NIELSEN THEORY AND REIDEMEISTER TORSION
BANACH CENTER PUBLICATIONS, VOLUME 49
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1999

EVALUATION SUBGROUPS AND G-SEQUENCES

MOO HA WOO

Department of Mathematics Education
Korea University
Seoul 136-701, South Korea
E-mail: woomh@kuccnx.korea.ac.kr

1. Introduction. This paper is an expository paper on G-sequences. Gottlieb [2,3,4] defined and studied the Gottlieb groups $G_n(X)$ of the homotopy groups $\pi_n(X)$. The importance of Gottlieb groups is that these subgroups have many applications in topology, especially, in fibration theory, in fixed point theory, and in the theory of identification of spaces. For instance, the nontriviality of the Gottlieb group of a finite complex suffices to ensure the vanishing of the Euler characteristic [2], the triviality of the n-th Gottlieb group of a space ensures that every fiber space over an $(n+1)$-dimensional sphere with the space as fiber has a cross section [3] or the condition that the image of an element of the evaluation subgroup under the Hurewicz homomorphism becomes a generator of the homology group ensures that the space is homotopy equivalent to S^n [3].

Though several authors [6, 7, 11, 13, 14, 15, 18, 19, 21] have studied and generalized $G_n(X)$, few $G_n(X)$ are known. The exactness of homotopy sequences plays an important role in computing homotopy groups. If we can construct an exact sequence containing the Gottlieb groups, then this sequence can be used in computing these groups.

In order to define such a sequence, we introduce some subgroups of the homotopy groups $\pi_n(A), \pi_n(X)$ and $\pi_n(X,A)$. First we consider the Gottlieb group $G_n(A)$ as a subgroup of $\pi_n(A)$. Next if we consider the Gottlieb group $G_n(X)$ as a subgroup of $\pi_n(X)$, it does not contain the image of $G_n(A)$ under the homomorphism $i_* : \pi_n(A) \to \pi_n(X)$. Thus we need to introduce a slightly larger subgroup $G_n(X,A)$ of $\pi_n(X)$ containing the image of $G_n(A)$ (see [6]). As a subgroup of $\pi_n(X,A)$, we introduce a subgroup $G_n^{rel}(X,A)$ such that $G_n(A)$, $G_n(X,A)$ and $G_n^{rel}(X,A)$ make a G-sequence (see [8,11]).

1991 *Mathematics Subject Classification*: 55P45.

Key words and phrases: evaluation subgroup, generalized evaluation subgroup, relative evaluation subgroup, G-sequence, ω-homology.

This work was supported by TGRC-KOSEF and BSRI-1409..

The paper is in final form and no version of it will be published elsewhere.

In this paper, we examine the difference of $G_n(X)$ and $G_n(X,A)$, the existence of nonexact G-sequences and the conditions for exactness of the G-sequence of a CW-pair. We also show some applications of G-sequences and introduce a new homology.

2. Evaluation subgroups. Gottlieb [2,3] introduced the Gottlieb group $G_n(X) = \{[f] \in \pi_n(X) \mid \exists \text{ map } H : X \times I^n \to X \text{ such that } [H|_{x_0 \times I^n}] = [f] \text{ and } H|_{X\times u} = 1_X \text{ for } u \in \partial I^n\}$ as a subgroup of $\pi_n(X)$. He showed this subgroup is equal to the image of $\omega_* : \pi_n(X^X, 1_X) \to \pi_n(X, x_0)$, where $\omega : X^X \to X$ is the evaluation map.

In particular $G_1(X)$ is equal to the Jiang group [5] of X.

Gottlieb also showed that

$$G_n(S^n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2Z \subset Z = \pi_n(S^n) & \text{if } n \text{ is odd and } n \neq 1,3,7, \\ Z = \pi_n(S^n) & n = 1,3,7. \end{cases}$$

Since then, many authors studied and generalized $G_n(X)$, for instance, G. E. Lang, J. Siegel, K. L. Lim, J. Oprea, J. Kim, K. Y. Lee and M. H. Woo and Varadarajan. In [6], the author and J. Kim generalized $G_n(X)$ to $G_n(X,A)$. This subgroup is defined by $G_n(X,A) = \{[f] \in \pi_n(X) | \exists \text{ map } H : A \times I^n \to X \text{ such that } [H|_{x_0\times I^n}] = [f] \text{ and } H|_{A\times u} = i \text{ for } u \in \partial I^n\}$ and is equivalent to the image $\omega_* : \pi_n(X^A, i) \to \pi_n(X, x_0)$, where X^A is the space of maps from A to X, $i : A \to X$ is the inclusion and $\omega : X^A \to X$ is the evaluation map.

