

WHAT IS A DISK?

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Introduction. This paper should be considered as a companion report to F.W. Gehring's survey lectures "Characterizations of quasidisks" given at this Summer School [7]. Notation, definitions and background results are given in that paper. In particular, D is a simply connected proper subdomain of \mathbf{R}^2 unless otherwise stated and D^* denotes the exterior of D in $\overline{\mathbf{R}}^2$.

Many of the characterizations of quasidisks have been motivated by looking at properties of euclidean disks. It is therefore natural to go back and ask if any of the original properties in fact characterize euclidean disks. We follow the procedure in Gehring's lectures and look at four different categories of properties:

1. Geometric properties,
2. Conformal invariants,
3. Injectivity criteria,
4. Extension properties.

As we shall see, the answers are not equally easy to obtain and not always positive. There are, in fact, still many interesting open questions.

1. Geometric properties

Reflection property

CHARACTERIZATION 1.1. *D is a half plane if and only if there exists a euclidean isometry f of $\overline{\mathbf{R}}^2$ which maps D onto its exterior D^* and is the identity on ∂D .*

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PROOF (sufficiency). Fix $z_1 \in D$ and let $\zeta \in \partial D$, $z_1, \zeta \neq \infty$. Then

$$|f(z_1) - \zeta| = |f(z_1) - f(\zeta)| = |z_1 - \zeta|.$$

Thus $\partial D \subset L \cup \{\infty\}$ where L is the perpendicular bisector of the segment $[z_1, f(z_1)]$. On the other hand let $z \in L$ and consider the broken line $\gamma = [z_1, z] \cup [z, f(z_1)]$. Then γ must meet ∂D and also $\gamma \cap \partial D \subset \gamma \cap L$ which implies that $z \in \partial D$. ■

Reversed triangle inequality property. If D is a disk or half plane, then

$$\min_{j=1,2} \text{dia}(\gamma_j) \leq |z_1 - z_2|$$

for any two points $z_1, z_2 \in \partial D$ where γ_1, γ_2 are the components of $\partial D \setminus \{z_1, z_2\}$. In particular, D satisfies Ahlfors's well known two point inequality [3] with constant 1. But this inequality also holds if D is a Reuleaux triangle, [6] and [15], and hence cannot be used to characterize a disk or half plane. However for the reversed triangle inequality [7] we have the following characterization.

CHARACTERIZATION 1.2. *A Jordan domain $D \subset \mathbf{R}^2$ is a disk or half plane if and only if*

$$|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \leq |z_1 - z_3||z_2 - z_4|$$

for each ordered quadruple of points $z_1, z_2, z_3, z_4 \in \partial D \setminus \{\infty\}$.

PROOF. For the sufficiency, choose three points on $\partial D \setminus \{\infty\}$ and let C be the line or circle through these points. To complete the proof we shall show that each $z \in \partial D \setminus \{\infty\}$ lies on C .

Fix a point $z \in \partial D \setminus \{\infty\}$ different from the three points given above and label these four points z_1, z_2, z_3, z_4 so that they constitute an ordered quadruple in ∂D with $z_2 = z$. Next choose a Möbius transformation f such that

$$f(z_1) = 0, \quad f(z_3) = 1, \quad f(z_4) = \infty.$$

Then $f(C)$ is the real axis \mathbf{R} and it suffices to show that

$$f(z_2) \in (0, 1) \subset \mathbf{R}.$$

By our hypothesis, the reversed triangle inequality yields

$$\begin{aligned} 1 &\geq |(z_1, z_4, z_2, z_3)| + |(z_2, z_1, z_3, z_4)| = |(0, \infty, f(z_2), 1)| + |(f(z_2), 0, 1, \infty)| \\ &= |f(z_2)| + |1 - f(z_2)| \geq 1. \end{aligned}$$

This means that we have equality throughout and hence that $f(z_2) \in (0, 1)$.

