LOGIC, ALGEBRA, AND COMPUTER SCIENCE
BANACH CENTER PUBLICATIONS, VOLUME 46
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1999

A CONSTRUCTIVE PROOF THAT EVERY 3-GENERATED ℓ-GROUP IS ULTRASIMPLICIAL

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Abstract. We discuss the ultrasimplicial property of lattice-ordered abelian groups and their associated MV-algebras. We give a constructive proof of the fact that every lattice-ordered abelian group generated by three elements is ultrasimplicial.

1. Preliminaries. The infinite-valued calculus of Łukasiewicz stands to MV-algebras as the classical two-valued calculus stands to boolean algebras. Indeed, as proved by Chang [Cha58], the latter coincide with the subclass of MV-algebras satisfying the equation $x \oplus x = x$. The ultrasimplicial property of MV-algebras is a generalization of the fundamental fact that every boolean algebra is generated by the limit of the direct system of its finite partitions. Accordingly, this property is a prerequisite for such results as the joint refinability of MV-algebraic partitions, or even—assuming the appropriate σ-closure conditions—for the definition of an MV-algebraic notion of entropy.

Using the categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit (see [Mun86]), one has a natural counterpart of the ultrasimplicial

1991 *Mathematics Subject Classification*: 03B50, 20K05.

The first named author is partially supported by CNR-GNSAGA Project on Symbolic Computation.

The paper is in final form and no version of it will be published elsewhere.

property for any such group G—with or without strong unit—to the effect that every finite set of positive elements of G is positively generated by an *independent* set B of positive elements (see [Ell79] and [Han83]).

In [Mun88], from the algebraic analysis of disjunctive normal forms in the infinite-valued calculus, it is proved that every free abelian lattice-group is ultrasimplicial. Thus, since Handelman proved that the ultrasimplicial property is preserved under formation of quotients by order-ideals [Han83, Theorem 3(ii)], it follows that *every* abelian lattice-group, as well as every MV-algebra, is ultrasimplicial. However, Handelman's proof is nonconstructive, and gives no hint on how to construct the independent set B.

Aim of this paper is to give a *constructive* proof, for the case of 3-generated lattice-groups, corresponding to 2-generated MV-algebras. We shall work throughout in the framework of partially ordered groups.

2. Definitions. A *partially ordered abelian group* is an abelian group G, together with a submonoid G^+ of G, such that G^+ generates G as a group and $G^+ \cap (-G^+) = \{0\}$. A *lattice-ordered abelian group* (ℓ-group, for short) is a partially ordered abelian group in which the order induced by

$$a \le b \quad \text{iff} \quad b - a \in G^+$$

is a lattice order. Equivalently, an ℓ-group is a structure $(G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and $+$ distributes over $\vee$ and $\wedge$. The words *ℓ-subgroup* and *ℓ-homomorphism* have their standard universal algebraic meaning, with respect to the signature $(+, -, 0, \vee, \wedge)$. An *ℓ-ideal* is an ℓ-subgroup J of G which is *convex* in G (i.e., $a \le b \le c$ in G and $a, c \in J$ imply $b \in J$). Thus, ℓ-ideals are precisely the same as kernels of ℓ-homomorphisms.

An ℓ-group is *simplicially ordered* iff it is isomorphic to a finite power $\mathbb{Z}^m$ of the integers, with componentwise order. A partially ordered abelian group G is *ultrasimplicially ordered* iff it can be expressed as the union of an increasing chain

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

of simplicially ordered groups, satisfying $G_i^+ \subseteq G_{i+1}^+$ for every i. As shown in [Han83, Proposition 1], the ultrasimplicial property has the following equivalent reformulation:

(U) for every $a_1, \dots, a_r \in G^+$ there exist $b_1, \dots, b_s \in G^+$ that are linearly independent over $\mathbb{Z}$ and span $a_1, \dots, a_r$ with integer coefficients ≥ 0.

It is well known that the free ℓ-group over n generators is the ℓ-subgroup $\mathrm{F}\ell(n)$ of $\mathbb{R}^{(\mathbb{R}^n)}$ generated by the projection functions $\mathbf{x}_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, \dots, n$. Equivalently, $\mathrm{F}\ell(n)$ is the ℓ-group of all homogeneous piecewise-linear functions with integer coefficients. These are defined as follows: a *homogeneous piecewise-linear function with integer coefficients* (a hpli function, for short) is a continuous function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ for which there exist finitely many elements $\mathbf{f}_1, \dots, \mathbf{f}_m$ of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that, for every $u \in \mathbb{R}^n$, there exists an index i with $\mathbf{f}(u) = \mathbf{f}_i(u)$.

