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PERIODIC SEGMENTS AND NIELSEN NUMBERS

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Abstract. We prove that the Poincaré map $\varphi_{(0,T)}$ has at least $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$ fixed points (whose trajectories are contained inside the segment W) where the homeomorphism $\tilde{h}$ is given by the segment W.

1. Introduction. In [9] Roman Srzednicki introduced the geometric method for detecting periodic solutions in nonautonomous periodic differential equations based on the notion of periodic isolating blocks (or periodic isolating segments considered in [11], [13]). The method is based on the Lefschetz Fixed Point Theorem and the Ważewski Retract Theorem. It calculates the fixed point index of the Poincaré map inside the segment from the Lefschetz number of the homeomorphism $\tilde{h}$ given by the segment (see Remark 2). The problem of using other topological invariants (like the Nielsen numbers) to get more information on the solutions inside the segment was proposed in [9], [12]. In this note we study Srzednicki's method from the point of view of the Nielsen fixed point theory ([2], [3]). We prove that the relative Nielsen number (defined in [5]) of $\tilde{h}$ is a lower bound for the number of fixed points of the Poincaré map whose trajectories are contained inside the segment. First we recall the definition of the periodic isolating segments and some basic facts from Nielsen theory.

Assume that X is a metric space and $\varphi : D \to X$ is a continuous mapping, $D \subset \mathbb{R} \times X \times \mathbb{R}$ is an open set. We will denote by $\varphi_{(\sigma,t)}$ the function $\varphi(\sigma, \cdot, t)$. φ is called a local process if the following conditions are satisfied:

(1) $\forall \sigma \in \mathbb{R}, x \in X : \{t \in \mathbb{R} : (\sigma, x, t) \in D\}$ is an interval,
(2) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,0)} = \mathrm{id}_X$,
(3) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$,

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If $D = \mathbb{R} \times X \times \mathbb{R}$, we call φ a (global) process. For $(\sigma, x) \in \mathbb{R} \times X$ the set

$$\{(\sigma + t, \varphi_{(\sigma,t)}(x) \in \mathbb{R} \times X : (\sigma, x, t) \in D\}$$

is called the trajectory of (σ, x) in φ. If T is a positive number such that

(4) $\forall \sigma, t \in \mathbb{R} \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$

we call φ a T-periodic local process. In this paper φ denotes always a T-periodic process. A local process φ on X determines a local flow Φ on $\mathbb{R} \times X$ by the formula

$$\Phi_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)}(x)).$$

Remark 1. The differential equation

$$(*) \qquad \dot{x} = f(x, t)$$

such that f is regular enough to guarantee the uniqueness of solutions of the Cauchy problems associated to $(*)$ generates a local process as follows. For $x(t_0, x_0; \cdot)$ the solution of $(*)$ such that $x(t_0, x_0; t_0) = x_0$ we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0, x_0; t_0 + \tau).$$

If f is T-periodic with respect to t then φ is a T-periodic local process and in order to determine all T-periodic solutions of the equation $(*)$ it suffices to look for fixed points of $\varphi_{(0,T)}$ (called the Poincaré map).

Now we introduce the notion of periodic (isolating) segment. By $\pi_1 : [0, T] \times X \to [0, T]$ and $\pi_2 : [0, T] \times X \to X$ we denote the projections and for a subset $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in X : (t, x) \in Z\}.$$

Let (W, W^-) be a pair of subsets of $[0, T] \times X$. We call W a periodic isolating segment over $[0, T]$ (for the equation $(*)$) and W^- the exit set of W if:

(i) W and W^- are compact ENR's, $W_0 = W_T$ and $W_0^- = W_T^-$,

(ii) there exists a homeomorphism

$$h : [0, T] \times (W_0, W_0^-) \longrightarrow (W, W^-)$$

such that $\pi_1 = \pi_1 \circ h$,

(iii) for every $\sigma \in [0, T)$ and $x \in \partial W_\sigma$ there exists a $\delta > 0$ such that for every $t \in (0, \delta)$ either $\varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x) \in \operatorname{int} W_{\sigma+t}$.

