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THE CONLEY INDEX THEORY: A BRIEF INTRODUCTION

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Abstract. A brief introduction to the Conley index theory is presented. The emphasis is the fundamental ideas of Conley's approach to dynamical systems and how it avoids some of the difficulties inherent in the study of nonlinear systems.

1. Introduction. Trying to pin down the first appearance of a scientific idea or method is a tricky subject, however, it is fairly safe to say that the paper of Conley and Easton [6], with its abstract discussion of isolating blocks and its use of cohomology to relate the boundary dynamics to the asymptotic interior dynamics, represents an important marker in the beginning of what is now referred to as the Conley Index Theory. As they indicate in the introduction the work was inspired by problems from differential equations, in particular, celestial mechanics. The usefulness of this approach was quickly recognized and exploited by Conley and Smoller in a series of papers in which they investigated the existence of shock waves [7, 8, 9, 10]. Both the theory and the range of applications of the index have grown considerably in the intervening quarter century. For this author, the close tie between an abstract mathematical framework for dynamics and applied problems has been one of the motivations to study the index theory.

There is no single reference source to the current state of the theory. The best insight to the general framework and philosophy remains Conley's monograph "Isolated Invariant Sets and the Morse Index" [4]. However, the theory has progressed since its publication. Salamon [43] simplified many of the proofs presented in [4] and earlier works. The eighties saw a variety of developments. Rybakowski [42] extended the theory to semiflows on noncompact spaces. Franzosa proved the existence of Conley's connection matrices [15, 16, 17, 18, 41, 39]. Reineck related transition matrices to the existence of co-dimension one

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connecting orbits [38, 24, 19]. Robbin and Salamon and later Mrozek extended the theory to the setting of discrete dynamics [40, 35, 37, 11, 44, 14] and Floer [13] adapted Conley's continuation ideas to develop what is now called Floer Homology. More recently, progress has been made in understanding the structure of an invariant set from its Conley index [12, 23, 25, 26, 27, 28, 29, 45, 1, 2]. Expanding on ideas of Conley [5] in the late seventies, a singular perturbation theory that incorporates the index is being explored [32, 20]. Finally, it has become evident that the Conley index theory provides a numerically cheap method for obtaining rigorous results about dynamics [30, 31, 46].

Most if not all of the above mentioned topics will be touched upon in this volume, along with examples of how these ideas can be used in applications. With this in mind, this short note will only provide a brief introduction to the fundamental points of the theory. The goal is to give an impression of why Conley's approach is so important in the study of concrete dynamical systems and to provide definitions of the basic tools used. A more complete survey of the theory can be found in [3].

2. Isolating neighborhoods. In order to emphasize the ideas and avoid technical difficulties we will consider the index theory in its simplest setting, that of flows $\varphi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ or homeomorphisms $f : \mathbf{R}^n \to \mathbf{R}^n$. $S \subset \mathbf{R}^n$ is an *invariant set* if

$$\varphi(\mathbf{R}, S) := \bigcup_{t \in \mathbf{R}} \varphi(t, S) = S \quad \text{or} \quad f(S) = S.$$

Much of dynamical systems involves the study of the existence and structure of invariant sets. The difficulty is three-fold:

1. invariant sets can be extremely complicated (chaotic dynamics and fractal structures);
2. the structure of invariant sets can change dramatically with respect to perturbations (bifurcation theory, normal forms, catastrophe theory);
3. the bifurcation points need not be isolated (structurally stable systems are not dense).

The fact that for a fixed family of dynamical systems, all three of these problems must be considered simultaneously, makes the analysis of nonlinear systems difficult. Conley's approach is an attempt to circumvent these issues by avoiding the direct study of invariant sets. In his own words [5]

> "... many significant properties of the flow are reflected in the existence of isolating neighborhoods, or perhaps more accurately, in the companion isolated invariant set ... This is true in some generality of those properties which are stable under perturbation."

