are exact except the first kernel which is $C^k = C^k(M, \rho)$, a chain complex of constant functions with values in the locally flat bundle E. The vertical complexes in the diagram (42) are exact except the first kernel which is $\Omega_k = \Omega_k(X, \rho)$. Therefore the diagram (42) may be extended to an exact diagram in both directions:

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & & & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & \Omega_0^\infty & \xrightarrow{d} & \Omega_1^\infty & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega_n^\infty & \to & 0 \\
 & & & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & & & \downarrow{\scriptstyle\epsilon} & & \\
0 & \to & C^0 & \xrightarrow{\epsilon'} & C^{0,0} & \xrightarrow{d} & C^{0,1} & \xrightarrow{d} & \dots & \xrightarrow{d} & C^{0,n} & \to & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & & & \downarrow{\scriptstyle\partial} & & \\
0 & \to & C^1 & \xrightarrow{\epsilon'} & C^{1,0} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & \dots & \xrightarrow{d} & C^{1,n} & \to & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & & & \downarrow{\scriptstyle\partial} & & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & & & \downarrow{\scriptstyle\partial} & & \\
0 & \to & C^n & \xrightarrow{\epsilon'} & C^{n,0} & \xrightarrow{d} & C^{n,1} & \xrightarrow{d} & \dots & \xrightarrow{d} & C^{n,n} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
\qquad (43)
$$

The exactness of horizontal rows follows from the fact that each intersection of charts is a contractible subspace (as an open star of a simplex). The exactness of the vertical column follows from the fact that the sheaf of germs of differential forms is fine. Therefore the result follows from the common algebraic fact that homology groups of the upper row are isomorphic to the homology groups of the left column.

5. The trace of the heat operator

5.1. Preliminaries. We are now ready to discuss some of the problems associated with the definition of the $\mathcal{A}$-torsion. Recall the space $\Omega_k(M, C^*[\pi])$ of smooth differential forms with values in the fibres of the locally flat bundle $E(M, C^*[\pi])$ generated by the regular representation of the group π in $C^*[\pi]$. This means that $\Omega_k(M, C^*[\pi])$ may be represented as the space of smooth sections of the bundle $\Lambda_k(M, C^*[\pi]) = \Lambda_k(M) \otimes E(M, C^*[\pi]$. The space $\Omega_k(M, C^*[\pi])$ is a module over the algebra $C^*[\pi]$. Using the Riemannian metric one can introduce a pre-Hilbert structure on $\Omega_k(M, C^*[\pi])$ with values in $C^*[\pi]$ in the sense of Pashke ([8]). The differential d commutes with the $C^*[\pi]$-module action as does its formal adjoint operator d^* and the corresponding Laplacian $\Delta = dd^* + d^*d$. Recall that the Sobolev completion $H^s\Omega_k(M, C^*[\pi])$ is defined using the formulas

$$(\omega_1, \omega_2)_s = ((1+\Delta)^s \omega_1, \omega_2) \in C^*[\pi], \quad \|\omega\|_s^2 = \|(\omega_1, \omega_2)_s\|,$$

so that each space $H^s\Omega_k(M, C^*[\pi])$ is isomorphic to the special Hilbert $C^*[\pi]$-module denoted $l_2(C^*[\pi])$ defined as the space of all sequences $\{x_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty x_k x_k^*$ converges in the algebra $C^*[\pi]$. Then the operator d is bounded from $H^{s+1}\Omega_k(M, C^*[\pi])$ to $H^s\Omega_k(M, C^*[\pi])$ and this gives a natural extension on d to an unbounded operator $d : H^s\Omega_k(M, C^*[\pi]) \to H^s\Omega_{k+1}(M, C^*[\pi])$ on the Hilbert $C^*[\pi]$-module $H^s\Omega_k(M, C^*[\pi])$. Here the subspace

$$H^s\Omega_k(M, C^*[\pi])$$

is the domain of definition of the extension of d. In a similar way one can define an extension of the Laplace operator

$$\Delta : H^{s+2}\Omega_k(M, C^*[\pi]) \to H^s\Omega_k(M, C^*[\pi]).$$

One may then consider the operator $\exp(-t\Delta)$ as a bounded operator from the Hilbert C^*-module $H^0\Omega_k(M, C^*[\pi])$ to itself.

This operator may therefore be represented as a matrix over $C^*[\pi])$. Our aim is to show that this matrix has a trace as follows. Given the canonical trace function on the C^*-algebra we define the trace of the operator $\exp(-t\Delta)$ by composition of the sum of the diagonal elements of the matrix and the trace function. To see that this works, consider the bounded operator $A = \Delta(1+\Delta)^{-1}$. This is a pseudodifferential operator of order 0. Then $\exp(-t\Delta) = f(A)$ for $f(\lambda) = \exp(\frac{-t\lambda}{1-\lambda})$. Next we observe that $f(A)$ is a smoothing operator, that is, a pseudodifferential operator of order $-\infty$. In this case $f(A)$ is a bounded operator

$$f(A) : H^{s+N}\Omega_k(M, C^*[\pi]) \to H^s\Omega_k(M, C^*[\pi])$$

for any sufficiently large N.

Lemma 7. *The operator $f(A)$ is represented by an infinite matrix (for any orthonormal basis) with convergent trace.*

Proof. One can define an adjoint operator by using the Hilbert module structure of $H^{s+1}\Omega_k(M, \mathcal{A})$ and $H^s\Omega_{k+1}(M, \mathcal{A})$. Thus define $d^{\#}$ by the formula

$$(d\omega_1, \omega_2)_s = (\omega_1, d^{\#}\omega_2)_{s+1}.$$

It is easy to check that

$$d^{\#} = d^*(1+\Delta)^{-1}.$$

Then one can express $\exp(-t\Delta)$ in terms of the bounded Laplace operator of the de Rham–Sobolev complex $\Delta_{\text{R–S}} = dd^{\#} + d^{\#}d$:

$$A = \exp(-t\Delta) = f(\Delta_{\text{R–S}}).$$

Then the operator A is a pseudodifferential operator of order $-N$ for sufficiently large N because

$$A = f(\Delta_{\text{R–S}}) = (1-\Delta_{\text{R–S}})^N h_t(\Delta_{\text{R–S}}),$$

where $h_t(\lambda) = (1-\lambda)^{-N} f_t(\lambda)$. Now we have that the trace of $A = (1-\Delta_{\text{R–S}})^N h_t(\Delta_{\text{R–S}})$, exists and that

$$\operatorname{tr}(1-\Delta_{\text{R–S}})^N h_t(\Delta_{\text{R–S}}) < C\|h_t(\Delta_{\text{R–S}})\|,$$

where the constant C does not depend on h. ∎

5.2. Construction of the trace for the cone of the de Rham homomorphism. The next step is to extend the previous construction to the cone of the de Rham homomorphism

$$\theta : H^s\Omega_k(M, C^*[\pi]) \to C^k(M, C^*[\pi]).$$

Recall the definition of the cone. Given the commutative diagram

$$\begin{array}{ccccccc} \to & H^s\Omega_k(M, C^*[\pi]) & \to & H^s\Omega_{k+1}(M, C^*[\pi]) & \to \\ & \downarrow & & \downarrow & \\ \to & C^k(M, C^*[\pi]) & \to & C^{k+1}(M, C^*[\pi]) & \to \end{array}$$