(iv) $W^- \cap ([0, T) \times X) = \{(\sigma, x) \in W : \sigma < T,\ \exists \delta > 0\ \forall t \in (0, \delta) : \varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}\}$.

In all practical applications isolating segments are manifolds with corners and all the necessary information can be obtained from the time periodic vector field f. The condition (iv) usually means that the vector $(1, f(t, x))$ is directed outward with respect to W. We call W a periodic segment over $[0, T]$ if conditions (i), (ii) and (iv) hold.

For a periodic segment W we define a homeomorphism

$$\tilde{h} : (W_0, W_0^-) \longrightarrow (W_T, W_T^-) = (W_0, W_0^-)$$

by $\tilde{h}(x) = \pi_2(h(T, \pi_2 h^{-1}(0,x)))$ for $x \in W_0$. A different choice of the homomorphism h in (ii) leads to a map which is homotopic to $\tilde{h}$ (compare [9]), hence the automorphism

$$\mu_W = \tilde{h}_* : H(W_0, W_0^-) \longrightarrow H(W_0, W_0^-)$$

induced by $\tilde{h}$ in singular homology is an invariant of the segment W.

Classical Nielsen fixed point theory (see [2], [3]) is concerned with the determination of the minimal number of fixed points for all maps in the homotopy class of a given map $f : X \to X$. The Nielsen number $N(f)$ provides a homotopy invariant lower bound for the number of fixed points of f. In many cases $N(f)$ is the best possible lower bound. The basis of Nielsen fixed point theory is the notion of fixed point class. Let X be a compact ENR, $f : X \to X$ be a map. The fixed point set $\mathrm{Fix} f = \{x \in X : x = f(x)\}$ splits into a disjoint union of fixed point classes—two fixed points are in the same class if and only if they can be joined by a path which is homotopic (rel end-points) to its own f-image. Each fixed point class F is an isolated subset of $\mathrm{Fix} f$, so its fixed point index $\mathrm{ind}(f, F) \in \mathbb{Z}$ is defined. A fixed point class is called essential if its index is non-zero. The Nielsen number $N(f)$ of f is defined as the number of essential fixed point classes. Every map homotopic to f must have at least $N(f)$ fixed points. The Nielsen number is a rather poor lower bound for the minimal number of fixed points for a self map $f : (X, A) \to (X, A)$. In [6] an extension of Nielsen theory to maps was begun. In this note we will use the Nielsen number of the closure $N(f, \mathrm{cl}(X \setminus A))$ introduced in [5]. We assume that (X, A) is a pair of compact ENR's, and $f : (X, A) \to (X, A)$ is a map. Let F be a fixed point class of $f : X \to X$. We say that F assumes its index in A if $\mathrm{ind}(f, F) = \mathrm{ind}(f|_A, F \cap A)$. The Nielsen number of the closure $N(f, \mathrm{cl}(X \setminus A))$ is defined as the number of the fixed point classes of f which do not assume its index in A (see [6], [7]). Note that all fixed point classes, whether essential or not, are counted in this definition. $N(f, \mathrm{cl}(X \setminus A)$ is a homotopy invariant (under homotopies of the form $H : (X \times [0,1], A \times [0,1]) \to (X, A)$) lower bound for the number of fixed points on $\mathrm{cl}(X \setminus A)$.

2. Main result. Our main result is the following

THEOREM 2. *Assume that W is a periodic segment (not necessarily isolating) over $[0,T]$. Put*

$$F_W = \{x \in X : \varphi_{(0,T)}(x) = x,\ \forall t \in [0,T] : \varphi_{(0,t)}(x) \in W_t\}.$$

Then $\mathrm{card} F_W \geq N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$.

