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CONNECTION MATRIX THEORY FOR DISCRETE DYNAMICAL SYSTEMS

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Abstract. In [C] and [F1] the connection matrix theory for Morse decomposition is developed in the case of continuous dynamical systems. Our purpose is to study the case of discrete time dynamical systems.

The connection matrices are matrices between the homology indices of the sets in the Morse decomposition. They provide information about the structure of the Morse decomposition; in particular, they give an algebraic condition for the existence of connecting orbit set between different Morse sets.

0. Introduction. One of the methods by which the Conley index theory studies isolated invariant sets is to decompose them into subinvariant sets (Morse sets) and connecting orbits between them. This structure is called a Morse decomposition of an isolated invariant set. A filtration of index pairs associated with a Morse decomposition can be used to find connections between Morse sets. The existence of such a filtration in the case of continuous dynamical systems has been proved in [CoZ], [Sal] and [F1]. In [BD] we have proved the existence of index triples and index filtrations for dynamical systems given by a homeomorphism (discrete dynamical systems).

Studying the topology of the sets in the filtration provides some information on the structure of the Morse decomposition. The principal tool for this purpose is the connection matrix. In [C] and [F1, F2] the connection matrix theory for Morse decompositions is developed for flows. We wish to investigate this theory for a homeomorphism. Once an index filtration has been found, most steps of the construction of the connection matrix in the continuous case can be carried over to the discrete case. The main difference is that the homology Conley index is not simply the homology of the index pair but the Leray reduction of its homology (see [M2] for more details).

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1. Index. Let X be a fixed locally compact metric space. Assume a discrete time dynamical system on X, i.e. a fixed homeomorphism $f : X \to X$, is given. We will use the convenient notation $xn = f^n(x)$ for any $x \in X$ and $n \in \mathbf{Z}$. If $A \subset X$ and $\Delta \subset \mathbf{Z}$, then $A\Delta := \{xn \ ; \ x \in A \text{ and } n \in \Delta\}$. For a given subset $N \subset X$ the sets $Inv^+N := \{x \in X \mid x\mathbf{Z}^+ \subset N\}$, $Inv^-N := \{x \in X \mid x\mathbf{Z}^- \subset N\}$, $InvN := Inv^+N \cap Inv^-N$ are called the *positively invariant, negatively invariant* and *invariant part* of N, respectively. A set A is called *invariant* if $Inv\ A = A$. Similarly A is *positively (negatively) invariant* if $Inv^+A = A$ ($Inv^-A = A$).

A subset A of N is called *positively invariant with respect to* N if $A \cap f^{-1}(N) \subset f^{-1}(A)$. A subset $S \subset X$ is an *isolated invariant set* if S is compact invariant set and there exists a compact neighbourhood N of S in X such that $S = Inv\ N$. In that case we call N an *isolating neighbourhood* (for S in X).

Definition (1.1). A pair $P = (P_1, P_2)$ of compact subsets of X is called an *index pair* for S in X if

(o) $P_2 \subset P_1$,
(i) $S = Inv(cl\ (P_1 - P_2)) \subset int(P_1 - P_2)$,
(ii) P_2 is positively invariant with respect to P_1,
(iii) $P_1 - P_2 \subset f^{-1}(P_1)$.

Now we recall the notion of Leray functor introduced by Mrozek ([M2], [M3]). Denote by ξ the category of graded vector spaces and linear maps of degree zero. A new category $Endo(\xi)$ of graded vector spaces with distinguished endomorphisms is defined as follows. Objects are pairs (E, e), where $E \in \xi$ and $e \in \xi(E, E)$. Morphisms from (E, e) to (F, f) are all maps $\Phi \in \xi(E, F)$ such that $\Phi \circ e = f \circ \Phi$. $Auto(\xi)$ is the full subcategory of $Endo(\xi)$ consisting of graded vector spaces with a distinguished isomorphism. The full subcategory of $Endo(\xi)$ consisting of all objects with finite dimensional components and their morphisms will be denoted by $Endo_0(\xi)$. For $(E, e) \in Endo(\xi)$ we define the generalized kernel of e as