By [Mun88], every free ℓ-group is ultrasimplicial. In [Han83, Theorem 3(ii)], it is claimed that the ultrasimplicial property is preserved under quotients, but the proof is—to say the least—nonconstructive. Let us note that Elliott's proof [Ell79] that every totally ordered group is ultrasimplicial provides also a recipe for constructing, given $a_1, \dots, a_r \in G^+$, elements $b_1, \dots, b_s$ satisfying property (U). The same holds for the proof in [Mun88], which can also be extended to finitely presented ℓ-groups (i.e., quotients of $\mathrm{F}\ell(n)$ by congruences generated by finitely many equations of the form $\mathbf{f} = \mathbf{g}$, for $\mathbf{f}, \mathbf{g} \in \mathrm{F}\ell(n)$); details of this extension are spelled out in [MP93].

In this paper we give a direct, effective proof that every ℓ-group generated by three elements is ultrasimplicial.

3. The main result. We need some preliminaries in piecewise-linear topology. Let $v_1, \dots, v_m \in \mathbb{Z}^n$; the *polyhedral cone* σ generated by $v_1, \dots, v_m$ is their positive hull. Explicitly:

$$\sigma = \langle v_1, \dots, v_m \rangle = \{r_1 v_1 + \dots + r_m v_m : r_1, \dots, r_m \in \mathbb{R}^+\},$$

where $\mathbb{R}^+$ is the set of real numbers ≥ 0. We say that σ is k-dimensional iff the linear space spanned by σ is k-dimensional. We set

$$\sigma^\vee = \{\mathbf{f} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) : \mathbf{f}(u) \geq 0, \text{ for every } u \in \sigma\}.$$

A *face* τ of σ is any set of the form $\tau = \sigma \cap \ker \mathbf{f}$, for some $\mathbf{f} \in \sigma^\vee$. Equivalently, a face of σ is a nonempty convex subset τ of σ such that every line segment in σ which has an interior point in τ lies entirely in τ. The *relative interior* $\operatorname{rel\,int} \sigma$ of σ is the topological interior of σ relative to the $\mathbb{R}$-vector space generated by σ. If σ is k-dimensional, then $u \in \operatorname{rel\,int} \sigma$ iff there exist vectors $v_1, \dots, v_k \in \sigma$ that are linearly independent over $\mathbb{R}$ and such that $u = a_1 v_1 + \dots + a_k v_k$, with $a_1, \dots, a_k \in \mathbb{R}^+ \setminus \{0\}$ (see, e.g., [Ful93, §1.2]).

A *simplicial cone* is a polyhedral cone of the form $\langle v_1, \dots, v_m \rangle$, with $v_1, \dots, v_m$ linearly independent over $\mathbb{R}$. A nonzero point $v \in \mathbb{Z}^n$ is *primitive* iff its coordinates are relatively prime. The *vertices* of a simplicial cone σ are the uniquely determined primitive points $v_1, \dots, v_m$ such that $\sigma = \langle v_1, \dots, v_m \rangle$.

A *unimodular cone* is a simplicial cone of the form $\langle v_1, \dots, v_m \rangle$, where $v_1, \dots, v_m \in \mathbb{Z}^n$ and there exist $v_{m+1}, \dots, v_n \in \mathbb{Z}^n$ such that $v_1, \dots, v_m, v_{m+1}, \dots, v_n$ constitute a free basis for $\mathbb{Z}^n$. A *fan* is a finite set Σ of polyhedral cones such that:

(i) every face of every cone of Σ belongs to Σ;
(ii) any two cones of Σ intersect in a common face.

The union of all cones of Σ is denoted by $|\Sigma|$; Σ is a *complete* fan iff $|\Sigma| = \mathbb{R}^n$. If all cones of Σ are unimodular, then Σ is said to be *unimodular*. Complete unimodular fans correspond to nonsingular projective toric varieties [Oda88], [Ful93], [Ewa96].

If Σ, Δ are fans, $|\Sigma| = |\Delta|$, and every cone of Δ is contained in some cone of Σ, then we say that Δ is a *subdivision* of Σ, and we write $\Delta \leq \Sigma$. In this case every cone of Σ is a union of cones of Δ. Any vertex of any cone of Σ is a *vertex* of Σ.