$G_n(X,A)$ always contains $G_n(X)$ and

$$G_n(X,A) = \begin{cases} G_n(X) & \text{if } A = X, \\ \pi_n(X) & \text{if } A = \{x_0\}. \end{cases}$$

In [8], Lee and the author introduced the subgroup $G_n^{rel}(X,A)$ of the relative homotopy group $\pi_n(X,A)$ which is defined by $G_n^{rel}(X,A) = \{[f] \in \pi_n(X,A) | \exists \text{ map } H : (X \times I^n, A \times \partial I^n) \to (X,A) \text{ such that } [H|_{x_0\times I^n}] = [f] \text{ and } H|_{X\times u} = 1_X \text{ for } u \in J^{n-1}\}$,where $J^{n-1} = I^{n-1} \times 1 \cup \partial I^{n-1} \times I$. They also showed this group is the image $\omega_* : \pi_n(X^A, A^A, i) \to \pi_n(X, A, x_0)$, where A^A is the subspace of X^A which consists of maps from A into itself.

Therefore the subgroups $G_n(X), G_n(X,A)$ and $G_n^{rel}(X,A)$ mentioned above are called *evaluation subgroups, generalized evaluation subgroups* and *relative evaluation subgroups* respectively.

In [22], Zhao used the generalized evaluation subgroups to solve a problem in fixed point theory. In order to apply the generalized evaluation subgroups to fixed point theory, we need to compute the generalized evaluation subgroups for many topological spaces. Especially we need the computations of $G_n(X,A)$ which are proper subgroups of $\pi_n(X)$ and contain $G_n(X)$ properly.

The following theorem is very useful to compute the generalized evaluation subgroups of some CW-pairs. We will consider the pair $(X \times Y, x_0 \times Y)$ (for a fixed $x_0 \in X$) which will be denoted $(X \times Y, Y)$.

THEOREM 2.1. $G_n(X \times Y, Y) \cong \pi_n(X) \oplus G_n(Y)$.

PROOF (sketch).

$$\begin{aligned}
&G_n(X\times Y, Y)\\
&\quad = \text{Image of } \omega_* : \pi_n((X\times Y)^Y, i_2) \to \pi_n(X\times Y)\\
&\quad = \text{Image of } \omega_* : \pi_n(X^Y, c)\times \pi_n(Y^Y, 1_Y) \to \pi_n(X)\times\pi_n(Y)\\
&\quad = (\text{Image of } \omega_* : \pi_n(X^Y, c)\to \pi_n(X))\times(\text{Image of } \omega_* : \pi_n(Y^Y, 1_Y)\to\pi_n(Y))\\
&\quad = \pi_n(X)\times G_n(Y)
\end{aligned}$$

where c is the constant map from Y to X and ω's are evaluation maps.

Gottlieb showed that $G_n(X\times X)$ is isomorphic to $G_n(X)\oplus G_n(X)$ [3]. It is also clear that $\pi_n(X\times X)$ is isomorphic to $\pi_n(X)\oplus\pi_n(X)$. The following diagram tells us the difference between the Gottlieb group, the generalized evaluation subgroups and homotopy groups.

$$\begin{tikzcd}
G_n(X\times X) \arrow[r, equal] \arrow[d, "\downarrow\cap"'] & G_n(X)\oplus G_n(X) \arrow[d, "\downarrow\cap"'] \\
G_n(X\times X, X) \arrow[r, equal] \arrow[d, "\downarrow\cap"'] & \pi_n(X)\oplus G_n(X) \arrow[d, "\downarrow\cap"'] \\
\pi_n(X\times X) \arrow[r, equal] & \pi_n(X)\oplus\pi_n(X),
\end{tikzcd}$$

COROLLARY 2.2. *The pair $(X\times X, X)$ has the proper n-th generalized evaluation subgroup if and only if $G_n(X)$ is a proper subgroup of $\pi_n(X)$.*

If we use Theorem 2.1 and the computations of $\pi_m(S^n)$ and $G_m(S^m)$, then we can compute the generalized evaluation subgroups of the pairs $(S^n\times S^m, S^m)$.

By the fact that $\pi_{n+1}(S^n)$ is cyclic of order 2 for every $n\geqq 3$ and computations of $G_n(S^n)$, we have the following.