$$gker\ e := \sum \{e^{-n}(0) \mid n \in N\}.$$

Put

$$L(E, e) := (E/gker\ e, e') \in Auto(\xi), \tag{1.2}$$

where $e' : E/gker(e) \ni [x] \longmapsto [e(x)] \in E/gker(e)$ is the induced endomorphism. Assume $\Phi : (E, e) \to (F, f)$ is a morphism. Let

$$\Phi' : E/gker(e) \ni [x] \to [\Phi(x)] \in F/gker(f)$$

denote the induced morphism. We put

$$L(\Phi) := \Phi'. \tag{1.3}$$

Thus we have defined a covariant functor

$$L : Endo_0(\xi) \to Auto(\xi)$$

called the *Leray functor.*

Let H^* be the functor of the Alexander–Spanier cohomology with rational coefficients.

If we consider an index pair (P_1, P_2), then the map $f_p : P_1/P_2 \to P_1/P_2$ given by:

$$f_p([x]) := \begin{cases} [f(x)] & \text{if } x, f(x) \in P_1 - P_2 \\ [P_2] & \text{otherwise} \end{cases}$$

is continuous (see e.g. [Szy], Lemma 4.3), and it induces an endomorphism $f^* : H^*(P_1, P_2) \to H^*(P_1, P_2)$. Therefore $((H^*(P_1, P_2), f^*) \in Endo(\xi)$. We denote also by H^* the extension of Alexander–Spanier cohomology functor to this category.

Definition (1.4). The *cohomological Conley index* of an isolated invariant set is defined as

$$CH(S) := LH^*(P),$$

where P is any index pair for S in X.

Due to [M2],Thm. 2.6 the above definition makes sense. Similarly we can define the homological index, which we denote by $HI(S)$.

2. Attractor-repeller pairs. For a given set $A \subset X$ the sets

$$\Omega^+(A) = \bigcap\{cl\ A[n, \infty)\ ;\ n \in N\}, \qquad \Omega^-(A) = \bigcap\{cl\ A(-\infty, -n]\ ;\ n \in N\}$$

are called the *positive and negative limit sets of* A.

Definition (2.1). Let Y be a compact, positively (negatively) invariant subset of X. The set $A \subset Y$ will be called an *attractor (repeller)* relative to Y if there exists a neighbourhood U of A in Y such that $\Omega^+(U) = A$ (respectively $\Omega^-(U) = A$).

For given subsets A, B of X we define the *connecting orbit set*:

$$C(A, B; X) := \{x \in X\ ;\ \Omega^-(x) \subseteq A \text{ and } \Omega^+(A) \subseteq B\}.$$

Let S be a compact isolated invariant set. If A is an attractor in S, then the set $A^* := \{x \in S\ ;\ \Omega^+(x) \cap A = \emptyset\}$ is a repeller in S (see [S], Prop. 3.4). It is called the repeller complementary to A in S. We call such a pair (A, A^*) an *attractor-repeller pair in* S.

Theorem (2.2) (see [BD], Thm. 3.3). *Let S be an isolated invariant subset of X and N an isolating neighbourhood of S. Assume that (A, A^*) is an attractor-repeller pair in S. Then there exists a filtration $P_2 \subset P_1 \subset P_0$ of compact subsets of N such that:*

(i) (P_0, P_2) *is an index pair for* S,
(ii) (P_0, P_1) *is an index pair for* A^*,
(iii) (P_1, P_2) *is an index pair for* A.

If (A, A^*) is an attractor-repeller pair in an isolated invariant set S such that $CH(S)$, $CH(A^*)$, $CH(A)$ are graded vector spaces with finite dimensional components (this assumption is satisfied e.g. when X is a compact ANR), then we can construct a long exact sequence relating the cohomology indices of S, A^* and A (see [M3]). Namely, there is a long exact sequence

$$0 \to H^0(P_0, P_1) \xrightarrow{i} H^0(P_0, P_2) \xrightarrow{j} H^0(P_1, P_2) \xrightarrow{\partial} H^1(P_0, P_1) \to \dots,$$

where i, j are induced by inclusions and (P_0, P_1, P_2) is a filtration given by Thm. 2.2. Applying the Leray functor we obtain an exact sequence of cohomology Conley indices

$$0 \to CH_0(A^*) \to CH_0(S) \to CH_0(A) \xrightarrow{\partial} CH_1(A) \to \dots$$

This sequence, called the *cohomology index sequence of the attractor-repeller pair*, provides an algebraic condition for the existence of connecting orbits. The map ∂ is called a *connection map*.