To show that the condition is necessary, we reverse the last part of the above argument. ■

REMARK. The task of proving that the condition with equality is a necessary one is given as an exercise on p. 82 in [1].

Local connectivity property. Langmeyer has established the characterization given below in her thesis [15]. Our proof is a modification of hers.

CHARACTERIZATION 1.3. *D is a disk or half plane if and only if it is 1-linearly locally connected, i.e., for each $z_0 \in \mathbf{R}^2$ and each $r > 0$*

- a. points in $D \cap \overline{B}(z_0, r)$ can be joined by an arc in the same set,
- b. points in $D \setminus B(z_0, r)$ can be joined by an arc in the same set.

PROOF (sufficiency). The hypotheses imply that D is a quasidisk and hence a Jordan domain [8]. In addition we may assume that D is bounded since the property of being 1-linearly locally connected is preserved under Möbius transformations [24].

Property a. implies that D is convex. To prove this fix $z_1, z_2 \in D$ and let α be an arc joining these points in D . It follows that every point in a bounded component of $\mathbf{R}^2 \setminus \{[z_1, z_2] \cup \alpha\}$ is in D . For consider a point ζ in such a domain, i.e., bounded by a segment $[z'_1, z'_2]$ of $[z_1, z_2]$ and a subarc α' of α . Let U be a disk with $z'_1, z'_2 \in \partial U$ and with radius so large that $\zeta \notin \overline{U}$. Since D has property a. there is an arc α'' in $D \cap \overline{U}$ joining z'_1 and z'_2 . Then ζ is in a component of $\mathbf{R}^2 \setminus \{\alpha' \cup \alpha''\}$ which lies in D and $\zeta \in D$ since D is simply connected. Thus $[z_1, z_2] \subset \overline{D}$ and this implies that \overline{D} is convex. Hence $D = \text{Int}(\overline{D})$ is also convex.

Next we claim that

$$(*) \quad \min\{|z_1 - z_0|, |z_3 - z_0|\} \leq \max\{|z_2 - z_0|, |z_4 - z_0|\}$$

for all $z_0 \in \mathbf{R}^2$ and all ordered quadruples of points $z_1, z_2, z_3, z_4 \in \partial D$. For suppose there exist five points for which (1) does not hold, and let r be a number such that

$$\max\{|z_2 - z_0|, |z_4 - z_0|\} < r < \min\{|z_1 - z_0|, |z_3 - z_0|\}.$$

Since D is convex, $\beta = [z_2, z_4]$ is a crosscut in D . Hence any arc in D joining points close to z_1 and z_3 must intersect $\beta \subset B(z_0, r)$ and cannot be in $D \setminus B(z_0, r)$, in contradiction to property b.

To finish up the proof choose $z_1, z_3 \in \partial D$ such that

$$|z_1 - z_3| = \text{dia}(D) = d$$

and let z_0 denote the midpoint of $[z_1, z_3]$. The perpendicular bisector of $[z_1, z_3]$ intersects ∂D in two points z_2 and z_4 lying in different components of $\partial D \setminus \{z_1, z_3\}$. Let z'_0 be the midpoint of $[z_2, z_4]$. Then

$$\frac{d}{2} \leq |z_1 - z'_0| = |z_3 - z'_0| \leq |z_2 - z'_0| = |z_4 - z'_0| \leq \frac{d}{2}$$

where the middle inequality follows from (1) above. We must have equality throughout, and z_2, z_4 are diametrically opposite points on the circle through z_0 with radius $d/2$. By (*) at most one of the four arcs $\widehat{z_1 z_2}, \widehat{z_2 z_3}, \widehat{z_3 z_4}, \widehat{z_4 z_1}$ of ∂D can have a point closer to z_0 than $d/2$. Hence we have two opposite arcs where this is not the case. They must both be on the circle $|z - z_0| = d/2$ since otherwise $\text{dia}(D) > d$. For any two diametrically opposite points on these arcs we can repeat the above argument with the perpendicular bisector. Therefore all the points on the two remaining arcs are also on the circle. ■