EXAMPLE 2.1.

$$G_{n+1}(S^n\times S^{n+1}, S^{n+1})\cong\begin{cases} 0, & n=1,\\ \mathbf{Z}\oplus\mathbf{Z}, & n=2,\\ \mathbf{Z}_2, & n\geqq 3 \text{ and } n \text{ is odd},\\ \mathbf{Z}_2\oplus\mathbf{Z}, & n=6,\\ \mathbf{Z}_2\oplus 2\mathbf{Z}, & n>3 \text{ and } n \text{ is even and } n\neq 6.\end{cases}$$

By the fact that $\pi_{n+2}(S^n)$ is cyclic of order 2 for every $n\geqq 3$ and computations of $G_{n+2}(S^{n+2})$, we have the following.

EXAMPLE 2.2.

$$G_{n+2}(S^n\times S^{n+2}, S^{n+2})\cong\begin{cases} \mathbf{Z}, & n=1,\\ \mathbf{Z}, & n=2,\\ \mathbf{Z}_2\oplus 2\mathbf{Z}, & n=3,\\ \mathbf{Z}_2, & n>3 \text{ and } n \text{ is even},\\ \mathbf{Z}_2\oplus\mathbf{Z}, & n=5,\\ \mathbf{Z}_2\oplus 2\mathbf{Z}, & n>5 \text{ and } n \text{ is odd}.\end{cases}$$

The generalized evaluation subgroups satisfy the homotopy invariance property (see [6]).

THEOREM 2.3. *Let (X, A) and (Y, B) be path-connected CW-pairs of the same homotopy type. If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then f_* carries $G_n(X, A)$ isomorphically onto $G_n(Y, B)$.*

Let $h_p : \pi_n(X) \xrightarrow{h} H_n(X; Z) \xrightarrow{\otimes \mathbf{Z}_p} H_n(X; \mathbf{Z}_p)$ be the mod p Hurewicz homomorphism as composition of the Hurewicz homomorphism h tensored with Z_p and $h_\infty : \pi_n(X) \to H_n(X; R)$ be the Hurewicz map where R is the field of rationals. In [3], Gottlieb shows that, for X a topological space with finitely generated homology,

(1) if n is an odd integer, then $G_n(X)$ is contained in the kernel of h_p, for any prime number p or ∞ provided $\chi(X) \neq 0$,

(2) if n is an even integer, then $G_n(X)$ is contained in the kernel of h_p, for any prime number p which does not divide $\chi(X)$.

Since $G_n(X)$ is contained in $G_n(X, A)$, we generalize the above results to $G_n(X, A)$. In order to generalize these, we studied the algebraic structure induced by $\phi : A \times S^n \to X$ affiliated to some $\alpha \in G_n(X, A)$, on the homology, which is a modification of Gottlieb methods, so that we obtained the following theorems. For the proofs, see [9].

THEOREM 2.4. *Let A have a finitely generated integral homology and $i : A \to X$ be the inclusion map which has a left homotopy inverse r. If n is an odd integer, then $G_n(X, A)$ is contained in the kernel of $r_* h_p$, for any prime number p or ∞ provided $\chi(A) \neq 0$, where $r_* : H_n(X; Z_p) \to H_n(A; Z_p)$ is induced by r.*

THEOREM 2.5. *Let A have a finitely generated integral homology and $i : A \to X$ be the inclusion map which has a left homotopy inverse r. Suppose p is a prime number which does not divide $\chi(A)$. Then $G_n(X, A)$ is contained in the kernel of $r_* h_p$, for even n.*

In [2], Gottlieb showed that $G_1(X)$ is contained in the center of $\pi_1(X)$.

THEOREM 2.6. *Let (X, A) be a CW-pair. Then $G_2^{rel}(X, A)$ is contained in the center of $\pi_2(X, A)$. Especially, if A is a connected aspherical polyhedron, then $G_1(X, A)$ is also contained in $Z(i_*(\pi_1(A)), \pi_1(X))$.*

PROOF (sketch). Let $[g] \in \pi_2(X, A)$, $[f] \in G_2^{rel}(X, A)$ and H be an affiliated map for $[f]$. If we define a homotopy

$$G : (I^2 \times I, \partial I^2 \times I, J^1 \times I) \to (X, A, x_0)$$

by

$$G((t_1, t_2), s) = \begin{cases} H(g(2t_1(1-s), t_2), (2t_1 s, t_2)) & \text{if } 0 \le t_1 \le \frac{1}{2}, \\ H(g(1-(2-2t_1)s, t_2), ((2-2t_1)s + 2t_1 - 1, t_2)) & \text{if } \frac{1}{2} \le t_1 \le 1, \end{cases}$$

then we have $[g][f] = [f][g]$. This completes the proof of the first part.

The remaining part is quite analogous to Theorem 10 of [7A, 1].