THEOREM (2.3). *If the connection map ∂ is nontrivial then $C(A^*, A; S)$ is nonempty.*

PROOF. To obtain a contradiction, suppose that $C(A^*, A; S)$ is empty. If $S = A \cup A^*$, there exist index pairs (P_0, P_1) and (P_0^*, P_1^*) for A and A^*, respectively, such that $P_0 \cap P_0^* = \emptyset$. In this case $(P_0 \cup P_0^*, P_1 \cup P_1^*)$ is an index pair for S. Therefore we have a pair of excisive triads: $(P_1 \cup P_1^*, P_1, P_1^*) \subset (P_0 \cup P_0^*, P_0, P_0^*)$ because P_i, P_i^* are open in $P_i \cup P_i^*$ for $i = 0, 1$. Thus we obtain the following exact cohomology sequence (see e.g. [D], Thm. 8.22), called the Mayer–Vietoris sequence:

$$\begin{gathered}\dots \leftarrow H^{n+1}(P_0 \cup P_0^*, P_1 \cup P_1) \xleftarrow{d^*} H^n(P_0 \cap P_0^*, P_1 \cap P_1) = H^n(\emptyset, \emptyset) \leftarrow \\ \xleftarrow{k^*} H^n(P_0, P_1) \oplus H^n(P_0^*, P_1^*) \xleftarrow{l^*} H^n(P_0 \cup P_0^*, P_1 \cup P_1^*) \leftarrow \\ \xleftarrow{d^*} H^{n-1}(P_0 \cap P_0^*, P_1 \cap P_1^*) = H^{n-1}(\emptyset, \emptyset) \leftarrow \dots\end{gathered}$$

By the exactness of this sequence it follows that

$$Ker\ l^* = Im\ d^* = 0, \qquad Im\ l^* = Ker\ k^* = H^*(P_0, P_1) \oplus H^*(P_0^*, P_1^*)$$

and, in consequence, $CH(S) \cong CH(A) \oplus CH(A^*)$, which gives $\partial = 0$, a contradiction. ∎

Since we need the Leray functor to maintain exactness of homological sequences, from now on we assume that X is a compact ANR, which is sufficient according to [M3].

3. Morse decompositions. Let $(P, <)$ be a finite partially ordered set. A subset $I \subset P$ is an *interval* if $\pi, \pi' \in I$ and $\pi < \pi'' < \pi'$ imply $\pi'' \in I$. The set of intervals is denoted by $I(<)$. An interval $I \subset P$ is an *attracting interval* if $\pi \in I$ and $\pi' < \pi$ imply $\pi' \in I$. The set of attracting intervals is denoted by $A(<)$.

An ordered collection $(I_1, \dots, I_n)$ of mutually disjoint intervals is called an *adjacent n-tuple of intervals* if:

(1) $\bigcup_{i=1}^n I_i \in I(<)$
(2) $\pi \in I_j$, $\pi' \in I_k$ and $j < k$ imply $\neg(\pi' < \pi)$.

The collection of adjacent n-tuples of intervals is denoted by $I_n(<)$.

We say that I and J are *noncomparable* if $(I, J) \in I_2(<)$ and $(J, I) \in I_2(<)$. Two elements $\pi, \pi' \in P$ are *adjacent* if $\{\pi, \pi'\} \in I(<)$.

DEFINITION (3.1). Let S be a compact, invariant subset of X. A *Morse decomposition of S* is a collection $M = \{M(\pi)\}_{\pi \in P}$ of mutually disjoint, compact, invariant subsets of S such that

$(*)$ if $\gamma \in S \setminus \bigcup_{\pi \in P} M(\pi)$, then there exists $\pi < \pi'$ with $\gamma \in C(M(\pi'), M(\pi); S)$.

Definition (3.2). For each $I \in I(<)$ we define the *Morse set* by

$$M(I) := \bigcup_{\pi \in I} M(\pi) \cup \bigcup_{\pi,\pi' \in I} C(M(\pi'), M(\pi); S).$$

Notice that if S is an isolated invariant set, then each Morse set is also an isolated invariant set.