A compact set $N \subset \mathbf{R}^n$ is an *isolating neighborhood* if

$$\operatorname{Inv}(N, \varphi) := \{x \in N \mid \varphi(\mathbf{R}, x) \subset N\} \subset \operatorname{int} N$$

or

$$\operatorname{Inv}(N, f) := \{x \in N \mid f^n(x) \subset N,\ n \in \mathbf{Z}\} \subset \operatorname{int} N$$

where int N denotes the interior of N. S is an *isolated invariant set* if $S = \mathrm{Inv}(N)$ for some isolating neighborhood N.

The most important property of an isolating neighborhood is that it is robust with respect to perturbation. More precisely we have the following proposition, the proof of which is fairly straightforward.

PROPOSITION 1. *Let N be an isolating neighborhood for a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$. There exists $\epsilon > 0$ such that if $||f(x) - g(x)|| < \epsilon$ for all $x \in \mathbf{R}^n$, then N is also an isolating neighborhood for g.*

A similar result holds for flows. Another way of viewing this result is in terms of multivalued maps (see [21, 36]). Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a homeomorphism. Consider a multivalued map $F_\epsilon : \mathbf{R}^n \overrightarrow{\to} \mathbf{R}^n$ defined by

$$F_\epsilon(x) := \{y \in \mathbf{R}^n \mid ||y - f(x)|| \leq \epsilon\}.$$

F_ϵ defines a dynamical system where a trajectory $\{x_n\}_{n=-\infty}^{\infty}$ satisfies the property that $x_{n+1} \in F_\epsilon(x_n)$. Given $N \subset \mathbf{R}^n$, $\mathrm{Inv}(N, F_\epsilon)$ is then the set of points which lie on trajectories which are contained in N. N is an *isolating neighborhood* for F_ϵ if

$$\{y \in \mathbf{R}^n \mid ||y - \mathrm{Inv}(N, F)|| \leq \epsilon\} \subset \mathrm{int}\, N.$$

The following result is a restatement of Proposition 1.

PROPOSITION 2. *Let N be an isolating neighborhood for a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$. There exists $\epsilon > 0$ such that N is an isolating neighborhood for the multi-valued map F_ϵ.*

What is more interesting is to view this proposition in the "opposite" direction. Recall that a continuous function $g : \mathbf{R}^n \to \mathbf{R}^n$ is a *continuous selector* for F_ϵ if $g(x) \in F_\epsilon(x)$ for all $x \in \mathbf{R}^n$.

PROPOSITION 3. *If N is an isolating neighborhood for F_ϵ, then N is an isolating neighborhood for any continuous selector of F_ϵ.*

This proposition indicates that working with multivalued maps (in the context of numerical analysis think of an approximation plus error bounds; in the context of experimental time series think of measurement plus error bounds) allows one to obtain a priori bounds on the sizes of perturbations that preserve isolation.

3. Decomposing invariant sets. As was mentioned earlier, Conley chose to concentrate on isolating neighborhoods because of their robustness with respect to perturbations. However, one cannot avoid the fundamental problem; in the end it is the associated invariant set that is of interest. The first step in resolving this problem is to obtain decompositions of isolated invariant sets that are consistent with the concept of isolating neighborhoods.

Recall that given a set $K \subset \mathbf{R}^n$ the *omega limit set* of K is given by

$$\omega(K) = \bigcap_{t>0} \mathrm{cl}(\varphi([t, \infty), K))$$

and the *alpha limit set* is

$$\alpha(K) = \bigcap_{t<0} \mathrm{cl}(\varphi((-\infty, t], K)).$$

Let S be an isolated invariant set. $A \subset S$ is an *attractor* in S if there exists a neighborhood U of A such that $\omega(U \cap S) = A$. Observe that this implies that A is also an isolated invariant set. The dual *repeller* of A is

$$R := \{x \in S \mid \omega(x) \cap A = \emptyset\}.$$

(A, R) is called an *attractor-repeller pair decomposition* of S. The local compactness of $\mathbf{R}^n$ and the continuity of the flow implies that R is also an isolated invariant set.

There are two fundamental theorems associated with attractor-repeller pair decompositions. The first indicates that the recurrent dynamics in S is contained entirely in $A \cup R$ and that off these sets the dynamics is gradient-like (for a proof see [43]).

