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THE WECKEN PROPERTY OF THE PROJECTIVE PLANE

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Abstract. A proof is given of the fact that the real projective plane P^2 has the Wecken property, i.e. for every selfmap $f : P^2 \to P^2$, the minimum number of fixed points among all selfmaps homotopic to f is equal to the Nielsen number $N(f)$ of f.

Let X be a compact connected polyhedron, and let $f : X \to X$ be a map. Let $MF[f]$ denote the minimum number of fixed points among all maps homotopic to f. The Nielsen number $N(f)$ of f is always a lower bound to $MF[f]$. A space X is said to have the Wecken property if

$$N(f) = MF[f] \quad \text{for all maps } f : X \to X.$$

See [Br] for information about our current knowledge of such spaces.

It is considered as a classical fact (cf. [J, §5]) that compact surfaces of non-negative Euler characteristic have the Wecken property. There are only seven such surfaces. The cases of the sphere, the disk, the annulus and the Möbius band are trivial. The Wecken property of the torus was first proved in [B1], later generalized to higher dimensional tori in [H1]. The Klein bottle was also treated in [B1], although the enumeration of homotopy classes of selfmaps was incomplete. A complete proof was given in the unpublished [Ha], see a sketch in [DHT, Theorem 5.8]. (The Wecken property of the Klein bottle is also a consequence of the result [HKW, Corollary 8.3] on solvmanifolds.) The case of the projective plane was only mentioned by Hopf at the end of [H1]. The purpose of this short note is to supply a proof for this case, to fill a gap in the literature.

Let S^2 be the unit sphere in the Euclidean 3-space. The map $p : S^2 \to P^2$ identifying antipodal pairs of points is the universal cover of the projective plane P^2. We know $\pi_1(P^2) = H_1(P^2) = \mathbb{Z}/2\mathbb{Z}$.

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Suppose $f : P^2 \to P^2$ is a map, with the induced homomorphism $f_* : H_1(P^2) \to H_1(P^2)$. The degree of a lift $\widetilde{f} : S^2 \to S^2$ of f is determined only up to sign, because there are two lifts. The absolute value of this degree will be denoted by d_f.

By the theory of covering spaces, if $f_* \neq 0$, i.e. if f induces the identity automorphism on $\pi_1(P^2)$, every lift $\widetilde{f}$ commutes with the antipodal map of S^2 hence has odd degree (see [GH, pp. 127, 229]), thus d_f is odd. When $f_* = 0$, the map f can be lifted to a map $f' : P^2 \to S^2$. In this case the degree of $\widetilde{f} = f' \circ p$ is 0, but the mod 2 degree of f' is well defined. It will be denoted $d'_f \in \mathbb{Z}/2\mathbb{Z}$.

The following information is available:

(1) The homotopy classification of self-maps of P^2 ([B2], [H2]; for a modern treatment see [O, Theorems III and VII]): If $f_* \neq 0$, the homotopy classes are classified by the d_f which takes value in odd natural numbers. If $f_* = 0$, the homotopy classes are classified by the d'_f in $\mathbb{Z}/2\mathbb{Z}$.

(2) The computation of the Nielsen number ([H1]): $N(f) = 1$ when $f_* = 0$ or when $f_* \neq 0$ and $d_f = 1$; and $N(f) = 2$ when $d_f > 2$.

Construction of good representatives. Let $e : R^2 \to S^2$ be the parametrization of S^2 by longitude and latitude, namely

$$e(\phi, \theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta).$$

The identifications under the map e are generated by the relations

$$\begin{cases} (\phi,\theta) \sim (\phi, \theta + 2\pi) & \text{for all } \phi, \theta; \\ (\phi,\theta) \sim (\phi + \pi, \pi - \theta) & \text{for all } \phi, \theta; \\ (\phi,\theta) \sim (\phi', \theta) & \text{for all } \phi, \phi' \text{ when } \cos\theta = 0. \end{cases}$$

The covering map $p : S^2 \to P^2$ introduces more relations:

$$\begin{cases} (\phi,\theta) \sim (\phi + \pi, -\theta) & \text{for all } \phi, \theta; \\ (\phi,\theta) \sim (\phi, \theta + \pi) & \text{for all } \phi, \theta. \end{cases}$$