3. The G-sequence and its exactness. The inclusion map i and the evaluation map ω induce the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \to & \pi_n(A^A) & \xrightarrow{i_*} & \pi_n(X^A) & \xrightarrow{j_*} & \pi_n(X^A, A^A) & \xrightarrow{\partial} & \pi_{n-1}(A^A) & \to & \cdots \\
 & & \downarrow \omega_* & & \downarrow \omega_* & & \downarrow \omega_* & & \downarrow \omega_* & & \\
\cdots & \to & G_n(A) & \xrightarrow{i_*} & G_n(X, A) & \xrightarrow{j_*} & G_n^{rel}(X, A) & \xrightarrow{\partial} & G_{n-1}(A) & \to & \cdots \\
 & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \\
\cdots & \to & \pi_n(A) & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \to & \cdots
\end{array}$$

where the top and bottom rows are exact and the middle sequence forms a chain complex. Thus we have the following.

Theorem 3.1. *$G_n(A), G_n(X, A)$ and $G_n^{rel}(X, A)$ form a sequence*

$$\cdots \to G_n(A) \xrightarrow{i_*} G_n(X, A) \xrightarrow{j_*} G_n^{rel}(X, A) \xrightarrow{\partial} G_{n-1}(A) \to \cdots$$

We call this middle sequence the *G-sequence* of a CW-pair (X, A). This sequence is not necessarily exact.

Example 3.1. The G-sequences of $(B^2 \vee S^1, S^1 \vee S^1)$ and (Z_p, S^7) are not exact, where $p : S^7 \to S^4$ is the Hopf bundle and Z_p is the mapping cylinder of p. For the proofs of these examples, see [11] and [16].

The following theorems give some conditions for the G-sequence to be exact.

Theorem 3.2. *Let (X, A) be a CW-pair and the inclusion $i : A \to X$ have a left homotopy inverse. Then the G-sequence of (X, A) is exact.*

Proof (sketch). Let us consider the G-sequence for (X, A)

$$\to G_{n+1}^{rel}(X, A) \xrightarrow{\partial} G_n(A) \xrightarrow{i_*} G_n(X, A) \xrightarrow{j_*} G_n^{rel}(X, A) \to$$

Since i has a left homotopy inverse, the G-sequence is exact at $G_n(A)$. Since $G_n(X, A) \cap i_*(\pi_n(A)) = i_*(G_n(A))$, we have $i_*(G_n(A)) = \ker(j_*|_{G_n(X,A)})$ and hence the G-sequence is exact at $G_n(X, A)$.

Since the inclusion $i : A \to X$ has a left homotopy inverse, the inclusion $\bar{i} : A^A \to X^A$ given by $\bar{i}(f) = f$ has a left homotopy inverse. Thus the diagram

$$\begin{array}{ccc}
\pi_n(X^A, 1_A) & \xrightarrow{\bar{j}_*} & \pi_n(X^A, A^A, 1_A) \\
\Big\downarrow \omega_* & & \Big\downarrow \omega_* \\
G_n(X, A) & \xrightarrow{j_*} & G_n^{rel}(X, A)
\end{array}$$

is commutative and $\bar{j}_* : \pi_n(X^A, i) \to \pi_n(X^A, A^A, i)$ is an epimorphism. So we have $G_n^{rel}(X, A) = j_* G_n(X, A)$. Thus the G-sequence is exact at $G_n^{rel}(X, A)$.

Corollary 3.3. *If (X, A) is a CW-pair and the inclusion $i : A \to X$ has a left homotopy inverse, then*

$$G_n(X, A) \cong G_n(A) \oplus G_n^{rel}(X, A) \quad \textit{for } n > 1.$$

Let B^n be the n-dimensional ball and S^{n-1} the boundary of B^n. Even if the inclusion map $i : S^{n-1} \to B^n$ does not have a left homotopy inverse, the G-sequence of (B^n, S^{n-1}) is exact [10]. From this result, we can show that if the inclusion $i : A \to X$ is null homotopic, then (X, A) has the exact G-sequence.

Let us consider the homotopy sequence

$$\cdots \xrightarrow{j_*} \pi_{n+1}(X,A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \to \cdots \xrightarrow{j_*} \pi_1(X,A) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

of a CW-pair (X,A). If the inclusion $i : A \to X$ is homotopic to a constant map, then i_* is the 0-homomorphism. Thus the connecting homomorphism ∂ is an epimorphism. This is also true for the G-sequence of (X,A).