Decomposition property

CHARACTERIZATION 1.4. D is a disk or half plane if and only if for each $z_1, z_2 \in D$ there exists a disk D' with

$$z_1, z_2 \in D' \subset D$$

PROOF (sufficiency). The hypotheses imply that D is a quasidisk and hence a Jordan domain [11]. Next we may assume that D is bounded. (If not we apply a suitable Möbius transformation.) We choose $z_j, z'_j \in D$ such that

$$|z_j - z'_j| \rightarrow \text{dia}(D) < \infty.$$

By hypothesis we get w_j, r_j such that

$$z_j, z'_j \subset B(w_j, r_j) \subset D$$

and by passing to subsequences we may assume

$$z_j \rightarrow z_0, z'_j \rightarrow z'_0, w_j \rightarrow w_0, r_j \rightarrow r_0.$$

Then $B(w_0, r_0) \subset \overline{D}$ and since D has no inner boundary $B(w_0, r_0) \subset D$. In particular $2r_0 \leq \text{dia}(D)$ but by construction $2r_0 \geq \text{dia}(D)$. We conclude that $B(w_0, r_0) = D$. ■

2. Conformal invariants. There are three different cases when a Jordan domain D together with some distinguished interior and/or boundary points has one conformal invariant. See [2] or [7].

- (i) Two interior points z_1, z_2 . The conformal invariant is the hyperbolic distance $h_D(z_1, z_2)$ between z_1 and z_2 .
- (ii) One interior point z_0 and the boundary points z_1, z_2 . The conformal invariant is $\omega(z_0, \alpha; D)$, the harmonic measure of the positively oriented boundary arc $\alpha = \widehat{z_1 z_2}$ evaluated at z_0 .
- (iii) Four positively oriented boundary points z_1, z_2, z_3, z_4 . The conformal invariant is the modulus of the quadrilateral $Q = D(z_1, z_2, z_3, z_4)$.

We will describe how each of these invariants can be used to characterize disks.

Hyperbolic bound property

CONJECTURE 2.1. D is a disk if and only if

$$h_D(z_1, z_2) \leq j_D(z_1, z_2)$$

for $z_1, z_2 \in D$ where j_D is the metric given by

$$j_D(z_1, z_2) = \log \left(\frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) \left(\frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right).$$

THEOREM 2.2 [10]. *The above conjecture is true if in addition, for one pair of points $w_1, w_2 \in \partial D$ with $|w_1 - w_2| = \text{dia}(D)$ there exist disks $D_1, D_2 \subset D$ with $w_1 \in \partial D_1$ and $w_2 \in \partial D_2$.*

Harmonic symmetry property

CHARACTERIZATION 2.3. *A Jordan domain D is a disk or a half plane if and only if there exist points $z_0 \in D$ and $z_0^* \in D^*$ such that if α and β are adjacent open arcs in ∂D with*

$$\omega(z_0, \alpha; D) = \omega(z_0, \beta; D),$$

then

$$\omega(z_0^*, \alpha; D^*) = \omega(z_0^*, \beta; D^*).$$

PROOF (sufficiency). We consider conformal mappings $f : B \rightarrow D$, $g : B^* \rightarrow D^*$ such that $f(0) = z_0$, $g(\infty) = z_0^*$. Then f, g have homeomorphic extensions to $\overline{B}, \overline{B}^*$ and by a preliminary rotation we may assume that $f(1) = g(1)$. Then

$$h = g^{-1} \circ f : \partial B \rightarrow \partial B$$

is a sense preserving homeomorphism such that $h(1) = 1$. We want to show that $h(z) = z$.