Then the cone $\text{Cyl}^{k,s}$ is defined as

$$\text{Cyl}^{k,s} = H^s\Omega_k(M, C^*[\pi]) \oplus C^{k-1}(M, C^*[\pi])$$

with boundary homomorphism d_c as

$$d_c = \begin{pmatrix} d & 0 \\ \theta & -d_0 \end{pmatrix}.$$

The homomorphism d_c differs from d by a finite dimensional operator. Hence the corresponding Laplace operator $d_c d_c^{\#} + d_c^{\#} d_c$ has the same properties as $\Delta_{\text{R–S}}$ including finite trace for the corresponding heat operator. On the other hand using the fact that θ induces an isomorphism of cohomology groups in the algebraic sense one concludes that the cone is an acyclic complex. This means that the Laplace operator Δ_c is invertible, that is, $\operatorname{Spec}(\Delta_c) \subset [\varepsilon, 1]$.

There is one more fact we need in order to obtain the zeta function. Since the torsion is defined for the complex $\mathrm{Cyl}^* \equiv \mathrm{Cyl}^{*,0}$ we need to work now with the Laplacian Δ_c of Cyl^* as an unbounded operator. This means that we need to understand whether the asymptotic behaviour of the trace of the heat operator for t near zero is the same as in the classical case:

$$\operatorname{tr}(\exp -t\Delta_c) = \sum_{m=0}^{n} t^{m-N/2} C_m + o(t^{n-N/2}) \tag{44}$$

We deal with this in the next section.

6. Complex powers of framed elliptic operators

6.1. Formulation of the problem. In [10] the construction of complex powers for elliptic (invertible) operators on a smooth manifold M was achieved. Here we shall extend this construction to a more general situation in two directions. For the first, we shall consider elliptic pseudodifferential operators with coefficients in an arbitrary C^*-algebra $\mathcal{A}$. For the second, we shall consider what we call framed pseudodifferential operators, which are (2×2) matrices

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{45}$$

where D_{11} is an elliptic pseudodifferential operator

$$D_{11} : \Gamma^\infty(M, \xi) \to \Gamma^\infty(M, \xi), \tag{46}$$

and D is a homomorphism

$$D : \Gamma^\infty(M, \xi) \oplus V \to \Gamma^\infty(M, \xi) \oplus V, \tag{47}$$

where V is a finitely generated projective Hilbert $\mathcal{A}$-module. Here we assume that other entries of the matrix D have the following form:

The map

$$D_{12} : V \to \Gamma^\infty(M, \xi) \tag{48}$$

should be a linear $\mathcal{A}$-homomorphism, while the map

$$D_{21} : \Gamma^\infty(M, \xi) \to V \tag{49}$$

should be a functional with smooth kernel. This means that the operator D_{21} can be written in the following form:

$$D_{21}(u)(x) = \int_M f(x)(u(x))dx, \tag{50}$$

where

$$f(x) : \xi \to V \tag{51}$$

is a smooth homomorphism from the vector bundle ξ to the module V. In addition we assume that D_{11} is an elliptic pseudodifferential operator with the properties of the operator A of Section 6 of Seeley [10], p. 298.

6.2. Construction of a parametrix. For definiteness we will assume that the operator D is selfadjoint and positive. The last means that the operator D is invertible as an unbounded operator

$$D : H^0\Gamma^\infty(M,\xi) \oplus V \to H^0\Gamma^\infty(M,\xi) \oplus V, \tag{52}$$

while the first means that

$$D_{11}^* = D_{11}, \quad D_{22}^* = D_{22}, \quad D_{12}^* = D_{21}. \tag{53}$$

In particular this means that

$$D_{12}(m)(x) = f^*(m)(x),\ x \in M. \tag{54}$$

We shall assume that homomorphism (54) belongs to the Sobolev space H^N, where N is sufficiently large. We denote by $\|D\|_{l,l'}$ the operator norm whenever

$$D : H^l\Gamma^\infty(M,\xi) \oplus V \to H^{l'}\Gamma^\infty(M,\xi) \oplus V.$$

Consider the parametrix $B_{11}(\lambda)$ for the operator D_{11}. It is a pseudodifferential operator whose symbol $\sigma(B_{11}(\lambda))$ in an arbitrary local chart satisfies the condition:

$$\sigma(B_{11}(\lambda))\sigma(D_{11} - \lambda) = I. \tag{55}$$

The local symbol of $B_{11}(\lambda)$ can be written as in ([10], (1)). Put

$$B(\lambda) = \begin{pmatrix} B_{11}(\lambda) & 0 \\ 0 & 0 \end{pmatrix}. \tag{56}$$

Following Theorem 1 from ([10]) we have:

THEOREM 2. *For any $l, K >> 0, 0 \le \varepsilon \le 1$ one has*

$$\|B(\lambda)(D - \lambda) - I\|_{l,l+K} \le C|\lambda|^{-1+\varepsilon}. \tag{57}$$

The constant C may depend on $l, K >> 0, 0 \le \varepsilon \le 1$, but not on λ for $\operatorname{Re}\lambda < 0$.