THEOREM 3.4. *Let (X,A) be a CW-pair. If the inclusion $i : A \to X$ is homotopic to a constant map, then the G-sequence of (X,A) is exact.*

PROOF (sketch). Consider the following commutative ladder:

$$\begin{array}{ccccccccccc} \cdots & \to & \pi_n(A) & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi_n(X,A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \to & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \to & G_n(A) & \to & G_n(X,A) & \to & G_n^{rel}(X,A) & \to & G_{n-1}(A) & \to & \cdots \end{array}$$

for $n \geq 1$. Since the inclusion $i : A \to X$ is homotopic to a constant map, $i_*|_{G_n(A)}$ is the 0-homomorphism and $j_*|_{G_n(X,A)}$ is a monomorphism. It is easy to show $\partial(G_n^{rel}(X,A)) = G_{n-1}(A)$ for $n \geq 1$. Thus we have an epimorphism $\partial|_{G_n^{rel}(X,A)}$ and hence the lower sequence on the commutative ladder is exact at $G_n(X,A)$ and $G_n(A)$. We also show that the G-sequence is exact at $G_n^{rel}(X,A)$. It is clear that

$$\text{image of } j_*|_{G_n(X,A)} \subset \text{kernel of } \partial|_{G_n^{rel}(X,A)}.$$

Since the inclusion $i : A \to X$ is homotopic to a constant map, we can show $G_n(X,A) = \pi_n(X)$. Thus we have

$$\begin{aligned} \text{kernel of } \partial|_{G_n^{rel}(X,A)} &\subset \text{kernel of } \partial = j_*(\pi_n(X)) \\ &= j_*(G_n(X,A)) = \text{image of } j_*|_{G_n(X,A)}. \end{aligned}$$

4. Applications of G-sequences. As an application of the exact G-sequences, we first show the computation of relative evauation subgroups of the pairs (S^n, S^{n-1}). Since $\pi_k(S^n, S^{n-1}) = 0$ for $1 < k < n$, we have $G_k^{rel}(S^n, S^{n-1}) = 0$. By exactness of the G-sequence of (S^n, S^{n-1}), we can estimate $G_n^{rel}(S^n, S^{n-1})$ as follows.

THEOREM 4.1. *$G_n^{rel}(S^n, S^{n-1}) \neq 0$ for all $n \geq 2$. In particular, if n is odd, then $G_n^{rel}(S^n, S^{n-1}) \cong \mathbf{Z}$ and if $n = 2, 4, 8$, then $G_n^{rel}(S^n, S^{n-1}) = \pi_n(S^n, S^{n-1}) \cong \mathbf{Z} \oplus \mathbf{Z}$.*

PROOF (sketch). By Theorem 3.4, the G-sequence of (S^n, S^{n-1}) is exact. If we consider the commutative ladder

$$\begin{array}{ccccccccc} 0 & \to & G_n(S^n, S^{n-1}) & \xrightarrow{j_*} & G_n^{rel}(S^n, S^{n-1}) & \xrightarrow{\partial} & G_{n-1}(S^{n-1}) & \xrightarrow{i_*} & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \pi_n(S^n) & \xrightarrow{j_*} & \pi_n(S^n, S^{n-1}) & \xrightarrow{\partial} & \pi_{n-1}(S^{n-1}) & \xrightarrow{i_*} & 0 \end{array}$$

and $G_n(S^n, S^{n-1}) = \pi_n(S^n) \cong \mathbf{Z}$, we have $G_n^{rel}(S^n, S^{n-1}) \neq 0$ for all $n \geq 2$.

Let n be odd. Then $G_{n-1}(S^{n-1}) = 0$ (see [3]). By exactness of the top row at $G_n^{rel}(S^n, S^{n-1})$, we see $G_n^{rel}(S^n, S^{n-1})$ is isomorphic to $\mathbf{Z}$.

For $n = 2, 4$ or 8, we have $G_{n-1}(S^{n-1}) = \pi_{n-1}(S^{n-1}) \cong \mathbf{Z}$. It is easy to show that $G_n^{rel}(S^n, S^{n-1}) = \pi_n(S^n, S^{n-1})$ using the above diagram and the short Five Lemma.

Next, we use the exact G-sequence to show that Theorems 2.4 and 2.5 are closely related to $G_n^{rel}(X, A)$. We know that there is a transformation $k : \pi_n(X, A) \to H_n(X, A)$ called the Hurewicz homomorphism. Define $k_p : \pi_n(X, A) \to H_n(X, A) \to H_n(X, A; Z_p)$ as composition of h tensored with Z_p.