For this let α, β be upper and lower halves of ∂B labeled so that $i \in \alpha$. By conformal invariance of harmonic measure

$$\omega(z_0, f(\alpha), D) = \omega(0, \alpha, B) = 1/2 = \omega(0, \beta, B) = \omega(z_0, f(\beta), D)$$

and by hypothesis

$$\begin{aligned} \omega(0, \overline{h}(\alpha), B) &= \omega(\infty, h(\alpha), B^*) = \omega(z_0^*, g(h(\alpha)), D^*) \\ &= \omega(z_0^*, f(\alpha), D^*) = \omega(z_0^*, f(\beta), D^*) \\ &= \omega(z_0^*, g(h(\beta)), D^*) = \omega(\infty, h(\beta), B^*) = \omega(0, \overline{h}(\beta), B). \end{aligned}$$

In particular, this implies $h(-1) = -1$ and, since h is sense preserving, we have $h(\alpha) = \alpha$.

Next let α, β be left and right halves of the upper half of ∂D labeled so that $e^{i\pi/4} \in \alpha$. Then as above

$$\omega(z_0, f(\alpha), D) = \omega(0, \alpha, B) = 1/4 = \omega(0, \beta, B) = \omega(z_0, f(\beta), D)$$

and

$$\omega(0, \overline{h}(\alpha), B) = \omega(z_0^*, f(\alpha), D^*) = \omega(z_0^*, f(\beta), D^*) = \omega(0, \overline{h}(\beta), B).$$

In particular we get that $h(i) = i$, and $h(\alpha) = \alpha$. Proceeding this way we see that

$$h(e^{2\pi it}) = e^{2\pi it}$$

for all $t \in [0, 1)$ of the form $t = 2^{-n}p$, $p \in \mathbf{Z}$, and by continuity $h(z) = z$. This implies that the conformal mappings f and g together define a homeomorphism φ of $\overline{\mathbf{R}^2}$ which is conformal in $B \cup B^*$ and hence in $\overline{\mathbf{R}^2}$. In particular φ is a Möbius transformation and $D = \varphi(B)$ is a disk or half plane. ■

Conjugate quadrilateral property. We recall that a quadrilateral $Q = D(z_1, z_2, z_3, z_4)$ consists of a Jordan domain $D \subset \overline{\mathbf{R}^2}$ together with four positively oriented points $z_1, z_2, z_3, z_4 \in \partial D$. Moreover $Q^* = D^*(z_4, z_3, z_2, z_1)$ is called the conjugate quadrilateral. Any quadrilateral Q can be mapped conformally onto a rectangle $R = R(0, m, m + i, i)$, and we call m the modulus of Q , $\text{mod}(Q) = m$.

CHARACTERIZATION 2.4. *A Jordan domain D is a disk or half plane if and only if for each pair of conjugate quadrilaterals Q and Q^* in D and D^* ,*

$$\text{mod}(Q^*) \leq 1$$

whenever $\text{mod}(Q) = 1$.

PROOF (sufficiency). Since

$$\text{mod}(D^*(z_4, z_3, z_2, z_1)) = \frac{1}{\text{mod}(D^*(z_3, z_2, z_1, z_4))},$$

the hypotheses actually imply that $\text{mod}(Q^*) = 1$ whenever $\text{mod}(Q) = 1$.

Let f and f^* be homeomorphic maps of \overline{D} and \overline{D}^* to the closed upper and lower half plane, respectively, which are conformal in D and D^* and normalized so that $w(0) = 0$, $w(1) = 1$, $w(\infty) = \infty$ where $w = f^* \circ f^{-1}$. If $z_j = f^{-1}(x_j)$ where $x_1 < x_2 < x_3$ and $x_4 = \infty$, then

$$\text{mod}(D(z_1, z_2, z_3, z_4)) = 1 \quad \text{if and only if} \quad x_2 = \frac{x_1 + x_3}{2}$$

while

$$\text{mod}(D^*(z_4, z_3, z_2, z_1)) = 1 \quad \text{if and only if} \quad w(x_2) = \frac{w(x_1) + w(x_3)}{2}$$

and thus

$$w\left(\frac{x_1 + x_3}{2}\right) = \frac{w(x_1) + w(x_3)}{2}.$$

Since $w(1) = 1$, $w(0) = 0$ we see that $w(x) = x$ for all $x \in \mathbf{R}$ of the form $x = 2^{-n}p$, $p \in \mathbf{Z}$, and by continuity $w(x) = x$ for all $x \in \mathbf{R}$. This implies that the maps f^{-1} and f^{*-1} coincide on \mathbf{R} and together define a conformal mapping in $\overline{\mathbf{R}}^2$, i.e., a Möbius transformation. It follows that D is a disk or half plane. ■