PROOF. One has

$$\begin{aligned} B(\lambda)(D-\lambda) - I &= \begin{pmatrix} B_{11}(\lambda) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_{11} - \lambda & D_{12} \\ D_{21} & D_{22} - \lambda \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} B_{11}(\lambda)(D_{11} - \lambda) - I & B_{11}(\lambda)D_{12} \\ 0 & -I \end{pmatrix}. \end{aligned} \tag{58}$$

The term $B_{11}(\lambda)(D_{11} - \lambda) - I$ of (58) can be estimated by Theorem 1 of [10], while the term $B_{11}(\lambda)D_{12}$ can be estimated from the inequality (see [10], p. 298)

$$\|B_{11}(\lambda)\|_{l,l+p} \le C_p|\lambda|^{-1+p/m} \tag{59}$$

and the fact that

$$\|D_{12}(m)\|_N < \infty. \ \blacksquare \tag{60}$$

The next result is the analogue of Corollary 1 of [10], p. 298 and is proved in a similar fashion from the preceding theorem.

COROLLARY. *The operator $(D-\lambda)^{-1}$ satisfies the estimate*

$$\|(D-\lambda)^{-1}\|_{0,0} \leq C'|\lambda|^{-1}$$

provided $|\lambda|$ is sufficiently large and in a region of the complex plane distant $O(|\lambda|)$ from the spectrum of D.

6.3. Complex powers and the zeta function. With these preliminary estimates established we can now define complex powers of D following the method of [10]. For simplicity we restrict to the case of greatest interest for us, namely where D is a framed second order elliptic differential operator with coefficients in $\mathcal{A}$. Specifically we have that D is positive selfadjoint with no spectrum in some small interval $[0, \lambda_0)$. We also assume that $\mathcal{A}$ is the group C^*-algebra of the fundamental group of M equipped with its canonical trace.

Now choose γ to be a contour in the complex plane consisting of $\gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1 := \{z = \rho e^{i\pi}; \infty \leq \rho \leq \epsilon\}$, $\gamma_2 := \{z = \epsilon e^{i\alpha}; \pi \geq \alpha \geq -\pi\}$ and $\gamma_3 := \{z = \rho e^{i(-\pi)}; \epsilon \leq \rho \leq \infty\}$ where $\epsilon < \lambda_0$ so that Γ avoids the spectrum of D. Then by the corollary to Theorem 2 of Subsection 6.2 we may set

$$D^{-s} = \frac{1}{2\pi i}\int_\gamma \lambda^{-s}(\lambda - D)^{-1} d\lambda \tag{61}$$

for $\operatorname{Re} s$ positive. For general s we define $D^{-s} = D^{-k}D^{-s+k}$ where k is an integer chosen so that $s-k$ has positive real part. We will use the method of [10] to determine the properties of the trace of D^{-s} (that is, of the zeta function of D). We claim that the estimates of Subsection 6.2 show that the analyticity properties in s of $\operatorname{tr} D^{-s}$ are exactly those of $\operatorname{tr}\, G(s)$ where $G(s) = \frac{1}{2\pi i}\int_\gamma \lambda^{-s}B(\lambda)d\lambda$.

LEMMA 8. *(i) For s in the region $\operatorname{Re} s > n/2$, $\operatorname{tr} G(s)$ is analytic.*
(ii) The difference $\operatorname{tr} D^{-s} - \operatorname{tr} G(s)$ is analytic in the region $\operatorname{Re} s > -1$.
(iii) The zeta function

$$\zeta_D(s) = \operatorname{tr}(D^{-s})$$

is well defined and analytic for $\operatorname{Re} s > n/2$.

PROOF. (i) We use the methods of [10]. Thus it suffices to work in a local chart and to consider the top symbol of $B(\lambda)$ denoted $b_{-2}(x,\xi,\lambda)$ where we have local coordinates $x \in R^n$, with $\xi \in R^n$ being the Fourier transform variable. Hence as in [10] we need only check the analytic behaviour in s of $\int_M k_s(x,x)$ where

$$k_s(x,x) = \frac{1}{2\pi i}\frac{1}{(2\pi)^n}\int d\xi \int_\gamma d\lambda \lambda^{-s} b_{-2}(x,\xi,\lambda).$$

Using the fact that the homogeneity in (ξ,λ) of b_{-2} is the same as in the classical case we see that $\int_M k_s(x,x)$ is well defined for $\operatorname{Re} s > n/2$ from which we obtain the first part of the lemma.

(ii) Consider the difference

$$\operatorname{tr} D^{-s} - \operatorname{tr} G(s) = \int_\gamma \lambda^{-s} \operatorname{tr}((\lambda - D)^{-1} - B(\lambda))d\lambda.$$

We denote by $\|.\|_{\mathrm{tr}}$ the trace norm so that

$$\begin{aligned} |\operatorname{tr}((\lambda - D)^{-1} - B(\lambda))| &\le \|(B(\lambda)(\lambda - D) - I)(\lambda - D)^{-1}\|_{\mathrm{tr}} \\ &\le \|(B(\lambda)(\lambda - D) - I)\|_{\mathrm{tr}} \|(\lambda - D)^{-1}\|. \end{aligned}$$

Using (58) for $C = B(\lambda)(\lambda - D) - I$ we see that, in a local chart, the operator $(C^*C)^{1/2}$ is pseudodifferential with the trace norm estimable in terms of an appropriate Sobolev norm $\|D\|_{l,l+K}$. This norm by Theorem 2 satisfies the estimate:

$$\|B(\lambda)(D - \lambda) - I\|_{l,l+K} \le C|\lambda|^{-1+\varepsilon}.$$

Combining this last fact with the Corollary to Theorem 2 completes the proof.

(iii) This follows from (i) and (ii).

We will also be interested in the heat semigroup which may be studied as follows. Introduce the path γ_0 which is the composition of two half lines

$$\gamma_\pm = \{x \pm i(x+1) | \ -1 \le x < \infty\}.$$

Then we may define, using Corollary 1 of Subsection 6.2,

$$e^{-tD} = \frac{1}{2\pi i} \int_{\gamma_0} (e^{-\lambda t}(\lambda - D)^{-1})dt. \tag{62}$$

Now $\mathrm{tr} e^{-tD} < \infty$ for $t > 0$ (see Subsection 5.2). We are interested in the asymptotic expansion of $\operatorname{tr}(e^{-tD})$ for small t. This is determined by the asymptotics of $\operatorname{tr} F(t)$ where

$$F(t) = \int_\Gamma e^{-t\lambda} B(\lambda) d\lambda.$$

The leading term in the asymptotic expansion of $\operatorname{tr} F(t)$ for small t is calculated by considering the symbol expansion of $B(\lambda)$ in a local chart in M. The relevant asymptotics is given by the top symbol of $B(\lambda)$ denoted $b_{-2}(x, \xi, \lambda)$ above. Corresponding to this top symbol we define a pseudodifferential operator which is given, in this local chart, by an $\mathcal{A}$-valued kernel $k_t(x, y)$ with $x, y \in R^n$ so that the asymptotics we need follows from the t-dependence of $\int_M k_t(x, x)dx$ where

$$k_t(x, x) = \frac{1}{2\pi i} \frac{1}{(2\pi)^n} \int d\xi \int_{\gamma_0} \exp(-\lambda t) b_{-2}(x, \xi, \lambda).$$