THEOREM 4.2. *Let A be a retract of a CW-complex X. Then $G_n(X, A) \subset \operatorname{Ker} r_* h_p$ and $G_n^{rel}(X, A) \subset \operatorname{Ker} k_p$ if and only if $G_n(X, A) \subset \operatorname{Ker} h_p$, where r is the retraction and h_p and k_p are the Hurewicz homomorphisms tensored with Z_p for all prime numbers p.*

PROOF (sketch). Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
\to & G_{n+1}^{rel}(X, A) & \to & G_n(A) & \xrightarrow{i_*} & G_n(X, A) & \xrightarrow{j_*} & G_n^{rel}(X, A) & \to \\
 & \downarrow{\scriptstyle k_p} & & \downarrow{\scriptstyle h_p} & & \downarrow{\scriptstyle h_p} & & \downarrow{\scriptstyle k_p} & \\
\to & H_{n+1}(X, A; Z_p) & \to & H_n(A; Z_p) & \xrightarrow{i_*} & H_n(X; Z_p) & \xrightarrow{j_*} & H_n(X, A; Z_p) & \to
\end{array}$$

Since j_* is surjective, the sufficiency is trivial.

Conversely, suppose $G_n(X, A) \subset \operatorname{Ker} r_* h_p$ and $G_n^{rel}(X, A) \subset \operatorname{Ker} k_p$. Then we have $j_* h_p(G_n(X, A)) = 0$ and hence $h_p(G_n(X, A)) \subset \operatorname{Ker} j_* = \operatorname{Im} i_*$. Therefore, for every $\alpha \in G_n(X, A)$, there is a $\beta \in H_n(A; Z_p)$ such that $i_*(\beta) = h_p(\alpha)$. Since $\beta = r_* i_*(\beta) = r_* h_p(\alpha) = 0$, we have $h_p(\alpha) = i_*(\beta) = 0$.

In general, the G-sequence is not exact but it is half exact. Thus we can think of the G-sequence of a CW-pair (X, A)

$$\xrightarrow{j_*^{n+1}} G_{n+1}^{rel}(X, A) \xrightarrow{\partial^{n+1}} G_n(A) \xrightarrow{i_*^n} G_n(X, A) \to \cdots \xrightarrow{j_*^2} G_1^{rel}(X, A) \xrightarrow{\partial^1} G_0(A) \xrightarrow{i_*^0} G_0(X, A)$$

as a chain complex. Finally we apply G-sequences to introduce a new homology. Let (X, A) be a CW-pair. The ω-homology

$$H_*^{\omega}(X, A) = \{H_{n+1}^{g\omega}(X, A), H_{n+1}^{r\omega}(X, A), H_n^{a\omega}(X, A)\}_{n \geq 0}$$

of (X, A) is defined to be

$$\begin{aligned}
H_{n+1}^{g\omega}(X, A) &= \text{Kernel of } j_*^{n+1}/\text{Image of } i_*^{n+1} \\
H_{n+1}^{r\omega}(X, A) &= \text{Kernel of } \partial^{n+1}/\text{Image of } j_*^{n+1} \\
H_n^{a\omega}(X, A) &= \text{Kernel of } i_*^n/\text{Image of } \partial^{n+1}
\end{aligned}$$

for $n \geq 0$.

By the definition, the ω-homology of a CW-pair is trivial if and only if its G-sequence is exact. Let $A = S^1 \vee S^1$ and $X = B^2 \vee S^1$. Then we have $H_1^{g\omega}(X, A) = Z$ (see Theorem 3.4 in [11]). Therefore there is a finite CW-pair with nontrivial ω-homology. There is an example of a finite CW-pair with nontrivial ω-homology in higher dimensions (see [16]). Let $X = K(Z, n)$ be an Eilenberg-McLane space. It is well known that $K(\pi, n)$ is an H-space if π is an abelian group. Thus $K(Z, n)$ is an H-space and $(K(Z, n), S^n)$ becomes a CW-pair. By a Gottlieb result [3], we have $G_n(K(Z, n)) \cong Z \cong \pi_n(K(Z, n))$. In [17], we showed the following.

EXAMPLE 4.1.

$$H_n^{g\omega}(K(Z, n), S^n) \cong \begin{cases} Z & \text{for } n \text{ even}, \\ Z_2 & \text{for } n \text{ odd and } n \neq 1, 3, 7, \\ 0 & \text{for } n = 1, 3, 7. \end{cases}$$

By Theorems 3.2 and 3.4, we have the following.

THEOREM 4.3. *Let (X, A) be a CW-pair. If the inclusion $i : A \to X$ has a left homotopy inverse or is homotopic to the constant map, then the ω-homology of (X, A) is trivial.*

EXAMPLE 4.2. If we identify S^k with $\{(x_1, \ldots, x_i, \ldots, x_{n+1}) \in S^n \mid x_i = 0$ for $i > k+1\}$ for $k < n$, then the inclusion map $i : S^k \to S^n$ is homotopic to a constant map and hence $H^{\omega}_*(S^n, S^k) = 0$ for $n > k > 0$.