Remark (3.3). An ordering on M satisfying condition $(*)$ is called *an admissible ordering*. The homeomorphism (discrete dynamical system) defines an extremal admissible ordering of M, denoted by $<_H$ and such that $\pi <_H \pi'$ iff there exists a sequence $\pi = \pi_0, \pi_1, \ldots, \pi_k = \pi'$ of distinct elements of P with $C(M(\pi_j), M(\pi_{j-1}); S) \neq \emptyset$ for each $j = 1, \ldots k$. Every admissible ordering of M is an extension of $<_H$.

Remark (3.4). If $(I, J) \in I_2(<)$ then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, where $IJ := I \cup J$.

Definition (3.5). An *index filtration* for the admissible ordering of M is a collection of compact sets $\mathcal{N} = \{N(I)\}_{I \in A(<)}$ such that

(i) for each $I \in A(<)$, $(N(I), N(\emptyset))$ is an index pair for the attractor $M(I)$,
(ii) for each $I_1, I_2 \in A(<)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

Remark (3.6). Let $J \in I(<)$ and assume that (I, J) is a decomposition of $K \in A(<)$. Then $I \in A(<)$ and $(N(K), N(I))$ is an index pair for the Morse set $M(J)$. Thus the index filtration defines an index pair for each Morse set.

For the Morse set $M(J)$ we will denote by $C(J)$ the singular chain complex of the quotient space $N(K)/N(I)$. Passing to homology we obtain $H(J)$, the singular homology of this space and applying the Leray functor we get $HI(J)$, the homological Conley index of the Morse set $M(J)$.

Let us formulate a natural

Proposition (3.7). *For any given admissible ordering of M there exists an index filtration.*

In [BD] we prove the existence of index filtrations for admissible orderings that are total orders. In the partially ordered case the construction can be made by adaptation of the proof of [F1], Thm. 3.8. It is quite clear, at least in the case when our dynamical system is a discretization of a continuous dynamical system.

Theorem (3.8) (see [F1], Section 4). *Given $\mathcal{N}$, an index filtration for the admissible ordering of M, there is a collection of chain complexes and chain maps satisfying:*

(i) *for each $I \in I(<)$ there is a chain complex C(I),*
(ii) *for each $(I, J) \in I_2(<)$ there are chain maps: $C(I) \xrightarrow{i} C(IJ) \xrightarrow{p} C(J)$ such that*
  (a) *i is injective and $p \circ i = 0$,*
  (b) *the chain map $\tilde{p} : C(IJ)/Im\ i \to C(J)$ defined by p induces an isomorphism in homology,*
  (c) *if I and J are noncomparable then $p \circ i = id\,|_{C(I)}$,*
  (d) *if $(I, J, K) \in I_3(<)$ then the following braid diagram commutes:*

$$\begin{tikzcd}
C(I) \arrow[dd, "i"] \arrow[dr, "i"] & & \\
& C(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
C(IJK) \arrow[dd, "p"] \arrow[dr, "p"] & & C(J) \\
& C(JK) \arrow[ur, "i"] \arrow[dl, "p"'] & \\
C(K) & &
\end{tikzcd}$$

This diagram of chain complexes and chain maps is called the *chain complex braid* of the index filtration and is denoted by $\mathcal{C}(\mathcal{N})$.

Passing to homology and applying the Leray functor we obtain the *homology index braid* of the admisssible ordering of the Morse decomposition, denoted by $\mathcal{H}I(<)$. It consists of graded modules $HI(I)$ for each $I \in I(<)$ and maps between graded modules:

$$i : HI(I) \to HI(J), \quad p : HI(IJ) \to HI(J), \quad \partial : HI(J) \to HI(I)$$

satisfying:

