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SYMBOLIC DYNAMICS FOR THE RÖSSLER FOLDED TOWEL MAP

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1. Main result. Let us define

$$f_\mu(x) := \mu x(1-x) \tag{1}$$

We consider the following *folded towel map* introduced by Rössler in [R]: $R(x,y,z) = (\bar{x},\bar{y},\bar{z})(x,y,z)$,

$$\bar{x}(x,y,z) = f_{3.8}(x) - a0.05(y+0.35)(1-2z) \tag{2}$$
$$\bar{y}(x,y,z) = a0.1[(y+0.35)(1-2z)-1](1-1.9x) \tag{3}$$
$$\bar{z}(x,y,z) = f_{3.78}(z) + a0.2y \tag{4}$$

where $a \in [-1,1]$. The case $a=1$ was considered by Rössler in [R].

Before we state the main result of this note we define the notion of *symbolic dynamics*.

Consider a continuous map $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose now that we have a family of compact mutually disjoint sets N_j for $j = 0,1,\ldots,l-1$.

We set $N = \bigcup_{j=0}^{l-1} N_j$. An invariant part of the set N is defined by

$$\mathrm{Inv}(N,F) := \bigcap_{i\in\mathbb{Z}} F^{-i}_{|N}(N) \tag{5}$$

Let $\Sigma_l := \{0,1,\ldots,l-1\}^{\mathbb{Z}}$, $\Sigma_l^+ := \{0,1,\ldots,l-1\}^{\mathbb{N}}$. Σ_l, Σ_l^+ are topological spaces with the Tikhonov topology. On Σ_l, Σ_l^+ we have the shift map σ given by

$$(\sigma(c))_i = c_{i+1}$$

For $i \in \mathbb{N}$ we define a map $\pi_i : \mathrm{Inv}(N,F) \to \{0,1,\ldots,l-1\}$ given by $\pi_i(x) = j$ iff $F^i(x) \in N_j$. Now we define a map $\pi : \mathrm{Inv}(N,F) \to \Sigma_l^+$ by $\pi(x) := (\pi_i(x))_{i\in\mathbb{N}}$. Such a

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map π is obviously continuous. The map π assigns to the point x the indices of the N_i-s its F-trajectory goes through. It is easy to see that

$$\pi \circ F = \sigma \circ \pi. \tag{6}$$

If F is also a homeomorphism, then the definition of π_i can be extended to all integers and the domain of π is Σ_l.

Definition 1. Let F and N_j be as above. We will say that F has *symbolic dynamics on l symbols* iff π is onto and the preimage of any periodic sequence from Σ_l^+ contains periodic points of F.

The main result of this note is the following

Theorem 1. *If $|a| \leq 1$ then R^2 has a symbolic dynamics on two symbols. If $|a| < 0.4$ then R^4 has a symbolic dynamics on four symbols.*

The proof of this theorem is based on the topological theorem from [Z1], which is presented in the next section.

2. Topological theorem. First we introduce some notations. Let $p \in \mathbb{R}^n$. By $x_i(p)$ we will denote the i-th coordinate of the point p. We will use the max norm on $\mathbb{R}^n$, so

$$|(x_1, \ldots, x_n)| := \max_i |x_i| \tag{7}$$

Let $Z \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$. Then we use the following notations $\operatorname{dist}(x, Z) = \inf\{|x-y| | y \in Z\}$, $B(x, \epsilon) = \{y | |x-y| < \epsilon\}$, $B(Z, \epsilon) = \{x | \operatorname{dist}(x, Z) < \epsilon\}$, $\operatorname{diam} Z = \sup_{x,y \in Z} |x-y|$.

By $\mathcal{C}$ we will denote a parallelogram in $\mathbb{R}^n$, so

$$\mathcal{C} := \{X \subset \mathbb{R}^n \mid X = \prod_{i=1}^{n} [x_{ai}, x_{bi}]\} \tag{8}$$

Definition 2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $X = [x_a, x_b]$ and $Y = [y_a, y_b]$. We say that X *f-covers Y* (*with a margin δ*) iff there exists $\delta > 0$ such that $[y_a - \delta, y_b + \delta]$ is contained either in $[f(x_a), f(x_b)]$ or in $[f(x_b), f(x_a)]$.