LEMMA 3.1. *Let Σ be a fan, $u \in |\Sigma|$. Then there exists $\sigma \in \Sigma$ such that $u \in \operatorname{rel\,int} \sigma$ and, for every $\sigma \neq \sigma' \in \Sigma$, we have $u \notin \operatorname{rel\,int} \sigma'$.*

PROOF. Let $\sigma = \bigcap \{\tau \in \Sigma : u \in \tau\}$. Any polyhedral cone is the disjoint union of the relative interior of its faces [Oda88, Appendix]. This applies to σ, and hence $u \in \operatorname{rel\,int} \sigma$. ∎

Let $\sigma = \langle u, v, w_1, \dots, w_r\rangle$ be an $(r+2)$-dimensional unimodular cone. Then $\tau = \langle u, v\rangle$ is a face of σ; we call $w = u + v$ the *Farey mediant* of τ. Let $\sigma' = \langle w, v, w_1, \dots, w_r\rangle$, $\sigma'' = \langle u, w, w_1, \dots, w_r\rangle$. Assume that Σ is a unimodular fan, with $\sigma \in \Sigma$. Define a subdivision Σ' of Σ by replacing each cone $\sigma \in \Sigma$ of which τ is a face by the two cones σ', σ'' obtained as above, along with all their faces. Then Σ' is a unimodular fan, and we say that Σ' is obtained by *starring* Σ along τ (see [Oda88, Proposition 1.26], or [Ewa96, Definition 6.1]).

Let u be a vertex of the complete unimodular fan Σ. The *Schauder hat* $\mathbf{u}$ (or $\mathbf{u}_\Sigma$, if we need to make explicit the dependence upon Σ) of Σ at u is the unique homogeneous piecewise-linear function $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\mathbf{u}(u) = 1$;
(ii) $\mathbf{u}(v) = 0$, for every vertex $v \neq u$ of Σ;
(iii) $\mathbf{u}$ is homogeneous linear on each cone of Σ.

Schauder hats exist for arbitrary vertices of Σ; indeed, the unimodularity of the cones of Σ automatically ensures that the coefficients of every linear piece of $\mathbf{u}$ are integers (see [Mun88, Proposition 1.3] for details). Hence $\mathbf{u}$ is a hpli function, and an element of $\mathrm{F}\ell(n)$.

Let Σ be a complete unimodular fan in $\mathbb{R}^n$; let $u_1, \dots, u_t$ display all vertices of Σ. Then the Schauder hats $\mathbf{u}_1, \dots, \mathbf{u}_t$ are linearly independent over $\mathbb{Z}$; let H_Σ be the subgroup of $\mathrm{F}\ell(n)$ generated —using the group operations only— by $\mathbf{u}_1, \dots, \mathbf{u}_t$. Note that H_Σ, with the order inherited from $\mathrm{F}\ell(n)$, is ℓ-isomorphic to $\mathbb{Z}^t$ as a simplicial group, but it is not an ℓ-subgroup of $\mathrm{F}\ell(n)$, since the lattice operations are different in the two structures. The *zero-set* of $\mathbf{f} \in \mathrm{F}\ell(n)$ is $Z\mathbf{f} = \{v \in \mathbb{R}^n : \mathbf{f}(v) = 0\}$. Let a fixed ℓ-ideal J of $\mathrm{F}\ell(n)$ be given; define

$$Z\Sigma = \bigcap\{Z\mathbf{f} : \mathbf{f} \in H_\Sigma \cap J\}.$$

The dependence of $Z\Sigma$ on J is tacitly understood.

Lemma 3.2. *For any $\mathbf{f} \in H_\Sigma$, we have $\mathbf{f} \in J$ iff $Z\mathbf{f} \supseteq Z\Sigma$.*

Proof. For the nontrivial direction, suppose $Z\mathbf{f} \supseteq Z\Sigma$. Since $H_\Sigma \cap J$ is a subgroup of the finitely generated free $\mathbb{Z}$-module H_Σ, we can find generators $\mathbf{g}_1, \dots, \mathbf{g}_h$ of $H_\Sigma \cap J$. Then

$$Z\Sigma = Z\mathbf{g}_1 \cap \cdots \cap Z\mathbf{g}_h \subseteq Z\mathbf{f}.$$