(1) $\ldots \to HI(I) \overset{i}{\to} HI(IJ) \overset{p}{\to} HI(J) \overset{\partial}{\to} HI(I) \to \ldots$ is exact,
(2) if I and J are noncomparable, then $p \circ i = id\,|_{HI(I)}$,
(3) if $(I, J, K) \in I_3(<)$ then the following braid diagram commutes:

$$\begin{tikzcd}
& \cdots \arrow[dl] \arrow[dr] & \\
HI(I) \arrow[dr, "i"] \arrow[dd, "i"] & & HI(K) \arrow[dl, "\partial"'] \arrow[dd, "\partial"] \\
& HI(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
HI(IJK) \arrow[dr, "p"] \arrow[dd, "p"] & & HI(J) \arrow[dl, "i"'] \arrow[dd, "\partial"] \\
& HI(JK) \arrow[dl, "p"'] \arrow[dr, "\partial"] & \\
HI(K) \arrow[dr, "\partial"] \arrow[dd, "\partial"] & & HI(I) \arrow[dl, "i"'] \arrow[dd, "i"] \\
& HI(IJ) \arrow[dl, "p"'] \arrow[dr, "i"] & \\
HI(J) \arrow[dr] & & HI(IJK) \arrow[dl] \\
& \cdots &
\end{tikzcd}$$

We denote $\mathcal{H}I(<_H)$ by $\mathcal{H}I(M)$.

4. The connection matrix. A sequence of chain maps $C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$ is *weakly exact* if

(1) i is injective and $p \circ i = 0$,
(2) $\tilde{p} : C_2/Im\ i \to C_3$ induces an isomorphism in homology.

THEOREM (4.1) (see [F2], Prop. 2.2). *If $C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$ is a weakly exact sequence of chain complexes, then there exists a connection homomorphism $\partial :\ H(C_1) \to H(C_3)$ such that the sequence*

$$\ldots \to H(C_1) \ \xrightarrow{i}\ H(C_2) \ \xrightarrow{p}\ H(C_3) \ \xrightarrow{\partial}\ H(C_1) \to \ldots$$

is exact.

DEFINITION (4.2). A *graded module braid over* $<$ is a collection $\mathcal{G}$ consisting of graded modules and maps between the graded modules satisfying:

(1) for each $I \in I(<)$ there is a graded module $G(I)$
(2) for each $(I, J) \in I_2(<)$ there are maps

$$i : G(I) \to G(IJ) \ \text{ of degree } 0,$$
$$p : G(IJ) \to G(J) \ \text{ of degree } 0,$$
$$\partial : G(J) \to G(I) \ \text{ of degree } -1,$$

which satisfy:

(a) $\ldots \to G(I) \ \xrightarrow{i}\ G(IJ) \ \xrightarrow{p}\ G(J) \ \xrightarrow{\partial}\ G(I) \to \ldots$ is exact,
(b) if I and J are noncomparable, then $p \circ i = id\,|_{G(I)}$,
(c) if $(I, J, K) \in I_3(<)$ then the following braid diagram commutes

$$\begin{tikzcd}
 & \ldots \arrow[dl] \arrow[dr] & \\
G(I) \arrow[dr, "i"] \arrow[dd, "i"'] & & G(K) \arrow[dl, "\partial"'] \arrow[dd, "\partial"] \\
 & G(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
G(IJK) \arrow[dr, "p"] \arrow[dd, "p"'] & & G(J) \arrow[dl, "i"'] \arrow[dd, "\partial"] \\
 & G(JK) \arrow[dl, "p"'] \arrow[dr, "\partial"] & \\
G(K) \arrow[dr, "\partial"] \arrow[dd, "\partial"'] & & G(I) \arrow[dl, "i"'] \arrow[dd, "i"] \\
 & G(IJ) \arrow[dl, "p"'] \arrow[dr, "i"] & \\
G(J) \arrow[dr] & & G(IJK) \arrow[dl] \\
 & \ldots &
\end{tikzcd}$$

Theorem (4.3) (see [F1], Sect. 4). *$\mathcal{HI}(<)$, the homology index braid of the admissible ordering of M is a graded module braid.*

Remark (4.4). A map $\theta : \mathcal{G} \to \mathcal{G}'$ between graded modules braids $\mathcal{G}$ and $\mathcal{G}'$ over $<$ is a collection of module homomorphisms $\theta(I) : G(I) \to G'(I)$, $I \in I(<)$, such that for each $(I, J) \in I_2(<)$ the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & G(I) \arrow[r, "i"] \arrow[d, "\theta(I)"] & G(IJ) \arrow[r, "p"] \arrow[d, "\theta(IJ)"] & G(J) \arrow[r, "\partial"] \arrow[d, "\theta(J)"] & G(I) \arrow[r] \arrow[d, "\theta(I)"] & \dots \\
\dots \arrow[r] & G'(I) \arrow[r] & G'(IJ) \arrow[r] & G'(J) \arrow[r] & G'(I) \arrow[r] & \dots
\end{tikzcd}$$

We say that $\mathcal{G}$ and $\mathcal{G}'$ are *isomorphic* if $\theta(I)$ is an isomorphism for each $I \in I(<)$.