Theorem 4. *Let (A, R) be an attractor-repeller pair decomposition of S. Then*

$$S = A \cup R \cup C(R, A)$$

where $C(R, A) := \{x \in S \mid \omega(x) \subset A,\ \alpha(x) \subset R\}$.

Furthermore, there exists a continuous function $V : S \to [0, 1]$ such that:

1. $R = V^{-1}(1)$;
2. $A = V^{-1}(0)$;
3. *if $x \in C(R, A)$ and $t > 0$ then $V(x) > V(\varphi(t, x))$.*

The second result begins to indicate how Conley's approach overcomes the three difficulties of dynamical systems discussed in Section 2. The notion of continuation is critical. Consider now a continuous family of dynamical systems

$$\varphi_\lambda : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n, \quad \lambda \in [-1, 1].$$

Let $N \subset \mathbf{R}^n$ be a compact set. Let $S_\lambda = \mathrm{Inv}(N, \varphi_\lambda)$. Two isolated invariant sets S_{λ_0} and S_{λ_1} are *related by continuation* or S_{λ_0} *continues to* S_{λ_1} if N is an isolating neighborhood for all φ_λ, $\lambda \in [\lambda_0, \lambda_1] \subset [-1, 1]$.

Theorem 5. *Attractor-repeller pair decompositions continue.*

Proof. Assume that S_0 is an isolated invariant set with isolating neighborhood N. Then, by Proposition 1 there exists $\lambda_S > 0$ such that N is an isolating neighborhood for all φ_λ, $\lambda \in [-\lambda_S, \lambda_S]$. Let (A_0, R_0) denote an attractor-repeller pair decomposition of S_0. Since A_0 and R_0 are isolated invariant sets, they have isolating neighborhoods $N_A \subset N$ and $N_R \subset N$. Furthermore, there exists $\lambda_A > 0$ and $\lambda_R > 0$ such that N_A is an isolating neighborhood for $\lambda \in [-\lambda_A, \lambda_A]$ and N_R is an isolating neighborhood for $\lambda \in [-\lambda_R, \lambda_R]$.

Let $\lambda_0 = \min\{\lambda_S, \lambda_A, \lambda_R\}$. Thus, for all $\lambda \in [-\lambda_0, \lambda_0]$, the $A_\lambda := \mathrm{Inv}(N_A, \varphi_\lambda)$ are related by continuation, the $R_\lambda := \mathrm{Inv}(N_R, \varphi_\lambda)$ are related by continuation, and the S_λ are related by continuation.

The final point that needs to be checked is that there exists $0 < \lambda_1 \leq \lambda_0$ such that for all $\lambda \in [-\lambda_1, \lambda_1]$, (A_λ, R_λ) is an attractor-repeller pair for S_λ. Assume not. Then there exists a sequence of parameter values $\lambda_n \to 0$ and points $x_n \in S_{\lambda_n}$ such that $x_n \notin A_{\lambda_n} \cup R_{\lambda_n}$ and $\omega(x_n) \notin A_{\lambda_n}$. Invoking the continuity of the family of flows and the fact that (A_0, R_0) is an attractor-repeller pair decomposition of S_0 gives a contradiction. Thus, there exists an interval $[-\lambda_1, \lambda_1]$ over which the attractor-repeller pair decomposition continues. ■

These remarks on continuation should make it clear that not only is the empty set an isolated invariant set, but in fact it plays an important role in continuation. For example, an isolated degenerate fixed point typically continues to the empty set. On a more general level, continuation alone is a statement about isolating neighborhoods, it is not a statement about the structure of the invariant sets. It is in this sense that the inherent difficulties of bifurcation theory are avoided.

The generalization of an attractor-repeller decomposition is as follows. A *Morse decomposition* of an isolated invariant set S is a finite collection of disjoint compact invariant sets called *Morse sets*,

$$\mathcal{M}(S) := \{M(p) \mid p \in \mathcal{P}\}$$

satisfying two properties. First, for any $x \in S$, both the alpha and omega limit sets of x lie in the Morse sets. Second, there exists a strict partial ordering on the indexing set $\mathcal{P}$ such that if $x \in S \setminus \bigcup M(p)$, $\alpha(x) \in M(p)$, and $\omega(x) \in M(q)$, then $p > q$. Such an ordering is called an *admissible order*. Admissible orders need not be unique, but the minimal order, i.e. that with the fewest relations, is called the *flow defined order*. Observe that it is the flow defined order which contains the most information, since it is most directly tied to the existence of heteroclinic orbits between Morse sets.