Now using the definition (55) we have that

$$b_{-2}(x, \xi, \lambda/t) = t b_{-2}(x, t^{1/2}\xi)$$

and so we deduce that $k_t(x, x)$ is given by

$$t^{-n/2} \frac{1}{2\pi i} \frac{1}{(2\pi)^n} \int d\xi \int_{\gamma_0} d\lambda \exp(-\lambda) b_{-2}(x, \xi, \lambda)$$

just as in the classical case. It follows then that the small time asymptotics of $\operatorname{tr} e^{-tD}$ is the same as in the classical case.

There are two (equivalent) expressions for the zeta function of D, one in terms of the trace of e^{-tD}, and the other in terms of the trace of D^{-s}. To see that these are equivalent we observe that for Re $s > n/2$

$$\frac{1}{\Gamma(s)} \operatorname{tr} \int_0^\infty \int_{\gamma_0} (e^{-\lambda t}(\lambda - D)^{-1}) d\lambda t^{s-1} dt = \operatorname{tr} \int_{\gamma_0} ((\lambda - D)^{-1}) \lambda^{-s} d\lambda.$$

We may replace γ_0 by γ in this last integral. So we have

$$\zeta_D(s) = \operatorname{tr} \left[\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tD} dt \right].$$

Finally we may show that this zeta function has no pole at $s = 0$. To prove this we again follow [10] (Theorem 4, p. 302). By Lemma 8 we know that it suffices to prove that tr$G(s)$ analytically continues to Re$s > -1$. However by the definition of $B(\lambda)$ we may appeal to the proof in [10] for this result. This means that we can introduce the definition: $\operatorname{Det}(D) = \exp(-\zeta'(0))$.

6.4. A property of determinants. In this subsection we prove an abstract result regarding the determinants of Laplacians which are framed pseudodifferential operators.

So let $C^{(*)}$ be a complex of the kind considered in this section with coboundary d and Laplacian $\Delta = dd^* + d^*d$ which is invertible. Suppose that $C^k = C_1^k \oplus C_2^k$ is a decomposition into submodules each of which is projective with $C_1^{(*)}$ being a subcomplex. Then d may be represented as a matrix $\begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}$ with respect to this decomposition. We suppose that $C_j^{(*)}$ is acyclic for $j = 1, 2$ and that Δ_j denotes the Laplacian for these complexes.

LEMMA 9. *Assume that these complexes are such that the determinants of the respective Laplacians exist as in the previous subsections. Then*

$$\operatorname{Det}(\Delta) = \operatorname{Det}(\Delta_1) \operatorname{Det}(\Delta_2).$$

PROOF. Let $d_u = \begin{pmatrix} d_1 & ua \\ 0 & d_2 \end{pmatrix}$ where u is a nonnegative real parameter. Let $\Delta_u = d_u^* d_u + d_u d_u^*$ be the corresponding Laplacian. Then $C^p = \operatorname{Ker} d_u \oplus \operatorname{Ker} d_u^*$ by acyclicity and so $\Delta_u^{-s} = (d_u^* d_u)^{-s} \oplus (d_u d_u^*)^{-s}$. Hence with $D = d_u^* d_u, D' = d_u d_u^*$, and Re$s > n/2$,

$$\zeta_{\Delta_u}(s) = \zeta_D(s) + \zeta_{D'}(s).$$

Now each of these terms can be handled in the same way. Thus

$$\zeta_D(s) = \operatorname{tr} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} tr(e^{-tD}) dt.$$

Hence

$$\frac{d}{du} \zeta_D(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{tr} \left(\frac{d}{du}(D) e^{-tD} \right) dt.$$

Then

$$\begin{aligned} \frac{d}{du} \zeta_D(s) &= -\frac{1}{\Gamma(s)} \int_0^\infty \operatorname{tr} \left[\frac{d}{du} D \int_{\gamma_0} (e^{-\lambda t}(\lambda - D)^{-1} d\lambda \right] t^s dt \\ &= -s \int_{\gamma_0} \operatorname{tr} \left[\frac{d}{du}(D)(\lambda - D)^{-1} \right] \lambda^{-s-1} d\lambda. \end{aligned}$$

Replacing γ_0 by γ in this last formula we find:

$$\frac{d}{du}\zeta'(s) = -s\int_\gamma \ln\lambda\lambda^{-s-1}\operatorname{tr}\left[\frac{d}{du}(D)(\lambda-D)^{-1}\right]d\lambda - \int_\gamma \operatorname{tr}\left[\frac{d}{du}(D)(\lambda-D)^{-1}\right]\lambda^{-s-1}d\lambda$$

So

$$\frac{d}{du}\zeta'(0) = -\int_\gamma \operatorname{tr}\left[\frac{d}{du}(D)(\lambda-D)^{-1}\right]\lambda^{-1}d\lambda = -\operatorname{tr}\left[\frac{d}{du}(D)D^{-1}\right]$$

Now

$$\frac{d}{du}(D)D^{-1} = \left(\frac{d}{du}(d_u^*)d_u + d_u^*\frac{d}{du}(d_u)\right)(d_u^*d_u)^{-1}$$

and using the acyclicity we know that d_u is an invertible operator onto its range so that

$$\frac{d}{du}(D)D^{-1} = \frac{d}{du}(d_u^*)(d_u^*)^{-1} + \frac{d}{du}(d_u)(d_u)^{-1}$$

and the trace of the right hand side of this last equation is zero. ■

6.5. Invertibility of the Laplacian on $\operatorname{Ker}\theta$. Consider the commutative diagram which connects the de Rham complex with the simplicial cochain complex

$$\begin{array}{ccccc} \Lambda_{p-1} & \xrightarrow{d_{p-1}} & \Lambda_p & \xrightarrow{d_p} & \Lambda_{p+1} \\ \downarrow\theta_{p-1} & & \downarrow\theta_p & & \downarrow\theta_{p+1} \\ C^{p-1} & \xrightarrow{\partial_{p-1}} & C^p & \xrightarrow{\partial_p} & C^{p+1} \end{array} \tag{63}$$