Definition (4.5). *A chain complex braid over* $<$ is a collection $\mathcal{C}$ consisting of chain complexes and chain maps satisfying:

(1) for each $I \in I(<)$ there is a chain complex $C(I)$,
(2) for each $(I, J) \in I_2(<)$ there are maps

$$i : C(I) \to C(IJ), \qquad p : C(IJ) \to C(J),$$

which satisfy:

(a) $C(I) \xrightarrow{i} C(IJ) \xrightarrow{p} C(J)$ is weakly exact,
(b) if I and J are noncomparable, then $p \circ i = id\,|_{C(I)}$,
(c) if $(I, J, K) \in I_3(<)$ then the following braid diagram commutes:

$$\begin{tikzcd}
C(I) \arrow[dr, "i"] \arrow[dd, "i"'] & & \\
& C(IJ) \arrow[dl, "i"'] \arrow[dr, "p"] & \\
C(IJK) \arrow[dr, "p"] \arrow[dd, "p"'] & & C(J) \\
& C(JK) \arrow[ur, "i"'] \arrow[dl, "p"] & \\
C(K) & &
\end{tikzcd}$$

Theorem (4.6) (see [F1], Sect. 4). *$\mathcal{C}(\mathcal{N})$, the chain complex braid of the index filtration, is a chain complex braid.*

Remark (4.7). Upon passing to homology, a chain complex braid defines a graded module braid. Namely, assume $\mathcal{C}$ is a chain complex braid over $<$. For each $I \in I(<)$ let $H(I)$ be the homology of the chain complex $C(I)$. If $(I, J) \in I_2(<)$ then we have a weakly exact sequence

$$C(I) \xrightarrow{i} C(IJ) \xrightarrow{p} C(J).$$

From Theorem 4.1 we obtain an exact homology sequence

$$\ldots \to H(I) \overset{i}{\to} H(IJ) \overset{p}{\to} H(J) \overset{\partial}{\to} H(I) \to \ldots .$$

Set $\mathcal{HC}$ equal to the collection consisting of the graded modules $H(I)$ for each $I \in I(<)$, along with the maps $i,\ p,\ \partial$ from the above sequence for each $(I,J) \in I_2(<)$.

Theorem (4.8) (see [F2], Prop. 2.7). *$\mathcal{HC}$ is a graded module braid.*

Definition (4.9). If $\mathcal{C}$ is a chain complex braid then we call $\mathcal{HC}$ the *graded module braid generated by* $\mathcal{C}$. Furthermore, if $\mathcal{G}$ is any graded module braid isomorphic to $\mathcal{HC}$ then we say that $\mathcal{G}$ is *chain complex generated*.

Let $C = \{C\triangle(\pi)\}_{\pi\in P}$ be a collection of graded modules. Let $I \subset P$. Then a map $\Delta : \bigoplus_{\pi\in I} C\triangle(\pi) \to \bigoplus_{\pi\in I} C\triangle(\pi)$ can be regarded as a matrix

$$\Delta = \begin{pmatrix} & \vdots & \\ \ldots & \triangle_{\pi,\pi'} & \ldots \\ & \vdots & \end{pmatrix}_{\pi,\pi'\in I} : \begin{pmatrix} \vdots \\ C\triangle(\pi) \\ \vdots \end{pmatrix}_{\pi\in I} \to \begin{pmatrix} \vdots \\ C\triangle(\pi) \\ \vdots \end{pmatrix}_{\pi\in I}$$

where each $\triangle_{\pi,\pi'}$ is a map from $C\triangle(\pi')$ to $C\triangle(\pi)$.

Definition (4.10).