As with attractor-repeller pairs, Morse decompositions admit Lyapunov functions.

THEOREM 6. *Let $\mathcal{M}(S) := \{M(p) \mid p \in \mathcal{P}\}$ be a Morse decomposition. Then, there exists a continuous function $V : S \to [0,1]$ such that;*

1. *for each $p \in \mathcal{P}$ and for every $x, y \in M(p)$, $V(x) = V(y)$;*
2. *if $x \in S \setminus \bigcup M(p)$, then $V(x) > V(\varphi(t,x))$ for all $t > 0$.*

The proof is a simple extension of the argument used for attractor-repeller pairs. Similarly, one can conclude that:

THEOREM 7. *Morse decompositions continue.*

As was mentioned at the beginning of this section, the fundamental problem with Conley's emphasis on isolating neighborhoods is that in the end one must be able to describe the associated invariant sets. Using the concept of a Morse decomposition $\mathcal{M}(S)$, the question of how to describe the dynamics of S can be posed as follows:

1. What is the structure of the dynamics of the individual Morse sets?
2. What is the structure of the connecting orbits between the Morse sets?

4. Ważewski's principle. Attractor-repeller pairs and, more generally, Morse sets provide robust decompositions of isolated invariant sets into isolated invariant subsets. However, at this point in the discussion we still have no mechanism for determining the structure of these sets from information associated with the isolating neighborhoods. This is the purpose of the Conley index which will be discussed in the next section. However, in an attempt to provide some intuition as to why the index carries information about the isolated invariant we will first consider the Ważewski principle.

Let $W \subset \mathbf{R}^n$. Define $W^0 = \{x \in W \mid \varphi(t,x) \notin W$ for some $t > 0\}$ and let $W^- = \{x \in W \mid \varphi([0,t),x) \not\subset W$ for all $t > 0\}$. Clearly, $W^- \subset W^0$. W is called a *Ważewski set* if the following conditions are satisfied:

1. If $x \in W$ and $\varphi([0,t],x) \subset \mathrm{cl}(W)$, then $\varphi([0,t],x) \subset W$,
2. W^- is closed relative to W^0.

THEOREM 8 (Ważewski [47]). *If W is a Ważewski set, then W^- is a strong deformation retract of W^0 and W^0 is open relative to W.*

The reader is referred to [4] for a proof, but the important point is that the strong deformation retract is constructed by carrying points along the flow lines to the boundary, i.e. it is the flow that is the retract.

Observe that if an isolating neighborhood N is a Ważewski set, then N^- lies in the boundary of N. Now assume that N^- is not a strong deformation retract of N. Then $N^0 \neq N$ which implies that there exists $x_0 \in N$ such that $\varphi([0,\infty),x) \subset N$. Since N is compact $\omega(x) \subset N$ and hence $\mathrm{Inv}(N) \neq \emptyset$.

The point is that for an appropriately chosen set one only needs to compare the topology of elements on the boundary that are leaving to the topology of the isolating set, to know if there must be a nontrivial invariant set.

It should also be noted that the Ważewski principle is extremely general. There are no assumptions concerning the topology of W and extremely mild assumptions on the behavior of the dynamics on W. For this reason it is easy to construct examples where the Ważewski principle is applicable and provides useful information and the Conley index does neither. The disadvantage of Ważewski's approach is that it is not robust with respect to perturbation. In particular, results obtained using the Ważewski principle at one parameter value need not persist at nearby parameter values.

5. The Conley index. As was indicated in the previous section, Ważewski's principle shows that knowledge about the dynamics on the boundary of a suitably chosen set can be sufficient to conclude that the associated invariant set is non-empty. In Section 3, it was argued that the important point of Conley's approach is the concept of continuation; a property that Ważewski's principle does not possess. In this section the index will be defined. We will begin with a presentation in the setting of flows, since the analogy with the Ważewski property is clear. We will then briefly discuss a method for extending the index to the case of maps.