In general, a map between pairs does not induce a homomorphism on ω-homology. Thus the following theorem gives a condition for a map between pars to induce a homomorphism on ω-homology [11].

THEOREM 4.4. *A map between CW-pairs with a right homotopy inverse induces a homomorphism on ω-homology.*

A triple (X, A, B) is called a CW-triple if (X, A), (X, B) and (A, B) are CW-pairs. It is well known that the homotopy sequence

$$\cdots \to \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A, B) \to \cdots$$

of a triple (X, A, B) is exact, where the boundary operator $\partial : \pi_n(X, A) \to \pi_{n-1}(A, B)$ is defined as the composite $\pi_n(X, A) \to \pi_{n-1}(A) \to \pi_{n-1}(A, B)$. A space X satisfying $G_n(X) = \pi_n(X)$ for all n is called a G-space [18]. It is clear every H-space is a G-space, but the converse is not true.

LEMMA 4.5. *Let (X, A, B) be a CW-triple. If A is a G-space and the inclusion i : $A \to X$ is homotopic to a constant map, then we have*

$$\begin{aligned} i_*(G^{rel}_n(A, B)) &\subset G^{rel}_n(X, B) \quad \text{for } n > 0, \\ j_*(G^{rel}_n(X, B)) &\subset G^{rel}_n(X, A) \quad \text{for } n > 0 \\ \partial_*(G^{rel}_n(X, A)) &\subset G^{rel}_{n-1}(A, B) \quad \text{for } n > 1. \end{aligned}$$

PROOF (sketch). If we consider the following commutative diagram

$$\begin{array}{ccc} \pi_n(A^B, B^B, i) & \xrightarrow{\bar{i}_*} & \pi_n(X^B, B^B, i) \\ \Big\downarrow{\scriptstyle \omega_*} & & \Big\downarrow{\scriptstyle \omega_*} \\ \pi_n(A, B, x_0) & \xrightarrow[i_*]{} & \pi_n(X, B, x_0), \end{array}$$

then we obtain $i_*(G^{rel}_n(A, B)) \subset G^{rel}_n(X, B)$.

Since A is a G-space and the inclusion $i : A \to X$ is homotopic to a constant map, (X, A) has the exact G-sequence, $G_n(X, A) = \pi_n(X)$ and $G_n(A) = \pi_n(A)$. Thus, we have the commutative ladder

$$\begin{array}{ccccccc} 0 & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) \to 0 \\ & & \Big\| & & \Big\uparrow & & \Big\| \\ 0 & \xrightarrow{i_*} & G_n(X, A) & \xrightarrow{j_*} & G^{rel}_n(X, A) & \xrightarrow{\partial} & G_{n-1}(A) \to 0 \end{array}$$

and hence $j_*(G^{rel}_n(X, B)) \subset G^{rel}_n(X, A)$.

Since the boundary operator $\partial_* : \pi_n(X,A) \to \pi_{n-1}(A,B)$ is the composite $\pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{j_*} \pi_{n-1}(A,B)$ and $G_{n-1}(A) \subset G_{n-1}(A,B)$, we obtain

$$\partial_*(G_n^{rel}(X,A)) = j_*\partial(G_n^{rel}(X,A)) \subset j_*(G_{n-1}(A)) \subset G_{n-1}^{rel}(A,B).$$

By Lemma 4.5, the relative evaluation subgroups $G_n^{rel}(A,B)$, $G_n^{rel}(X,B)$ and $G_n^{rel}(X,A)$ for a CW-triple (X,A,B) form a sequence

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial^{n+1}} & G_n^{rel}(A,B) & \xrightarrow{i_*^n} & G_n^{rel}(X,B) & \xrightarrow{j_*^n} & G_n^{rel}(X,A)\cdots & \xrightarrow{j_*^1} & G_1^{rel}(X,A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \pi_n(A,B) & \xrightarrow{i_*} & \pi_n(X,B) & \xrightarrow{j_*} & \pi_n(X,A)\cdots & \xrightarrow{j_*} & \pi_1(X,A)
\end{array}$$

on the assumption that A is a G-space and the inclusion from A to X is null homotopic. This will be called the *G-sequence* of a CW-triple (X,A,B).

The following theorem gives a condition for a CW-triple (X,A,B) to have the exact G-sequence.