(a) $\triangle$ is *upper triangular* if $\triangle_{\pi,\pi'} \neq 0$ implies $\pi \leq \pi'$,
(b) $\triangle$ is *strictly upper triangular* if $\triangle_{\pi,\pi'} \neq 0$ implies $\pi < \pi'$,
(c) $\triangle$ is *a boundary map* if each $\triangle_{\pi,\pi'}$ is of degree -1 and $\triangle^2 = 0$.

Now we assume that $\Delta : \bigoplus_{\pi\in P} C\triangle(\pi) \to \bigoplus_{\pi\in P} C\triangle(\pi)$ is an upper triangular boundary map. For each $I \in I(<)$ set $C\triangle(I) = \bigoplus_{\pi\in I} C\triangle(\pi)$, and for each $I,J \in I(<)$ let $\triangle(I,J) : C\triangle(J) \to C\triangle(I)$ be the map defined by the matrix

$$\begin{pmatrix} & \vdots & \\ \ldots & \triangle_{\pi,\pi'} & \ldots \\ & \vdots & \end{pmatrix}_{\pi\in I,\ \pi'\in J}$$

If $I \in I(<)$ then we denote $\triangle(I,I)$ by $\triangle(I)$.

Theorem (4.11) (see [F2], Prop. 3.2 and 3.3).

(a) *$\triangle(I)$ is an upper triangular boundary map for each $I \in I(<)$.*
(b) *$C\triangle(I)$ is a chain complex with boundary map $\triangle(I)$.*
(c) *If $(I,J) \in I_2(<)$ then the inclusion map $i : C\triangle(I) \to C\triangle(IJ)$ and the projection map $p : C\triangle(IJ) \to C\triangle(J)$ are chain maps.*

Theorem (4.12) (see [F2], Prop. 3.4). *Given an upper triangular boundary map $\triangle : \bigoplus_{\pi\in P} C\triangle(\pi) \to \bigoplus_{\pi\in P} C\triangle(\pi)$ the collection, denoted $\mathcal{C}\triangle$, consisting of the chain complexes $C\triangle(I)$ with boundary map $\triangle(I)$ for each $I \in I(<)$ and the chain maps i, p for each $(I,J) \in I_2(<)$, is a chain complex braid over $<$.*

Let $\mathcal{H}\triangle$ be a graded module braid generated by $\mathcal{C}\triangle$, i.e. $\mathcal{H}\triangle = \mathcal{HC}\triangle$. For simplicity from now on we assume that all modules are over a field.

DEFINITION (4.13). Given $\mathcal{G}$, a graded module braid over $<$, let

$$\triangle : \bigoplus_{\pi\in P} G(\pi) \to \bigoplus_{\pi\in P} G(\pi)$$

be an upper triangular map. If $\mathcal{H}\triangle$ is isomorphic to $\mathcal{G}$ then $\triangle$ is called a *connection matrix of* $\mathcal{G}$.

We denote the collection of connection matrices of $\mathcal{G}$ by $\mathcal{CM}(\mathcal{G})$.

THEOREM (4.14) (see [F2], Th. 3.8). *If $\mathcal{G}$ is a chain complex generated graded module braid then $\mathcal{CM}(\mathcal{G}) \neq \emptyset$.*

5. The connection matrices for Morse decompositions. Let $M = \{M(\pi)\}_{\pi\in P}$ be a Morse decomposition of the isolated invariant set S with admissible ordering $<$. Then $\mathcal{H}I(<)$ is the homology index braid of $<$. The existence of connection matrices was shown by Franzosa in the case of continuous dynamical systems (see [F1]). Now the same conclusion can be drawn for discrete dynamical systems.

THEOREM (5.1). *The set $\mathcal{CM}(\mathcal{H}I(<))$ is nonempty.*

PROOF. Let us start with the observation that if X is a compact ANR than the homology functor and the Leray functor commute (see [M3]). Now the theorem is an easy consequence of Proposition 3.7 and Theorem 4.14. ∎

We can now state the analogue of Theorem 2.2.