There are three crucial properties of the Conley index.

1. The Conley index is an index of isolating neighborhoods. If N and N' are isolating neighborhoods and $\mathrm{Inv}N = \mathrm{Inv}N'$, then

$$\text{Conley Index}(N) = \text{Conley Index}(N').$$

(Do not worry at the moment about what the index is.)

2. If Conley Index(N) is not trivial, then
$$\mathrm{Inv}(N) \neq \emptyset.$$
3. If N is an isolating neighborhood for a continuous family of dynamical systems φ_λ, $\lambda \in [0,1]$, then
$$\text{Conley Index}(N, \varphi_0) = \text{Conley Index}(N, \varphi_1).$$

Observe that the first property implies that one can just as well view the Conley index as an index of isolated invariant sets. It is useful to adopt both points of view. In the development of the theory it is typically easier to think of the isolated invariant set as being given and then derive the properties of the index. In applications one begins with an isolating neighborhood and then uses the index to draw conclusions about the associated invariant set. The second property is often referred to as the Ważewski property. The third property is the continuation property of the index and indicates that the index remains constant under the appropriate homotopy class of dynamical systems.

Returning to the theme of this article, that the Conley index allows one to circumvent the three essential difficulties of dynamical systems, the usefulness of this approach depends on how much information about the structure of the dynamics of the invariant set can be concluded from knowledge of the Conley index. Considerable progress has been made along these lines and the interested reader is referred to the references given in the introduction with [3] as a possible starting point. Of course, given that only topological techniques are being employed, one cannot hope that the results obtained using these methods will be as sharp as those that explicitly use the smooth structure of the dynamical system.

Turning now to the definition of the index, let S be an isolated invariant set. A pair of compact sets (N, L) with $L \subset N$ is an *index pair* for S if:

1. $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L))$ and $N \setminus L$ is a neighborhood of S.
2. L is *positively invariant* in N; that is, given $x \in L$ and $\varphi([0,t],x) \subset N$, then $\varphi([0,t],x) \subset L$.
3. L is an *exit set* for N; that is, given $x \in N$ and $t_0 > 0$ such that $\varphi(t_0,x) \notin N$, then there exists $0 \leq t_1 < t_0$ such that $\varphi(t_1,x) \in L$.

The *homotopy Conley index* of S is defined to be the homotopy type of the pointed topological space obtained by collapsing the exit set L to a point, i.e.
$$h(S) \sim (N/L, [L]).$$
Observe that if one chooses an index pair in which N is a Ważewski set and $L = W^-$, then $h(S)$ not being homotopically equivalent to a one point space is the same as assuming that L is not a strong deformation retract of N and hence $\mathrm{Inv}(N) \neq \emptyset$.

Having defined the index in the setting of flows we now turn to the case of maps. The approach that will be described here is due to Szymczak [44]. Again, one begins with the definition of an index pair. As the reader will notice it is the discretized version of that used for flows.

A pair of compact sets (N, L) is an *index pair* if:

1. $\mathrm{Inv}(\mathrm{cl}(N \setminus L)) \subset \mathrm{int}(N \setminus L)$,
2. $x \in L$ implies that $f(x) \notin N \setminus L$ (positive invariance),
3. $x \in N$ and $f(x) \notin N$, then $x \in L$.

Again, the quotient space N/L plays a crucial role. However, in the case of maps the topological type of N/L cannot play the role of an index. This can be seen from the simple example of $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x + 1$. Clearly, the only isolated invariant set is the empty set. Let $N = [0, 2]$ and $L = [1, 2]$. Then (N, L) is an index pair. Define $N' = \{-1\} \cup N$. Then (N', L) is also an index pair, but the topological types of N/L and N'/L are different. To reconcile these differences requires incorporating the action of the map on the space.