Extremal distance. The modulus of a quadrilateral can also be defined using families of curves. We shall now consider the more general situation where we have defined for any family Γ of curves in $\overline{\mathbf{R}}^2$ a modulus, $\text{mod}(\Gamma)$, which is conformally invariant.

CHARACTERIZATION 2.5 [25]. *D is a disk or half plane if and only if*

$$\text{mod}(\Gamma) \leq 2 \text{mod}(\Gamma_D)$$

whenever E, F are continua in D and Γ_D, Γ denote the families of all curves which join E and F in D and in $\overline{\mathbf{R}}^2$, respectively.

3. Injectivity criteria

Injectivity of analytic functions. Two different criteria for global injectivity of functions f analytic in a domain D are considered in [7]. The first involves the Schwarzian derivative S_f and the second the pre-Schwarzian derivative $T_f = f''/f'$. For each domain D we define two constants $\sigma(D)$ and $\tau(D)$ as follows:

$\sigma(D)$ is the supremum of the constants $a \geq 0$ such that f is injective whenever f is meromorphic and locally injective with $|S_f| \leq a\rho_D^2$ in D .

$\tau(D)$ is the supremum of the constants $b \geq 0$ such that f is injective whenever f is analytic and locally injective with $|T_f| \leq b\rho_D$ in D .

Here ρ_D denotes the hyperbolic density with curvature -1 in D . Then

$$0 \leq \sigma(D) \leq 1/2 \quad \text{and} \quad 0 \leq \tau(D) \leq 1/2$$

for all D ; [16] and [23]. Moreover both constants are positive if and only if D is a quasidisk. See [3], [8] and [4]

CHARACTERIZATION 3.1 [16], [19]. *D is a disk or half plane if and only if*

$$\sigma(D) = 1/2.$$

The situation for the pre-Schwarzian derivative is somewhat different.

THEOREM 3.2 [5]. *If D is a disk or half plane, then*

$$\tau(D) = 1/2.$$

But the converse of this result is not true.

THEOREM 3.3 [23]. *If f is conformal with $|zT_f(z)| \leq 1/2$ in B , then $\tau(f(B)) = 1/2$.*

The following is an immediate consequence of Theorem 3.3.

EXAMPLE 3.4. *If $D = f(B)$ where $f(z) = e^{z/2}$, then $\tau(D) = 1/2$.*

Injectivity of local quasi-isometries. We recall that f is a local L -quasi-isometry in D if each point in D has a neighborhood such that

$$\frac{1}{L} |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq L |z_1 - z_2|$$

for points z_1, z_2 in this neighborhood. Furthermore we denote by $L(D)$ the supremum of the numbers L such that f is injective whenever f is a local L -quasi-isometry in D .

$L(D) > 1$ if and only if D is a quasidisk ([9] and [18]). $L(D)$ is not known when D is a disk or half plane, much less that it can be used to characterize such domains. We have, however, the following results.