where

$$\theta(\omega)(\sigma) = \int_\sigma (1+\Delta)^{-N}\omega \tag{64}$$

for a sufficiently large positive number N. The diagram (64) can be completed with the insertion of the complex defined using the Sobolev norms

$$\begin{array}{ccccc} \Lambda_{p-1} & \xrightarrow{d_{p-1}} & \Lambda_p & \xrightarrow{d_p} & \Lambda_{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ H^s\Lambda_{p-1} & \xrightarrow{d_{p-1}} & H^s\Lambda_p & \xrightarrow{d_p} & H^s\Lambda_{p+1} \\ \downarrow\theta_{p-1} & & \downarrow\theta_p & & \downarrow\theta_{p+1} \\ C^{p-1} & \xrightarrow{\partial_{p-1}} & C^p & \xrightarrow{\partial_p} & C^{p+1} \end{array} \tag{65}$$

for nonnegative $s \geq 0$ whence the operators d are unbounded with domain of definition $H^\infty\Lambda_p = \Lambda_p$. Since θ is surjective and has an adjoint, each vertical homomorphism θ_p can be described in terms of a splitting of the diagram (65) in the following way. Put

$$\operatorname{Ker}_p^s = \operatorname{Ker}(H^s\Lambda_p \xrightarrow{\theta_p} C^p). \tag{66}$$

This submodule is closed in $H^s\Lambda_p$ and has an orthogonal complement, $\tilde{C}_s^p$, so

$$H^s\Lambda_p = \operatorname{Ker}_p^s \oplus C_s^p \tag{67}$$

with respect to the Hilbert space structure in $H^s\Lambda_p$.

Lemma 10. *The inclusion*

$$\operatorname{Ker}_p^\infty \subset \operatorname{Ker}_p^s \tag{68}$$

is onto a dense subspace.

Proof. One has a splitting

$$\Lambda_p = \mathrm{Ker}_p^\infty \oplus C_\infty^p \tag{69}$$

at least locally. Therefore we may write the inclusion mapping as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} : \mathrm{Ker}_p^\infty \oplus C_\infty^p \to \mathrm{Ker}_p^s \oplus C_s^p, \tag{70}$$

where A_4 is an isomorphism, A_2 is a continuous mapping from a finitely generated module to Ker_p^s and the image of A is dense. Then if $x \in \mathrm{Ker}_p^s$ there exists a sequence $(y_n, z_n) \in \mathrm{Ker}_p^\infty \oplus C_\infty^p$ such that $\lim A(y_n, z_n) = x$. Hence $\lim A_4(z_n) = 0$ and $\lim z_n = 0$. Hence $\lim A_2(z_n) = 0$ from which we deduce $\lim A_1(y_n) = x$. ∎

Recall that the differential d restricted to Ker_p^s,

$$\mathrm{Ker}_{p-1}^s \xrightarrow{d} \mathrm{Ker}_p^s \tag{71}$$

is an unbounded operator.

Theorem 3. *Consider the Hilbert structure on* Ker_p^s *induced by the Hilbert structure of* $H^s\Lambda_p$ *and the restriction* d_s *of* d *to* Ker_p^s*, the corresponding adjoint operator* δ_s *and the Laplace operator*

$$\Delta_s = d_s\delta_s + \delta_s d_s, \tag{72}$$

$$\Delta_s : \mathrm{Ker}_p^s \to \mathrm{Ker}_p^s\,. \tag{73}$$

Then the operator Δ_s *is an invertible unbounded operator.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \Lambda_{p-1} & \xrightarrow{d_{p-1}} & \Lambda_p & \xrightarrow{d_p} & \Lambda_{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ H^s\Lambda_{p-1} & \xrightarrow{d_{p-1}^{s,s-1}} & H^{s-1}\Lambda_p & \xrightarrow{d_p^{s-1,s-2}} & H^{s-2}\Lambda_{p+1}\,, \\ \downarrow \theta_{p-1} & & \downarrow \theta_p & & \downarrow \theta_{p+1} \\ C^{p-1} & \xrightarrow{\partial_{p-1}} & C^p & \xrightarrow{\partial_p} & C^{p+1} \end{array} \tag{74}$$

$s - 2 \geq 0$. In (74) in the middle row the homomorphisms d are bounded.

Lemma 11. *The vertical homomorphisms in (74) induce an isomorphism of homology groups.*

Proof. Because the operators d form an elliptic complex there are operators

$$B_p : H^{s-1}\Lambda_p \to H^s\Lambda_{p-1} \tag{75}$$

such that

$$d_{p-1}^{s,s-1}B_p : +B_{p+1}d_p^{s-1,s-2} = 1 + S : H^{s-1}\Lambda_p \to H^{s-1}\Lambda_p, \tag{76}$$

where S is a smoothing operator. The inclusion $\Lambda_p \to C^p$ in the diagram (74) induces an isomorphism of cohomology groups, hence θ_p^* is surjective. Let us prove that θ_p^* is injective. Let $x \in H^{s-1}\Lambda_p$ be closed, $d_p^{s-1,s-2}x = 0$. Then using (76) one has

$$d_{p-1}^{s,s-1}B_p x = x + S(x), y = S(x) \in \Lambda_p. \tag{77}$$

Assume that $\theta_p(x)$ gives the trivial cohomology class. Then for y we have $y = d_{p-1}(z)$, that is, $d_{p-1}^{s,s-1}(z) = y$. Hence

$$d_{p-1}^{s,s-1}(B_p x - z) = x. \quad \blacksquare \tag{78}$$

COROLLARY 1. *The complex*

$$\mathrm{Ker}_{p-1}^{s} \xrightarrow{d_{p-1}^{s,s-1}} \mathrm{Ker}_{p}^{s-1} \xrightarrow{d_{p}^{s-1,s-2}} \mathrm{Ker}_{p+2}^{s-2} \tag{79}$$

is acyclic.

The proof needs some additional facts. Consider the operator Δ_0 from (73) for $s = 0$. Introduce new Hilbert space structures on the Hilbert modules Ker_p^s by setting:

$$\langle \omega_1, \omega_2 \rangle_s = ((1+\Delta_0)^s \omega_1, \omega_2)_0. \tag{80}$$

LEMMA 12. *The new Hilbert structure (80) has a norm equivalent to the old norm on* Ker_p^s.