Note that the set F_W of fixed points of the Poincaré map whose trajectories are contained in W was first considered in [9].

REMARK 3. (1) It follows that $\mathrm{cl}(W_0 \setminus W_0^-) = W_0$ but in general $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$ is not equal to $N(\tilde{h})$.

(2) In [9] it was proved that if W is a periodic isolating segment over $[0,T]$ then F_W is compact and open in the set of fixed points of $\varphi_{(0,T)}$ and the fixed point index of $\varphi_{(0,T)}$ in F_W is given by

$$\mathrm{ind}(\varphi_{(0,T)}, F_W) = \mathrm{Lef}(\mu_W).$$

This result has many applications in detecting periodic solutions and chaos in nonautonomous periodic differential equations (see [9], [10], [11], [12], [13]).

(3) If W_0 is contractible then $\mathrm{Lef}(\mu_W) \neq 0$ iff $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-)) = 1$ and $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-)) = 0$ if $\mathrm{Lef}(\mu_W) = 0$, so in this case our Theorem 1 does not give more information than results in [9].

We will need some notions related to W. Assume that W is a periodic segment over $[0, T]$. Put $S^1 = \mathbb{R}/T\mathbb{Z}$ and by $[t]$ denote the equivalence class of $t \in \mathbb{R}$ in S^1. By T-periodicity of φ the local flow Φ on $\mathbb{R} \times X$ induces a local flow $\widetilde{\Phi}$ with $S^1 \times X$ as the phase space. Put

$$\widetilde{W} = \{([t], x) \in S^1 \times X : x \in W_t, t \in [0, T]\}.$$

If W is isolating then by the condition (i), the set $\widetilde{W}$ is an isolating block in the usual sense in the theory of isolated invariant sets (see [8]). The exit set $\widetilde{W}^-$ of that isolating block is equal to $\{([t], x) : x \in W_t^-, t \in [0, T]\}$.

Proof of Theorem 1. Define a map

$$\tau = \tau_W : W_0 \ni x \longrightarrow \sup\{t \geq 0 : \forall s \in [0, t] : \widetilde{\Phi}_s([0], x) \in \widetilde{W}\} \in [0, \infty].$$

τ is continuous (by the argument in a proof of Ważewski Theorem, [8]). For $s \in [0, T]$ we define a homeomorphism

$$h_{s,T} : (W_s, W_s^-) \to (W_0, W_0^-)$$

by $h_{s,T}(x) = \pi_2(h(T, \pi_2 h^{-1}(s, x)))$. Note that $h_{0,T} = \tilde{h}$. Consider a homotopy $H : (W_0, W_0^-) \times [0, 1] \to (W_0, W_0^-)$ given by

$$H(x, t) = \begin{cases} h_{\tau(x),T}(\varphi_{0,\tau(x)}(x)), & \tau(x) \leq (1-t)T, \\ h_{(1-t)T,T}(\varphi_{(0,(1-t)T)})(x)), & \tau(x) \geq (1-t)T. \end{cases}$$

Put $H_t(x) = H(x, t)$. It is easy to check that $H_t(x) = \tilde{h}(x)$ for $x \in W_0^-$ and $H_1 = \tilde{h}$. By the homotopy property of the Nielsen number of the closure we have $N(H_0, \mathrm{cl}(W_0 \setminus W_0^-)) = N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$. In order to prove Theorem 1 it suffices to show that $\mathrm{card}(F_W) \geq N(H_0, \mathrm{cl}(W_0, W_0^-))$. This follows from

LEMMA 4. *Assume that $K \subset W_0$ is a fixed point class of H_0. If K does not assume its index in W_0^- then $F_W \cap K$ is non-empty.*

PROOF. Suppose that $F_W \cap K$ is empty. It is easy to check that $\mathrm{Fix}(H_0) = F_W \cup \mathrm{Fix}(\tilde{h}|_{W_0^-})$, so $K \subset W_0^-$. Put

$$U = \{x \in W_0 : \tau(x) < T\}.$$

It follows that U is open in W_0, $W_0^- \subset U$ and $H_0(U) \subset W_0^-$, so by the commutativity of the fixed point index we obtain

$$\mathrm{ind}(H_0, K) = \mathrm{ind}(\tilde{h}|_{W_0^-}, K),$$

a contradiction.