Definition 3. Let $X = \prod_{i=1}^{n} [x_{ai}, x_{bi}]$. For $i \in \{1, \ldots, n\}$ we define the i-th upper and lower edge of X respectively by

$$U_i(X) = \{p \in X \mid x_i(p) = x_{bi}\} \tag{9}$$

$$D_i(X) = \{p \in X \mid x_i(p) = x_{ai}\} \tag{10}$$

Definition 4. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $X = \prod_{i=1}^{n} [x_{ai}, x_{bi}]$ and $Y = \prod_{i=1}^{n} [y_{ai}, y_{bi}]$. Let $1 \leq i \leq n$. We say that X *F-covers Y in i direction* (*with a margin δ*) iff there exists $\delta > 0$ such that one of the two following conditions hold

$$[y_{ai} - \delta, y_{bi} + \delta] \subset [\max x_i(F(D_i(X))), \min x_i(F(U_i(X)))] \tag{11}$$

$$[y_{ai} - \delta, y_{bi} + \delta] \subset [\max x_i(F(U_i(X))), \min x_i(F(D_i(X)))] \tag{12}$$

Definition 5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $\delta > 0$, $X = \prod_{i=1}^{n} [x_{ai}, x_{bi}]$ and $Y = \prod_{i=1}^{n} [y_{ai}, y_{bi}]$. Let $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ be a sequence of integers. We say that *X F-covers Y in $(i_1, i_2, \ldots, i_k)$-direction* (*with a margin δ*) if the following conditions hold:

- for every $l = 1, \dots, k$ X F-covers Y in i_l direction with margin δ,
- for every j not in the sequence $i_1, i_2, \dots, i_k$ we have

$$x_j(F(X)) \subset [y_{aj} + \delta, y_{bj} - \delta] \tag{13}$$

To illustrate the notions introduced above let us consider the following example. Let $n = 3$ and $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ and the segments X_i, Y_i for $i = 1, 2$ be such that X_i f_i-covers Y_i with margin $\delta < 1$ and $f_3(x) = 0$ for $x \in \mathbb{R}$. We set $X_3 = Y_3 = [-1, 1]$, $X = X_1 \times X_2 \times X_3$, $Y = Y_1 \times Y_2 \times Y_3$. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by $F(x_1, x_2, x_3) = (f_1(x_1), f_2(x_2), f_3(x_3))$. It is easy to see that the set X F-covers Y in $(1, 2)$-direction with margin δ. Consider now a perturbation $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ of F such that $|\tilde{F} - F|_{|X} < \delta$. Then it is easy to see that X $\tilde{F}$-covers Y in $(1, 2)$-direction.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Let us fix a sequence $1 \le i_1 < i_2 < \dots < i_k \le n$. Let $\{e_i\}$ be the canonical basis in $\mathbb{R}^n$. Then we will call the linear subspace spanned by $\{e_{i_1}, \dots, e_{i_k}\}$ a *topologically expanding direction* (with respect to F). The reason for this name will be clear from theorem 2.

Definition 6. Let $X, Y \in \mathcal{C}$. We will say that there exists an *F-transition of length m* from X to Y iff there exists a sequence of sets $\{N_j\}_{j=0,1,\dots,l} \subset \mathcal{C}$ and a sequence of integers $\{m_j\}_{j=0,\dots,l-1}$, such that

$$N_0 \subset X, \quad N_l = Y$$
$$N_j \ F^{m_j}\text{-covers } \ N_{j+1} \text{ in } (i_1, \dots, i_k)\text{-direction, for } j = 0, \dots, l-1$$
$$m_0 + m_1 + \dots m_{l-1} = m$$

We will use the graphical notation $X \overset{F^m}{\Longrightarrow} Y$.

Suppose now that we have a family of sets $N_j \in \mathcal{C}$ for $j = 0, 1, \dots, l-1$ and an integer m such that

$$N_j \cap N_k = \emptyset, \quad \text{for } j, k = 0, \dots, l-1, \ j \ne k \tag{14}$$
$$N_j \overset{F^m}{\Longrightarrow} N_k, \quad \text{for } j, k = 0, \dots, l-1 \tag{15}$$

We set $N = \bigcup_{j=0}^{l-1} N_j$. The following theorem is proved in [Z1]

Theorem 2. *Let the family of sets $\{N_j\}_{j=0,\dots,l-1} \subset \mathcal{C}$ satisfy (14)–(15). Then $\Sigma_l^+ = \pi(Inv(N, F^m))$. The preimage of any periodic sequence from Σ_l^+ contains periodic points of F^m. If we additionally suppose that F is a homeomorphism, then $\Sigma_l = \pi(Inv(N, F^m))$.*

3. Proof for $|a| \le 1$. Our aim is to apply theorem 2 to R to obtain theorem 1. As topologically expanding directions we set e_1, e_3.