THEOREM 4.6. *Let (X,A,B) be a CW-triple such that A is a G-space. If the inclusion maps $i : A \to X$ and $i : B \to A$ are homotopic to a constant map, then (X,A,B) has the exact G-sequence.*

PROOF (sketch). By Theorem 3.4, the CW-pairs $(A,B),(X,B)$ and (X,A) have the exact G-sequences. Since A is a G-space, we also have $G_n(A) = G_n(A,B) = \pi_n(A)$ for $n \geq 0$. Thus we have the commutative diagram

$$\begin{array}{ccccccc}
G_n(B) \xrightarrow{i_{2*}} & G_n(X,A) = G_n(X,B) & \xrightarrow{j_{3*}} & G_n^{rel}(X,A) & \xrightarrow{\partial_*} & G_{n-1}^{rel}(A,B) & \\
i_{1*}\searrow \quad i_{3*}\nearrow & & j_{2*}\searrow \quad \nearrow j_* & & \searrow\partial_3 \quad \nearrow j_{1*} & & \\
G_n(A,B) = G_n(A) & & G_n^{rel}(X,B) & & G_{n-1}(A,B) = G_{n-1}(A) & & \\
\partial_3\nearrow \quad j_{1*}\searrow & & i_*\nearrow \quad \searrow\partial_2 & & \nearrow i_{1*} \quad \searrow i_{3*} & & \\
G_{n+1}^{rel}(X,A) \underset{\partial_*}{\to} & G_n^{rel}(A,B) & \underset{\partial_1}{\to} & G_{n-1}(B) \underset{i_{2*}}{\to} G_{n-1}(X,A) = G_{n-1}(X,B) & & &
\end{array}$$

consisting of these four sequences arranged in the form of overlapping sine curves. If we chase the diagram as in the analogous situation in homology theory, we can easily prove that the G-sequence of (X,A,B) is exact (see p. 163 or p. 208 of [20]).

COROLLARY 4.7. *Let (X,A,B) be a CW-triple. If A is contractible, then (X,A,B) has exact G-sequence.*

EXAMPLE 4.3. For $0 < l < k < n$ and $k = 3$ or 7, the triple (S^n, S^k, S^l) has exact G-sequence.

References

[1] R. F. BROWN, *The Lefschetz Fixed Point Theorem*, Scott-Foresman, Chicago, 1971.

[2] D. H. GOTTLIEB, *A certain subgroup of the fundamental group*, Amer. J. Math. 87 (1965), 840–856.

[3] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. 91 (1968), 729–756.

[4] D. H. Gottlieb, *Covering transformations and universal fibrations*, Illinois J. Math. 13 (1969), 432-437.

[5] B. J. Jiang, *Estimation of the Nielsen numbers*, Chinese Math. 5 (1964), 330-339.

[6] J. R. Kim and M. H. Woo, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. 21 (1984), 109–120.

[7] G. E. Lang, Jr. *Evaluation subgroups of factor spaces*, Pacific J. Math. 42 (1972), 701–709.

[8] K. Y. Lee and M. H. Woo, *The relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. 25 (1988), 149–160.

[9] K. Y. Lee and M. H. Woo, *Homology and generalized evaluation subgroups of homotopy groups*, J. Korean Math. Soc. 25 (1988), 333–342.

[10] K. Y. Lee and M. H. Woo, *Exact G-sequences and relative G-pairs*, J. Korean Math. Soc. 27 (1990), 177–184.

[11] K. Y. Lee and M. H. Woo, *The G-sequence and ω-homology of a CW-pair*, Topology Appl. 52 (1993), 221–236.

[12] K. Y. Lee and M. H. Woo, *The generalized evaluation subgroups of product spaces relative to a factor*, Proc. Amer. Math. Soc. 124 (1996), 2255–2260.

[13] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc. Ser. A 32 (1982), 349–357.

[14] N. Oda, *The homotopy set of the axes of pairings*, Canad. J. Math. 42 (1990), 856–868.

[15] J. Oprea, *Gottlieb groups, group actions, fixed points and rational homotopy*, preprint.

[16] J. Z. Pan, X. Shen and M. H. Woo, *The G-sequence of a map and a CW-pair with nontrivial ω-homology*, to appear.

[17] J. Pak and M. H. Woo, *A remark on G-sequences*, Math. Japonica, to appear.

[18] J. Siegel, *G-spaces, W-spaces, H-spaces*, Pacific J. Math. 31 (1969), 209–214.

[19] K. Varadarajan, *Generalized Gottlieb groups*, J. Indian Math. Soc. 33 (1969), 141–164.

[20] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

[21] M. H. Woo and Y. S. Yoon, *T-spaces by the Gottlieb groups and its duality*, J. Austral. Math. Soc. (Series A) 58 (1995), 1–11.

[22] X. Zhao, *Realization of fixed point sets*, Acta Math. Sinica 12 (1996), 71–76.