THEOREM (5.2). *If $\triangle \in \mathcal{CM}(\mathcal{H}I(M))$, π and π' are adjacent in $<$ and the entry $\triangle_{\pi,\pi'} \neq 0$ then $C(M(\pi'), M(\pi); S) \neq \emptyset$.*

PROOF. If π and π' are adjacent in $<$ then they are adjacent in $<_H$. $\triangle_{\pi,\pi'} \neq 0$ implies $\pi <_H \pi'$. Therefore by the definition of the homeomorphism ordering there is a sequence $\pi = \pi_0, \ldots, \pi_n = \pi'$ of distinct elements of P with $C(M(\pi_j), M(\pi_{j-1}); S) \neq \emptyset$ for each $j = 1, \ldots, n$. Since π and π' are adjacent in $<_H$, we see that $n = 1$. This gives us $C(M(\pi'), M(\pi); S) \neq \emptyset$. ∎

EXAMPLE (5.3). This example is adapted from [M3]. Let $D \subset R^2$ be a square and $f_0 : D \to D$ be a continuous map as indicated in Fig. 1. Extend f_0 to a homeomorphism $f : S^2 \to S^2$ with a repelling point r outside of D.

Take $M_1 := Inv(D_7 \cup D_8)$, $M_2 := Inv(D_1 \cup D_2)$, $M_3 := \{r\}$. It is easy to check that $M = \{M_1, M_2, M_3\}$ is a Morse decomposition of $S = S^2$ with the homeomorphism ordering $(1 < 2 < 3)$ and $\mathcal{N} = \{N_i\}_{i=0}^3$ with $N_0 = \emptyset, N_1 = D_7 \cup D_8 \cup P$ (P is the sum of the striped areas), $N_2 = D_1 \cup D_2 \cup D_7 \cup D_8, N_3 = S^2$ is an index filtration for M. Moreover, a simple verification shows that

$$HI_k(M_1) = \begin{cases} Q^2 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$HI_k(M_2) = \begin{cases} Q & \text{for } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$HI_k(M_3) = \begin{cases} Q & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

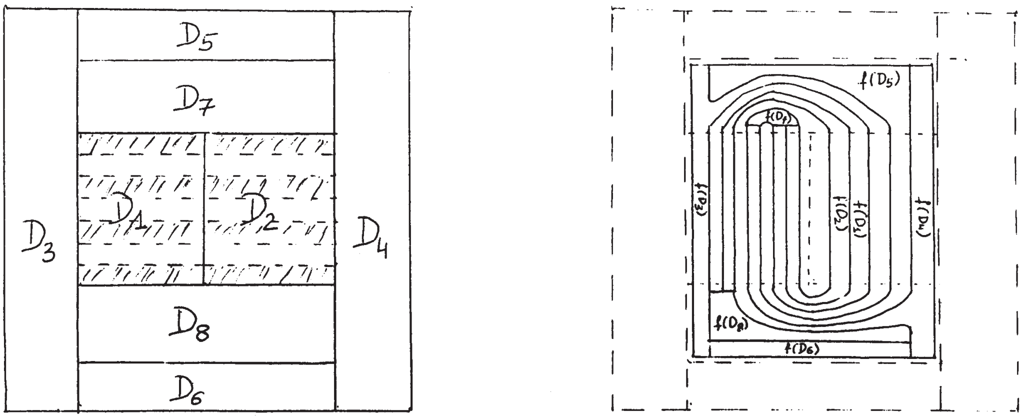

Fig. 1

Let us compute the connection matrix of the above Morse decomposition. Because we have chosen field coefficients, the connection matrix is strictly upper triangular (see Def. 4.13). $HI_1(M_{21})$ is easily seen to be trivial; therefore so must be $H\Delta_1(12)$. Since the homology indices $HI_1(M_1)$ and $HI_0(M_1)$ are nontrivial, it follows that $\Delta_{12} \neq 0$. It is not difficult to see that it is the only nonzero entry of Δ. Thus $\mathcal{CM}(M;Q)$ consists of one matrix and it is of the form

$$\begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ indicates the only nonzero entry. Then since M_2 and M_1 are adjacent in the homeomorphism ordering it follows that $C(M_2, M_1)$ is nonempty.

Remark (5.4). Similar results have been obtained independently by David Richeson. Although his approach gives more detailed conditions for the existence of the connecting orbits, we think that in several cases it is sufficient to use our methods.

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