Let us set $\epsilon = 0.022$, $\epsilon_1 = 0.01$, $\epsilon_2 = 0.02$ and define

$$x_{max} := \max_{x \in [0,1]} f_{3.8}(x) - \epsilon_1 = 3.8/4 - \epsilon_1 = 0.94 \tag{16}$$
$$x_{min} := f_{3.8}(x_{max}) = 0.21432 \tag{17}$$
$$y_{max} := 0.1095 \tag{18}$$
$$z_{max} := \max_{z \in [0,1]} f_{3.78}(z) - \epsilon = 3.78/4 - \epsilon = 0.923 \tag{19}$$
$$z_{min} := f_{3.8}(z_{max}) = 0.26864838 \tag{20}$$

Let $D := [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times [z_{min}, z_{max}]$.

We show that

$$|\bar{y}| < y_{max}, \quad \text{for } (x,y,z) \in D \tag{21}$$

We have

$$|\bar{y}| \le 0.1|[(y+0.35)(1-2z)-1](1-1.9x)| \le$$
$$0.1|(y_{max}+0.35)(1-2z_{max})-1||1-1.9x_{max}| <$$
$$0.1(0.46 \cdot 0.846 + 1)0.786 = 0.1 \cdot 1.38916 \cdot 0.786 < 0.1092$$

We show now

$$|\bar{x}(x,y,z) - f_{3.8}(x)| < \epsilon_2, \quad \text{for } (x,y,z) \in D \tag{22}$$
$$|\bar{z}(x,y,z) - f_{3.78}(z)| < \epsilon, \quad \text{for } (x,y,z) \in D \tag{23}$$

(23) follows immediately from (4). To get (22) we compute

$$|\bar{x}(x,y,z) - f_{3.8}(x)| \le |0.05(y+0.35)(1-2z)| \le$$
$$0.05(y_{max}+0.35)|1-2z_{max}| < 0.05 \cdot 0.46 \cdot 0.846 < 0.0195$$

We set

$$Z_0 := [0.295, 0.5], \quad Z_2 := [0.809, 0.922]$$

It is easy to check that

$$Z_0 \ f_{3.78}\text{-covers } Z_2 \text{ with margin } \epsilon \tag{24}$$
$$Z_2 \ f_{3.78}\text{-covers } Z_0 \text{ with margin } \epsilon \tag{25}$$

For $(x,y,z) \in [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times Z_0$ we have

$$|\bar{x}(x,y,z) - f_{3.8}(x)| \le |0.05(y+0.35)(1-2z)| \le$$
$$0.05(y_{max}+0.35)|1-2 \cdot 0.295| < 0.05 \cdot 0.46 \cdot 0.41 < 0.0095$$

Hence

$$|\bar{x}(x,y,z) - f_{3.8}(x)| < \epsilon_1, \quad \text{for } (x,y,z) \in [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times Z_0 \tag{26}$$

We set

$$X_0 = [0.2347, 0.5], \qquad X_1 = [0.5, 0.7653] \tag{27}$$
$$X_2 = [0.6927, 0.94]. \tag{28}$$

Observe that X_1 is the image of X_0 under the reflection $x \to 1-x$.

It is easy to check that

$$X_0,\ X_1 \text{ both } f_{3.8}\text{-cover } X_2 \text{ with a margin } \epsilon_1 \tag{29}$$
$$X_2 \ f_{3.8}\text{-covers } X_0 \cup X_1 \text{ with a margin } \epsilon_2 \tag{30}$$

We set

$$N_{00} = X_0 \times [-y_{max}, y_{max}] \times Z_0 \tag{31}$$
$$N_{10} = X_1 \times [-y_{max}, y_{max}] \times Z_0 \tag{32}$$
$$N_2 = X_2 \times [-y_{max}, y_{max}] \times Z_2 \tag{33}$$

From (21)–(26), (29) and (29) it follows that N_{00} and N_{10} R-cover N_2 in $(1,3)$-direction, and N_2 R-covers in $(1,3)$-direction both N_{00} and N_{10}.