Simple examples show that in many cases it is possible that $\mathrm{card}(F_W) < N(\tilde{h})$ but we have the following

Corollary 5. *If* $N(\tilde{h}|_{W_0^-}) = 0$ *then*

$$\mathrm{card}(F_W) \geq N(\tilde{h}).$$

Proof. Let $K \subset W_0$ be a fixed point class of $\tilde{h}$ which does not assume its index in W_0^-. Then $K \cap W_0^-$ is either empty or the union of fixed point classes of $\tilde{h}|_{W_0^-}$ (see [7]), so $\mathrm{ind}(\tilde{h}|_{W_0^-}, K \cap W_0^-) = 0$ (because $N(\tilde{h}|_{W_0^-}) = 0$). This means that K is an essential fixed point class of $\tilde{h}$. The proof is complete by Theorem 1.

Remark 6. (1) The Nielsen number of the closure $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$ in Theorem 1 cannot be replaced by the relative Nielsen number $N(\tilde{h}, W_0, W_0^-)$ defined in [6]. For example consider the flow generated by the system

$$\dot{z} = \bar{z}^n.$$

Fix $T > 0$. From the phase portrait one can deduce the existence of a periodic segment W over $[0,T]$ such that $W_0 = W_t$ is a regular $2(n+1)$-gon, W_0^- consists of $n+1$ disjoint contractible parts. In this example $\tilde{h} = \mathrm{id}_{(W_0, W_0^-)}$. One can check (see [6]) that $N(\tilde{h}, W_0, W_0^-) = n+1$ and the map after time T has exactly one fixed point in W_0. Note that $N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-)) = 1$. On the other hand, it follows by our proof of Theorem 1 that $\mathrm{card} F_W \geq N(\tilde{h}, W_0 \setminus W_0^-)$, where $N(\tilde{h}, W_0 \setminus W_0^-)$ denotes the Nielsen number of the complement defined in [14] (it is a lower bound for the number of fixed points of $\tilde{h}$ in $W_0 \setminus W_0^-$). By Theorem 2.9 in [14] we have $N(\tilde{h}, W_0 \setminus W_0^-) \leq N(\tilde{h}, \mathrm{cl}(W_0 \setminus W_0^-))$. One can check using Theorem 4.1 in [14] that in our example $N(\tilde{h}, W_0 \setminus W_0^-) = 0$.

(2) Note that our proof of Theorem 1 contains also the proof of Srzednicki's theorem. In fact if W is a periodic isolating segment then the homotopy H shows that

$$\mathrm{Lef}(\tilde{h}) = \mathrm{Lef}(H_1) = \mathrm{Lef}(H_0).$$

Moreover

$$\mathrm{Lef}(H_0) = \mathrm{ind}(\varphi_{(0,T)}, F_W) + \mathrm{ind}(\tilde{h}, \mathrm{Fix}(\tilde{h}|_{W_0^-})),$$

so Srzednicki's result follows, because by the commutativity of the fixed point index

$$\mathrm{ind}(\tilde{h}, \mathrm{Fix}(\tilde{h}|_{W_0^-})) = \mathrm{Lef}(\tilde{h}|_{W_0^-}).$$