PROOF. The old norm on Ker_p^s is written as

$$(\omega_1, \omega_2)_s = ((1+\Delta)^s \omega_1, \omega_2)_0. \tag{81}$$

The operator

$$\Delta = d\delta + \delta d \tag{82}$$

can be written using the matrix presentation of d defined by the splitting (67) for $s = 0$:

$$d = \begin{pmatrix} d_0 & a \\ 0 & b \end{pmatrix}. \tag{83}$$

Hence

$$\Delta = \begin{pmatrix} d_0 d_0^* + d_0^* d_0 + aa^* & * \\ * & * \end{pmatrix}, \tag{84}$$

where the stars denote finite dimensional bounded operators. Therefore

$$(1+\Delta)^s = \begin{pmatrix} (1 + d_0 d_0^* + d_0^* d_0)^s + F & * \\ * & * \end{pmatrix}, \tag{85}$$

where F denotes a selfadjoint nonnegative differential operator of order less than $2s$. The old Sobolev s-norm is given on Ker_p^s by

$$(\omega,\omega)_s = ((1+\Delta)^s\omega,\omega)_0 = (((1+d_0d_0^*+d_0^*d_0)^s+F)\omega,\omega)_0 = (((1+\Delta)^s+F)\omega,\omega)_0 \tag{86}$$

which evidently is equivalent to the norm (80). $\blacksquare$

We return now to complete the proof of the corollary. The definition (72) of Δ_0 implies immediately that d and δ_0 commute with Δ_0. Consider then the acyclic complex (79) with bounded differentials and new Hilbert space structure

$$\mathrm{Ker}_{p-1}^{\langle s \rangle} \xrightarrow{d_{p-1}^{s,s-1}} \mathrm{Ker}_{p}^{\langle s-1 \rangle} \xrightarrow{d_{p}^{s-1,s-2}} \mathrm{Ker}_{p+2}^{\langle s-2 \rangle}. \tag{87}$$

Hodge theory says that if $\hat{\delta}_p$ is the adjoint of $d_p^{s-1,s-2}$ in (87) then the bounded operator

$$\hat{\delta}_p d_p^{s-1,s-2} + d_{p-1}^{s,s-1}\hat{\delta}_{p-1} : \mathrm{Ker}_p^{\langle s-1 \rangle} \to \mathrm{Ker}_p^{\langle s-1 \rangle} \tag{88}$$

is an isomorphism.

On the other hand one has

$$\hat{\delta}_p d_p^{s-1,s-2} + d_{p-1}^{s,s-1}\hat{\delta}_{p-1} = \Delta_0(1+\Delta_0)^{-1} = (1+\Delta_0)^{-1}\Delta_0. \tag{89}$$

Now (89) means that Δ_0 is an invertible unbounded operator. For arbitrary $s \geq 0$ the arguments are similar. ∎

7. The $\mathcal{A}$-torsion and L^2-torsion. With all of the technical preliminaries accounted for in the previous section we come to the main point of the paper. We let $\mathcal{A}$ have its canonical trace tr. We assume that the Sobolev index $s = 0$. Recall that we have the two complexes:

$$\mathrm{Cone}(\theta|_{V^*(M,\theta)}) \tag{90}$$

and

$$U^*(M,\theta) = \overline{\mathrm{Ker}(\theta)}. \tag{91}$$

Each of these is acyclic and the coboundary for

$$\mathrm{Cyl}^* = \mathrm{Cone}(\theta|_{V^*(M,\theta)}) \oplus U^*(M,\theta),$$

has the matrix form

$$d_c = \begin{pmatrix} d & 0 \\ * & d_0 \end{pmatrix}.$$

As the Laplacians of d_0 and d are invertible on their respective complexes we may apply Lemma 9 to conclude that on j-cochains

$$\mathrm{Det}(\delta_c d_c) = \mathrm{Det}(\delta_0 d_0)\,\mathrm{Det}(\delta d). \tag{92}$$

Now define the $\mathcal{A}$-torsion for the cone complex of the de Rham homomorphism with coboundary d by the formula:

$$T(\mathrm{Cone}) = \exp\left[\frac{1}{2}\sum_j (-1)^j (\zeta_{D_j})'(0)\right]$$

where D_j is the operator δd restricted to j-cochains. Similarly the $\mathcal{A}$-torsion $T(\mathrm{Ker})$ of the complex $\mathrm{Ker}(\theta)$ is given by the same formula with D_j replaced by $(\delta_0 d_0)$. Finally, the $\mathcal{A}$-torsion of the full complex Cyl^* is the product:

$$T(M,\mathcal{A},\mathrm{tr}) = T(\mathrm{Cone})T(\mathrm{Ker}) = \exp\left[\frac{1}{2}\sum_j (-1)^j (\zeta_{D_j^c})'(0)\right]$$

where $D_j^c = \delta_j^c d_j^c$.

Notice that this definition avoids the need to discuss the asymptotic behaviour of the trace of the heat operator for large time (the so-called decay problem) but at a price. This will become evident in the next section when we relate this definition to that of the L^2-torsion. For this we need some preliminary discussion which is contained in the next subsection.

7.1. Asymptotics at zero of the spectral density function. Consider the complex $C^*(M, C^*[\pi])$. If we equip $C^*[\pi]$ with the trace arising from the regular representation then this complex may be completed in its natural Hilbert space structure so that it becomes a free Hilbert $\mathcal{U}$-module $C^*(M,\mathcal{U})$ where $\mathcal{U}$ is the von Neumann algebra generated by π in the regular representation. The trace now extends to the commutant of this $\mathcal{U}$-action. Define the spectral resolution $\partial_j^*\partial_j = \int_0^\infty \lambda dE_\lambda$ and using the fact that the spectral projections lie in the commutant of the $\mathcal{U}$-action, the spectral density

$F_j(\lambda) = \mathrm{tr}(E^j_\lambda) - \mathrm{tr}(E^j_0)$. Finally we have the Fuglede-Kadison determinant $\mathrm{Det}\partial_j = \mathrm{Det}(\partial_j^*\partial_j)^{1/2}$ where

$$\mathrm{Det}(\partial_j^*\partial_j) = \exp\Big[\int_0^\infty \ln\lambda dF_j(\lambda)\Big].$$

We say that the complex $C^*(M, C^*[\pi])$ is of regular determinant type whenever $\mathrm{Det}\partial_j$ is nonvanishing for all j. This occurs precisely when the integral in the exponent in the definition of the determinant converges. A sufficient condition for this is that the manifold has decay, that is, there is a $\beta_j > 0$ with $F_j(\lambda) \le C\lambda^{\beta_j}$ for some $C > 0$ and all j.