We want to apply theorem 2 to R, $m = 2$ and the sets N_{00}, N_{10}, but $N_{00} \cap N_{10} \neq \emptyset$. We overcome this problem by observing that there exist sets $\tilde{N}_{00} \subset \operatorname{int} N_{00}$ and $\tilde{N}_{10} \subset \operatorname{int} N_{10}$ such that $\tilde{N}_{00}$ and $\tilde{N}_{10}$ R-cover N_2 in $(1,3)$-direction. We have

$$\tilde{N}_{00} \cap \tilde{N}_{10} = \emptyset \tag{34}$$

$$\tilde{N}_{00} \overset{R^2}{\Longrightarrow} \tilde{N}_{00}, \tilde{N}_{10} \quad \tilde{N}_{10} \overset{R^2}{\Longrightarrow} \tilde{N}_{00}, \tilde{N}_{10} \tag{35}$$

and hence by theorem 2 we get theorem 1 for $|a| = 1$. ∎

4. Proof for small $|a|$. As in the previous section we want to apply theorem 2 to obtain theorem 1. As topologically expanding directions we take again e_1, e_3.

We set

$$y_{max} := 0.12 \tag{36}$$

Let $D := [0,1] \times [-y_{max}, y_{max}] \times [0,1]$. It is easy to see that

$$|\bar{y}(x,y,z)| < 0.15|a|, \quad \text{for } (x,y,z) \in D \tag{37}$$

Namely

$$|\bar{y}(x,y,z)| < |a|0.1|(y_{max} + 0.35) + 1| < 0.15|a|$$

So to have $|\bar{y}| < y_{max}$, we impose on a the following condition

$$|a| < 0.8 \tag{38}$$

We have

$$|\bar{x}(x,y,z) - f_{3.8}(x)| < |a|0.05 \cdot 0.5 \cdot 1 = 0.025|a| \tag{39}$$

$$|\bar{z}(x,y,z) - f_{3.78}(x)| \leq |a|0.2 \cdot y_{max} < 0.025|a| \tag{40}$$

We define

$$X_0 = [0.235, 0.5], \quad X_1 = [0.5, 0.765] \tag{41}$$

$$Z_0 := [0.3, 0.5], \quad Z_1 := [0.5, 0.7] \tag{42}$$

It is easy to check that

$$X_0, X_1 \ f_{3.8}\text{-covers } [0.7, 0.94] \quad \text{with margin } 0.01 \tag{43}$$

$$[0.7, 0.94] \ f_{3.8}\text{-covers } X_0 \cup X_1 \quad \text{with margin } 0.01 \tag{44}$$

To obtain the sequence of coverings starting from Z_0 and Z_1 we define

$$Z_1^a = [0.81, 0.93] \tag{45}$$

$$Z_2^a = [0.26, 0.5] \supset Z_0 \tag{46}$$

$$Z_3^a = [0.74, 0.93] \tag{47}$$

It is easy to verify that with margin 0.01 the following covering relations hold:

$$Z_0, Z_1 \overset{f_{3.78}}{\Longrightarrow} Z_1^a \overset{f_{3.78}}{\Longrightarrow} Z_2^a \overset{f_{3.78}}{\Longrightarrow} Z_3^a \overset{f_{3.78}}{\Longrightarrow} Z_0 \cup Z_1 \tag{48}$$

Let us define the sets

$$N_{ij} = X_i \times [-y_{max}, y_{max}] \times Z_j, \quad \text{for } i,j = 0,1 \tag{49}$$

Now if $|a| < 0.4$ then $0.025|a| < 0.01$. From the above considerations we obtain the following covering relations:

$$N_{ij} \overset{R^4}{\Longrightarrow} N_{00} \cup N_{10} \cup N_{01} \cup N_{11}, \quad i,j = 0,1 \tag{50}$$

Using the above relations we obtain the symbolic dynamics for R^4 on four symbols referring to the sets N_{ij}, which finishes the proof of theorem 1 for $|a| < 0.4$. ■

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