THEOREM 3.5 [13]. *If D is a disk or half plane, then $L(D) \geq 2^{1/4}$.*

LEMMA 3.6. *If D is convex, then $L(D) \leq 2^{1/2}$.*

PROOF. Let $z_0 \in D$, choose $z_1 \in \partial D$ so that $|z_1 - z_0| = \text{dist}(z_0, \partial D)$, and let T be the corresponding support plane for D at z_1 . Without loss of generality we may assume that $z_1 = 0$ and $T = \{z : \Re(z) > 0\}$. Then for each $L > 2^{1/2}$,

$$f(z) = \frac{|z|}{L} \exp(iL^2 \arg(z))$$

is a local L -quasi-isometry in D which is not injective. ■

4. Extensions

Comparable Dirichlet integral property

CHARACTERIZATION 4.1. *Let D be a Jordan domain in \mathbf{R}^2 . Then D is a disk or half plane if and only if*

$$\int_{D^*} |\text{grad } u^*|^2 dm = \int_D |\text{grad } u|^2 dm$$

for each pair of functions u and u^ which are harmonic in D and D^* , respectively, with continuous and equal boundary values.*

PROOF (sufficiency). By the conjugate quadrilateral characterization we are done if we can prove that

$$\text{mod}(D(z_1, z_2, z_3, z_4)) = \text{mod}(D^*(z_4, z_3, z_2, z_1))$$

for all positively oriented quadruples $z_1, z_2, z_3, z_4 \in \partial D$.

For this choose u harmonic in D and continuous in \bar{D} with boundary values 0 and 1, respectively, on the Jordan arcs $\widehat{z_2 z_3}$ and $\widehat{z_4 z_1}$. Next let f be the canonical mapping of

Q onto the rectangle $R = R(0, m, m + i, i)$ and set $v = u \circ f^{-1}$. Then

$$\int_0^m |\operatorname{grad} v| \, dx \geq 1$$

for $0 < y < 1$, and with equality if $v = x/m$. Thus

$$1 \leq \int_0^1 \left(\int_0^m |\operatorname{grad} v| \, dx \right) dy \leq \left(\int_R |\operatorname{grad} v|^2 \, dm \right)^{1/2} \operatorname{mod}(Q)^{1/2}$$

and

$$\frac{1}{\operatorname{mod}(Q)} = \inf_v \int_R |\operatorname{grad} v|^2 \, dm = \inf_v \int_D |\operatorname{grad} v|^2 |f'|^2 \, dm = \inf_u \int_D |\operatorname{grad} u|^2 \, dm.$$

Similarly

$$\frac{1}{\operatorname{mod}(Q^*)} = \inf_{u^*} \int_{D^*} |\operatorname{grad} u^*|^2 \, dm.$$

Hence our assumption implies that

$$\operatorname{mod}(Q^*) = \operatorname{mod}(Q),$$

and the result follows as remarked above. ■

REMARK. Since

$$\operatorname{mod}(D(z_2, z_3, z_4, z_1)) = \frac{1}{\operatorname{mod}(D(z_1, z_2, z_3, z_4))},$$

the result is implied by

$$\int_{D^*} |\operatorname{grad} u^*|^2 \, dm \leq \int_D |\operatorname{grad} u|^2 \, dm.$$

Extension of functions with bounded Dirichlet integral

CHARACTERIZATION 4.2. D is a disk or half plane if and only if each locally integrable function u with bounded Dirichlet integral in D has an extension to a locally integrable function v with bounded Dirichlet integral in \mathbf{R}^2 such that

$$\int_{\mathbf{R}^2} |\operatorname{grad} v|^2 \, dm \leq 2 \int_D |\operatorname{grad} u|^2 \, dm.$$

PROOF (sufficiency). The hypotheses imply that D is a quasidisk and hence a Jordan domain. See [12] or [14].