It is fairly easy to check that given an index pair (N, L), f induces a continuous map $F_{N,L} : N/L \to N/L$ called the *index map*. Thus, one has the pair $(N/L, F_{N,L})$. Choosing a different index pair for S, (N', L'), produces a different quotient space and index map $F_{N',L'} : N'/L' \to N'/L'$. To obtain a Conley index a class of maps that determines equivalence between these two pairs $(N/L, F_{N,L})$ and $(N'/L', F_{N',L'})$ needs to be defined. A minimal condition is that given $g : N/L \to N'/L'$ a continuous function, $g \circ F_{N,L} = F_{N',L'} \circ g$. Let $q : N/L \to N'/L'$ be another continuous function such that $q \circ F_{N,L} = F_{N',L'} \circ q$. Finally, let n and m be nonnegative integers. Then the pairs (g, n) and (q, m)) are equivalent (denoted by $\sim$) if and only if there exists a nonnegative integer k such that the following diagram commutes:

$$\begin{array}{ccc} N/L & \xrightarrow{F_{N,L}^{n+k}} & N/L \\ F_{N,L}^{m+k} \downarrow & & \downarrow q \\ N/L & \xrightarrow{g} & N'/L' \end{array}$$

Szymczak's definition of the Conley index is the pair $(N/L, F_{N,L})$ up to the equivalence class induced by $\sim$. More precisely $(N/L, F_{N,L})$ and $(N'/L', F_{N',L'})$ are equivalent if there exist maps $g : N/L \to N'/L'$, $h : N'/L' \to N/L$ and nonnegative integers n, m such that $(g \circ h, n + m) \sim (id_{N/L}, 0)$ and $(h \circ g, m + n) \sim (id_{N'/L'}, 0)$.

As a trivial, but perhaps illuminative example, the reader may wish to prove the equivalence of $(N/L, F_{N,L})$, $(N'/L', F_{N',L'})$, and $(N''/L'', F_{N'',L''})$ where $N'' = L'' = [0, 2]$ and $F_{N'',L''}$ is the constant map.

In a recent paper Franks and Richeson [14] recast this equivalence relation in the language of shift equivalence.

6. Continuation. It has been stressed at several points that the continuation of the Conley index is one of its most important properties, to a large extent because this is what allows the user to avoid hard analytic estimates that rule out global and local bifurcations. There are two ways in which the continuation property has been used. The first approach is similar in spirit to the use of degree theory. One begins by constructing a homotopy of the dynamics of interest to a simple example in which the index can be explicitly computed. The continuation property then guarantees that the Conley index is the same for the original system. Finally, an abstract index result (for example, nontrivial index implies a nonempty invariant set) is used to draw a conclusion about the system of interest.

A more recent use of the continuation property has come about through numerical computations. In this case the idea is as follows. One approximates the dynamics through numerical computations. For this numerically generated dynamical system one can find isolating neighborhoods and index pairs, and compute for instance the homology of the Conley index. If the numerical approximation is sufficiently close to the original system, then by Proposition 1 numerically generated isolating neighborhoods are in fact true isolating neighborhoods, and therefore, the Conley indices for the numerical system are also the Conley indices for the original system of interest.

Observe that to apply this second approach a stronger continuation result than that of Proposition 1 is needed. In particular, what is needed is an a priori estimate for the range of parameter values to which the continuation result applies. This is the reason for the inclusion of Proposition 3. In practice, one performs the numerical computations keeping track of error bounds. If ϵ represents the bound, the multivalued map F_ϵ, defined by choosing ϵ balls about the numerically computed images under the dynamics will contain the true dynamical system as a continuous selector. Examples of how these ideas can be carried out in practice can be found in the references cited in the introduction.

It is worth mentioning that even on a theoretical level the multivalued map approach provides a useful framework in which to consider proofs of the continuation properties of the index theory [22].

Finally, this approach to continuation can also be applied to the experimental setting where the dynamical system is not known, but rather all one has is data with associated error estimates. Using time delay reconstruction techniques one can build a multivalued dynamical system, essentially a map whose images are large enough to contain the data points along with the possible experimental errors. This multivalued map can then be analyzed with the goal of extracting isolating neighborhoods, index pairs and computing the Conley indices [33, 34].

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