7.2. A homotopy equivalence. In this subsection we construct a homotopy equivalence between the identity and the zero map for the chain complex of Hilbert modules:

$$\mathrm{Ker}^0_{p-1} \xrightarrow{d_{p-1}} \mathrm{Ker}^0_p \xrightarrow{d_p} \mathrm{Ker}^0_{p+1}\,. \tag{93}$$

In (93) the operators d_p should be considered as closed unbounded operators whose domains are Ker^1_p. Then the corresponding bounded chain complex (79) can be described in terms of (93) using the natural isomorphism of chain complexes

$$\begin{array}{ccccc} \mathrm{Ker}^s_{p-1} & \xrightarrow{d^{s,s-1}_{p-1}} & \mathrm{Ker}^{s-1}_p & \xrightarrow{d^{s-1,s-2}_p} & \mathrm{Ker}^{s-2}_{p+1} \\ \Big\downarrow j_{p-1} & & \Big\downarrow j_p & & \Big\downarrow j_{p+1} \\ \mathrm{Ker}^0_{p-1} & \xrightarrow{\tilde d_{p-1}} & \mathrm{Ker}^0_p & \xrightarrow{\tilde d_p} & \mathrm{Ker}^0_{p+1} \end{array} \tag{94}$$

where the vertical isomorphisms are $j_{p-1} = (1+\Delta_0)^{s/2}$, $j_p = (1+\Delta_0)^{(s-1)/2}$, $j_{p+1} = (1+\Delta_0)^{(s-2)/2}$, ... The operators in the bottom row are

$$\tilde d_p = (1+\Delta_0)^{-1/2} d_p. \tag{95}$$

The upper row is acyclic. This means that there are bounded operators

$$\tilde T_p : \mathrm{Ker}^{s-1}_p \to \mathrm{Ker}^s_{p-1} \tag{96}$$

such that

$$\tilde d^{s,s-1}_{p-1}\tilde T_p + \tilde T_{p+1}\tilde d^{s-1,s-2}_p = 1 : \mathrm{Ker}^{s-1}_p \to \mathrm{Ker}^{s-1}_p\,. \tag{97}$$

Put

$$T_p = (1+\Delta_0)^{(-p+s)/2}\tilde T_p (1+\Delta_0)^{(p-s)/2} : \mathrm{Ker}^0_p \to \mathrm{Ker}^0_{p-1}\,. \tag{98}$$

If the $\tilde T_p$ are pseudodifferential operators of order -1, then the T_p are pseudodifferential operators of order -1. Hence the range of T_p belongs to the domain of d_{p-1}.

Thus we construct $\tilde T_p$ as a pseudodifferential operator (or at least operators which preserve order on the Sobolev scale of spaces of large order). Then the complex (79) is acyclic. Hence there is a splitting of each space Ker^{s-1}_p into the direct sum

$$\mathrm{Ker}^{s-1}_p = V_p \oplus W_p, \tag{99}$$

such that the operator $d^{s,s-1}_p$ can be represented as a matrix

$$d^{s,s-1}_p = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} : V_p \oplus W_p \to V_{p+1} \oplus W_{p+1}. \tag{100}$$

The splitting (100) can be defined with projectors which are pseudodifferential operators since the projector can be constructed as a holomorphic function on the spectrum of the operator $(d^{s,s-1}_p)^* d^{s,s-1}_p$. The last has its spectrum with zero as isolated point as a consequence of acyclicity. On the other hand one has almost a chain homotopy. This means that there are pseudodifferential operators D such that

$$d_{p-1}^{s,s-1} D_p + D_{p+1} d_p^{s-1,s-2} = 1 + S : \mathrm{Ker}_p^{s-1} \to \mathrm{Ker}_p^{s-1}, \tag{101}$$

where S is a smoothing operator. If

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, \tag{102}$$

then put $U_3 = -d^{-1}S_1$, $U_4 = -d^{-1}S_2$, $U_1 = U_2 = 0$ and put $T = D + U$.

7.3. The $\mathcal{A}$-torsion in the presence of decay. The discussion of the previous subsection established that the orthogonal complement $V^*(M,\theta)$ of $U^*(M,\theta) = \mathrm{Ker}(\theta)$ is chain homotopic to the full de Rham complex $\Omega^*(M,\mathcal{A})$ so that by results of Gromov and Shubin [4] the decay of these two complexes is the same. Equally the decay of the complexes $\mathrm{Cone}(\theta|_{V^*(M,\theta)})$ and

$$\mathrm{Cyl}^k = H^0\Omega_k(M, C^*[\pi]) \oplus C^{k-1}(M, C^*[\pi])$$

is the same. Our aim is to establish the relationship between the C^*-torsion for Cyl^k and the von Neumann torsion when M has the decay property. To do this we first note that as $V^*(M,\theta)$ and $C^*(M, C^*[\pi])$ are isomorphic modules their completions $\bar{V}^*(M,\theta)$ and $C^*(M,\mathcal{U})$ are isomorphic as $\mathcal{U}$-modules. It follows from results of [4] that if one has decay so too does the other. Henceforth we assume that $C^*(M,\mathcal{U})$ has decay.

To go further we first observe that relative to the decomposition

$$\mathrm{Cyl}^k = \mathrm{Cone}(\theta|_{V^*(M,\theta)} \oplus \overline{\mathrm{Ker}(\theta)},$$

the coboundary d_c for Cyl^* has the matrix form

$$d_c = \begin{pmatrix} d & 0 \\ * & d_0 \end{pmatrix}.$$

As the Laplacians of d_0 and d are invertible on their respective complexes we may apply Lemma 9 to conclude that on j-cochains

$$\mathrm{Det}(\delta_c d_c) = \mathrm{Det}(\delta_0 d_0)\,\mathrm{Det}(\delta d). \tag{103}$$