Let $f_m : P^2 \to P^2$ be the map induced by the map $F_m : R^2 \to R^2$ defined below:

$$F_0(\phi,\theta) = (0, \tfrac{\pi}{2}); \qquad F_1(\phi,\theta) = (\phi + 1, \theta);$$
$$F_2(\phi,\theta) = (\phi, 2\theta + \tfrac{\pi}{2}); \qquad F_m(\phi,\theta) = (m\phi, -\theta), \quad \text{for odd } m \geq 3.$$

It is easy to see that $f_{m*} \neq 0$ and $d_{f_m} = m$ for odd m, and that $f_{m*} = 0$ for even m. Furthermore, $d'_{f_0} = 0$ and $d'_{f_2} = 1$. These maps represent all the homotopy classes. The maps f_m, $m \leq 2$, have a unique fixed point, so they have the minimum number of fixed points.

For $m = 2n+1 \geq 3$, Hopf claimed at the end of [H1] that in the homotopy class there is a map with two fixed points. We now give an explicit construction.

For a real number x, we denote by $[x]$ the greatest integer less than or equal to x. Define maps $u, v : R \to R$ by

$$u(\phi) = \begin{cases} \operatorname{sgn}\sin\phi \cdot \frac{2n(2n+1)}{\pi} \left|\phi - \frac{h\pi}{n}\right|, & \text{if } \left|\phi - \frac{h\pi}{n}\right| \leq \frac{\pi}{2n(2n+1)} \\ & \text{where } h = [\frac{n\phi}{\pi} + \frac{1}{2}], \\ \operatorname{sgn}\sin\phi, & \text{otherwise}; \end{cases}$$

$$v(\theta) = |\arcsin(\cos\theta)| = |\theta - k\pi - \tfrac{\pi}{2}|, \qquad \text{where } k = [\tfrac{\theta}{\pi}].$$

Obviously $u(\phi+\pi) = -u(\phi)$ and $v(\theta+\pi) = v(\theta)$. Define a function $F : R^2 \to R^2$ by

$$F(\phi,\theta) = \begin{cases} ((2n+1)\phi, -\theta + u(\phi)v(\theta)), & \text{if } [\frac{n\phi}{\pi}] \text{ is a multiple of } n; \\ \left(\frac{h\pi}{n} + \frac{\pi}{2n}, -\theta + 2u(\phi)v(\theta)\right), & \text{if } h = [\frac{n\phi}{\pi}] \text{ is not a multiple of } n, \\ & |u(\phi)| = 1 \text{ and } \theta - [\frac{\theta}{\pi}]\pi \le \frac{2\pi}{3}; \\ ((2n+1)\phi, -\theta + 2u(\phi)v(\theta)), & \text{otherwise.} \end{cases}$$

If we write $F(\phi,\theta) = (\Phi(\phi,\theta), \Theta(\phi,\theta))$, then it is clear that

$$\begin{cases} \Phi(\phi,\theta+\pi) = \Phi(\phi,\theta), \\ \Theta(\phi,\theta+\pi) = \Theta(\phi,\theta) - \pi, \end{cases} \qquad \begin{cases} \Phi(\phi+\pi,-\theta) \equiv \Phi(\phi,\theta) + \pi \pmod{2\pi}, \\ \Theta(\phi+\pi,-\theta) = -\Theta(\phi,\theta), \end{cases}$$

and that $\cos\Theta(\phi,\theta) = 0$ if $\cos\theta = 0$. The function $\Theta(\phi,\theta)$ is continuous, but $\Phi(\phi,\theta)$ has discontinuities where limits from different directions differ by multiples of 2π. Thus F induces a continuous map $S^2 \to S^2$, hence also a map $f : P^2 \to P^2$. It is not difficult to see that $d_f = 2n+1 = m$.

Let Q denote the arc in P^2 which is the image of the set $\{(\phi,0) \mid \frac{\pi}{n} \le \phi \le \pi\} \subset R^2$. We have $f(Q) = Q$, and all fixed points of f lie in Q, except the one represented by $(0,\frac{\pi}{2}) \in R^2$.

By shrinking the arc Q to a point, we obtain a map $\bar{f} : P^2 \to P^2$ with exactly two fixed points, as desired.

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