Now consider a pair of functions u and u^* harmonic in D and D^* , respectively, with continuous and equal boundary values and suppose that

$$\int_D |\operatorname{grad} u|^2 \, dm < \infty.$$

By the assumption, u has an extension to a function v such that

$$\int_{D^*} |\operatorname{grad} v|^2 \, dm + \int_D |\operatorname{grad} u|^2 \, dm \leq 2 \int_D |\operatorname{grad} u|^2 \, dm$$

whence

$$\int_{D^*} |\operatorname{grad} v|^2 \, dm \leq \int_D |\operatorname{grad} u|^2 \, dm.$$

Next by the Dirichlet Principle,

$$\int_{D^*} |\text{grad } u^*|^2 dm \leq \int_{D^*} |\text{grad } v|^2 dm.$$

Thus

$$\int_{D^*} |\text{grad } u^*|^2 dm \leq \int_D |\text{grad } u|^2 dm$$

and the result follows from the Remark above. ■

Extension of conformal mappings

CHARACTERIZATION 4.3. *D is a disk or half plane if and only if every conformal self mapping of D has a conformal extension to $\overline{\mathbf{R}^2}$.*

Our proof depends on the following characterization for Jordan domains.

LEMMA 4.4. *D is a Jordan domain if and only if every conformal self mapping of D has a homeomorphic extension to \overline{D} .*

PROOF (sufficiency). By hypothesis, each conformal self mapping of D has a homeomorphic extension to \overline{D} . We shall show that D is a Jordan domain, or equivalently, by Theorem 2.6 of [21], that ∂D is locally connected and has no cut points.

To prove that ∂D is locally connected, we consider a conformal map $g : B \rightarrow D$. By Theorem 1.7 of [21] there exists $z_0 \in \partial B$ such that

$$\lim_{r \rightarrow 1} g(rz_0) = w_0.$$

Next let $z_1 \in \partial B$ and set

$$h(z) = \frac{z_1}{z_0} z.$$

Then by our hypothesis,

$$f = g \circ h \circ g^{-1} : D \rightarrow D$$

has a homeomorphic extension $f^* : \overline{D} \rightarrow \overline{D}$. In particular,

$$\lim_{r \rightarrow 1} g(rz_1) = \lim_{r \rightarrow 1} g \circ h(rz_0) = \lim_{r \rightarrow 1} f^* \circ g(rz_0) = f^*(w_0) = w_1.$$

Since z_1 was an arbitrary point in ∂B , we conclude that g has a radial limit at each point $z \in \partial B$. Then Proposition 2.23 of [21] implies that there exists a point $z_0 \in \partial B$ at which g has in fact a limit and, arguing as above, we see that g has a limit at each point $z \in \partial B$. Thus g has a continuous extension to \overline{D} and ∂D is locally connected by Theorem 2.1 of [21].

Finally by Theorem IV.11.1 of [20], there exists $w_0 \in \partial D$ which is not a cut point of ∂D . But then ∂D is cut point free. For let w_1 be any point in $\partial D \setminus \{w_0\}$ and choose $z_0, z_1 \in \partial B$ such that $g(z_0) = w_0$ and $g(z_1) = w_1$. As before, $w_1 = f^*(w_0)$,

$$\partial D \setminus \{w_1\} = f^*(\partial D) \setminus \{f^*(w_0)\} = f^*(\partial D \setminus \{w_0\})$$

is connected and hence w_1 is not a cut point of ∂D . ■

Proof of Characterization 4.3. For the sufficiency it is enough to prove that D is a disk when D is bounded since D is a Jordan domain by Lemma 4.4.

Now pick any disk $U \subset D$ with two boundary points $z_1, z_2 \in \partial D$ and choose $w_1, w_2 \in \partial D$ such that $|w_1 - w_2| = \text{dia}(D)$. Then because D is a Jordan domain, there exists a conformal self mapping f of D such that $f^*(z_j) = w_j$ for $j = 1, 2$ where f^* is the homeomorphic extension of f of \bar{D} . By our hypothesis, f^* is the restriction of a Möbius transformation to D and $f(U)$ must be a disk in D with diameter equal to $\text{dia}(D)$. It follows that $f(U) = D$. ■

To be in accordance with the characterization of quasidisks as quasiconformal extension domains [22] we can reformulate the above result as follows.

CHARACTERIZATION 4.5. *D is a disk or half plane if and only if each K -quasiconformal self mapping of D has a K -quasiconformal extension to $\bar{\mathbf{R}}^2$.*

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