Next we note that $\mathrm{Cone}(\theta|_{V^*(M,\theta)})$ is a free $C^*[\pi]$ module to which we may apply the methods of [2]. Thus we have, on j-cochains,

$$d = \begin{pmatrix} d^V & 0 \\ \theta & -\partial \end{pmatrix},$$

and also decompositions

$$V^*(M,\theta) = (\mathrm{Ker}\, d^V)^\perp \oplus H^V \oplus \overline{\mathrm{Im}\, d^V}$$

and

$$C^*(M, C^*[\pi]) = (\mathrm{Ker}\,\partial)^\perp \oplus H^C \oplus \overline{\mathrm{Im}\,\partial}$$

where H^V and H^C denote the kernels of the respective Laplacians. Relative to these decompositions d has the matrix form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d^V & 0 & 0 & 0 & 0 & 0 \\ \theta_{11} & 0 & 0 & 0 & 0 & 0 \\ \theta_{21} & \theta_{22} & 0 & 0 & 0 & 0 \\ \theta_{31} & \theta_{32} & \theta_{33} & -\partial & 0 & 0 \end{pmatrix}.$$

From this one deduces that the range of δ decomposes as

$$(\operatorname{Ker} d_1)^\perp \oplus H^V \oplus W^V$$

where $W^V = \overline{\delta \operatorname{Im} d^V}$ while the range of d decomposes as

$$\overline{d(\operatorname{Ker} d_1)^\perp} \oplus H^C \oplus \overline{\operatorname{Im}\ d^V}.$$

Relative to these decompositions d is lower triangular:

$$d = \begin{pmatrix} d_1 & 0 & 0 \\ * & \theta_{22} & 0 \\ * & * & d_2 \end{pmatrix}$$

where $d_1 = d|_{(\operatorname{Ker} d^V)^\perp}$ and $d_2 = d|_{W^V}$. Using properties of the Fuglede-Kadison determinant [2] we deduce that

$$\operatorname{Det}(d) = \operatorname{Det}(d_1)\operatorname{Det}(d_2)\operatorname{Det}(\theta_{22}). \tag{104}$$

Now consider the complex ${H^V}^\perp \oplus {H^C}^\perp$ with decomposition

$$(\operatorname{Ker} d^V)^\perp \oplus \overline{\operatorname{Im} d^V} \oplus (\operatorname{Ker} \partial)^\perp \oplus \overline{\operatorname{Im} \partial}$$

and coboundary

$$d' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d^V & 0 & 0 & 0 \\ \theta_{11} & 0 & 0 & 0 \\ 0 & \theta_{33} & -\partial & 0 \end{pmatrix}.$$

Now $\operatorname{Det}(d') = \operatorname{Det}(d_1)\operatorname{Det}(d_2)$ and we wish to relate this to $\operatorname{Det}(d^V)\operatorname{Det}(\partial)$. While it is not necessarily true that $\operatorname{Ker}\delta$ is the direct sum of $\operatorname{Ker}\delta_V$ and $\operatorname{Ker}\partial^*$ it is nevertheless isometric to this direct sum by partial isometries U_j on j-cochains. Then $\operatorname{Det}(U_{j+1}^* d_j' U_j) = \operatorname{Det} d_j'$ and it is not hard to see that $\operatorname{Det}(U_{j+1}^* d_j' U_j)$ is of the form

$$\begin{pmatrix} d^V & 0 \\ * & -\partial \end{pmatrix}$$

relative to the direct sum $\operatorname{Ker}\delta_V \oplus \operatorname{Ker}\partial^*$. Hence we have

$$\operatorname{Det}(d') = \operatorname{Det}(d^V)\operatorname{Det}(\partial). \tag{105}$$

Now let $D_j = \delta_{c_j} d_{c_j}$ where d_{c_j} is the coboundary for Cyl^j. Combining (103), (104), (105) we conclude that the $\mathcal{A}$-torsion

$$T(M, \mathcal{A}, \mathrm{tr}) = \exp\left[\frac{1}{2}\sum_j (-1)^j (\zeta_{D_j})'(0)\right]$$

is the quotient

$$T(M, \mathcal{A}, \mathrm{tr}) = T^{(2)}(M)/T^{RF}(C)$$

of the torsions for the de Rham and simplicial complexes. Here, with $\tilde{\theta}_j = \theta_j|_{H^V}$ and d^M the coboundary on the de Rham complex,

$$T^{(2)}(M) = \exp\left[\frac{1}{2}\sum_j (-1)^j (\zeta_{\delta_j^M d_j^M})'(0) + \tilde{\theta}_j^* \tilde{\theta}_j\right]$$

is the L^2-analytic torsion while

$$T^{RF}(C) = \prod_{j=0}^{n} \mathrm{Det}(\partial_j^* \partial_j)^{(-1)^j/2}.$$

is the L^2-RF torsion of the simplicial complex.

Concluding remarks. The Laplacian for $C^*(M, C^*[\pi])$ is a matrix over the algebraic group algebra. In the case of a free abelian fundamental group the projection onto the kernel of the Laplacian may be shown to be a matrix over the algebraic group algebra (an ingredient in the proof is the fact that the kernel is a free module). The same is therefore true for the isomorphic $\mathcal{A}$-module $V^*(M, \theta)$. Thus for the full de Rham complex the projection onto the kernel of the Laplacian is a matrix over $\mathcal{A}$. In this case therefore we may discuss the property of decay in the C^*-algebra context.

If this is true more generally then it opens the way to studying the $\mathcal{A}$-torsion for the de Rham complex twisted by any traceable representation of the fundamental group for which decay holds. Some solvable discrete groups have many such representations. Then we may ask if these $\mathcal{A}$-torsion invariants contain significant information about the pair M, π.

References

[1] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. 6 (1996), 751–859.

[2] A. L. Carey and V. Mathai, *L^2 torsion invariants*, J. Funct. Anal. 110 (1992), 377–409.

[3] J. Dodziuk, *De Rham–Hodge theory for L^2-cohomology of infinite coverings*, Topology 16 (1977), 157–165.

[4] M. Gromov and M. Shubin, *Von Neumann spectra near zero*, Geom. Funct. Anal. 1 (1991), 375–404.

[5] J. Lott, *Heat kernels on covering spaces and topological invariants*, J. Diff. Geom. 35 (1992), 471–510.

[6] W. Luck and M. Rothenberg, *Reidemeister torsion and the K-theory of von Neumann algebras*, Math. Gott. Heft 31 (1991), 1–64.

[7] V. Mathai, *L^2 analytic torsion*, J. Funct. Anal. 107 (1992), 369–386; *L^2 analytic torsion and locally symmetric spaces*, preprint.

[8] W. Pashke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. 182 (1973), 443–468.

[9] D. B. Ray and I. M. Singer, *R-Torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. 7 (1971), 145–210.

[10] R. T. Seeley, *Complex powers of an elliptic operator*, Proc. Sympos. Pure Appl. Math. 10 (1967), 288–388.