(3) Corollary 1 gives a possibility of applications of the results in [4] to study periodic points of the Poincaré map even if $\varphi_{(0,T)}$ is defined only locally and W_0 is not invariant under $\varphi_{(0,T)}$. Suppose that W is a periodic (not necessarily isolating) segment such that $N(\tilde{h}^n|_{W_0^-}) = 0$ for any $n \in \mathbb{N}$. The asymptotic Nielsen number $N^\infty(\tilde{h})$ of $\tilde{h}$ is defined by (compare [4])

$$N^\infty(\tilde{h}) = \mathrm{Growth}_{n\to\infty} N(\tilde{h}^n),$$

where

$$\mathrm{Growth}_{n\to\infty} a_n = \max\{1, \limsup_{n\to\infty} |a_n|^{1/n}\}.$$

By W^n we denote the segment over $[0, nT]$ given by

$$W^n = \{(kT+t, x) \in [0, nT] \times X : k \in \{0, ..., n-1\}, t \in [0,T], x \in W_t\}.$$

It follows by Corollary 1 that
$$\mathrm{card} F_{W^n} \geq N(\tilde{h}^n).$$
In particular if $N^\infty(\tilde{h}) > 1$ then there are infinitely many periodic solutions whose trajectories are contained in the segment W. For example suppose that there exists a segment such that W_0 is the space obtained from the closed 2-dimensional disk B of radius one by removing the interiors of three disjoint disks inside B (thus W_0 is a disk with three holes). Let W_0^- be equal to S^1. Assume that $\tilde{h}$ is the homeomorphism H'_P defined in [1, p. 236]. The suspension of H'_P and the braid related with it are given in Fig. 2.2 in [1]. In this case $N^\infty(\tilde{h}) > 5/2$ (see [4]), so we have infinitely many subharmonic solutions. Note that the sequence $\mathrm{Lef}(\mu_W^n)$ is periodic, so existence of infinitely many subharmonic solutions does not follow from Srzednicki's result.

References

[1] P. Boyland, *Topological methods in surface dynamics*, Topol. Appl. 58 (1994), 223–298.

[2] R. F. Brown, *On the Nielsen fixed point theorem for compact maps*, Duke Math. J. 36 (1969), 699–708.

[3] B. Jiang, *Lectures on Nielsen Fixed point Theory*, Contemp. Math. vol. 14, AMS Providence, 1983.

[4] B. Jiang, *Nielsen theory for periodic orbits and applications to dynamical systems*, Contemp. Math. vol. 152, 183–202, AMS Providence 1993

[5] H. Schirmer, *On the location of fixed point sets of pairs of spaces*, Topol. and its Appl. 30 (1988), 253–266.

[6] H. Schirmer, *A relative Nielsen number*, Pacific J. Math. 122 (1986), 65–72.

[7] H. Schirmer, *A Survey of Relative Nielsen Fixed Point Theory*, Contemp. Math. vol. 152, 291–309, AMS Providence 1993.

[8] C.C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. no 38, AMS, Providence R.I., 1978,

[9] R. Srzednicki, *Periodic and bounded solutions in block for time-periodic nonautonomous ordinary differential equations*, Nonlinear Anal. Theory Meth. Appl. 22 no 6 (1994), 707–737,

[10] R. Srzednicki, *On periodic solutions of planar differential equations with periodic coefficients*, J. Diff. Equat. 114 (1994), 77–100,

[11] R. Srzednicki and K. Wójcik, *A Geometric Method for Detecting Chaotic Dynamics*, J. Diff. Equat. Vol. 135, (1997), 66–82.

[12] R. Srzednicki, *A Geometric Method for the Periodic Problem in Ordinary Differential Equations*, Seminaire D'Analyse Moderne No.22, Eds.: G. Fournier, T. Kaczynski, Université de Sherbrooke 1992.

[13] K. Wójcik, *Isolating segments and symbolic dynamics*, to appear in Nonlinear Anal. Th. Meth. Appl.

[14] X. Z. Zhao, *A relative Nielsen number for the complement*, Lecture Notes in Math. vol. 1411, Springer-Verlag, 1989, 189–199.