Let $\mathbf{g} = |\mathbf{g}_1| + \cdots + |\mathbf{g}_h| \in J$. Since $Z\mathbf{g} \subseteq Z|\mathbf{f}|$, we have by compactness $0 \le |\mathbf{f}| \le m\mathbf{g}$ in $\mathrm{F}\ell(n)$, for some multiple $m\mathbf{g}$ of $\mathbf{g}$. Since J is convex, $|\mathbf{f}| \in J$, and since $-|\mathbf{f}| \le \mathbf{f} \le |\mathbf{f}|$, we have $\mathbf{f} \in J$. ∎

Lemma 3.3. *Let σ be a k-dimensional cone of Σ, and let $v_1, \dots, v_p \in \sigma \cap Z\Sigma$. Let V denote the $\mathbb{R}$-vector space spanned by $v_1, \dots, v_p$. Then $\sigma \cap Z\Sigma \supseteq \sigma \cap V$.*

Proof. Every $\mathbf{f} \in H_\Sigma$ is homogeneous linear on σ. ∎

Lemma 3.4. *If Δ is a unimodular subdivision of Σ, then $Z\Delta \subseteq Z\Sigma$.*

Proof. $H_\Sigma \subseteq H_\Delta$, and hence $H_\Sigma \cap J \subseteq H_\Delta \cap J$. ∎

LEMMA 3.5. *Let Σ be such that $Z\Sigma$ is a union of cones of Σ. Then the set*

$$\{\mathbf{u}/J : \mathbf{u} \text{ is a Schauder hat of } \Sigma \text{ and } \mathbf{u} \notin J\}$$

is linearly independent over $\mathbb{Z}$ in $\mathrm{F}\ell(n)/J$.

PROOF. Let $\mathbf{f} = a_1\mathbf{u}_1 + \cdots + a_t\mathbf{u}_t$, where $a_1, \dots, a_t \in \mathbb{Z}$ and $\mathbf{u}_1, \dots, \mathbf{u}_t$ are Schauder hats of Σ at distinct vertices of Σ. Assume $a_1 \neq 0$, $\mathbf{u}_1 \notin J$: we will show that $\mathbf{f} \notin J$. Since $u_1 \notin J$, by Lemma 3.2 there exists $v \in Z\Sigma \setminus Z\mathbf{u}_1$. Let σ be the cone of Σ to whose relative interior v belongs; as $v \in Z\Sigma$, and $Z\Sigma$ is a union of cones of Σ, we have $\sigma \subseteq Z\Sigma$. Since $\mathbf{u}_1$ is not identically 0 on σ, u_1 must be a vertex of σ. Since $\mathbf{f}(u_1) = a_1\mathbf{u}_1(u_1) = a_1 \neq 0$, then $Z\mathbf{f} \not\supseteq Z\Sigma$, and hence $\mathbf{f} \notin J$. ∎

THEOREM 3.6. *Let Σ be a complete unimodular fan, $|\Sigma| = \mathbb{R}^3$. Let J be an ℓ-ideal of $\mathrm{F}\ell(3)$. Then there exists a subdivision Δ of Σ with the following two properties:*

(i) Δ can be obtained from Σ via a finite sequence of starrings along 2-dimensional cones;
(ii) $Z\Delta$ is a union of cones of Δ.

4. Proof of Theorem 3.6. Let $\sigma \in \Sigma$, $k \in \{2,3\}$, $1 \leq j \leq k$. We say that σ *is of type (k, j) with respect to* Σ iff the following hold:

(1) σ is k-dimensional;
(2) $\sigma \cap Z\Sigma$ is j-dimensional;
(3) $\operatorname{rel\,int} \sigma \cap Z\Sigma \neq \emptyset$.

If either $\operatorname{rel\,int} \sigma \cap Z\Sigma = \emptyset$, or σ is 0- or 1-dimensional, then σ is of *no type*. Note that (3) is equivalent to:

(3′) $\sigma \cap Z\Sigma$ is not contained in a proper face of Σ.

Indeed, (3) clearly implies (3′), while the reverse direction follows from Lemma 3.3.

DEFINITION 4.1. For every $\sigma \in \Sigma$ of type (k, j), and every unordered pair u, v of distinct vertices of σ, we define the *badness* of (σ, u, v), denoted by $\mathrm{bad}_\Sigma(\sigma, u, v)$, as follows:

(a) if $k = j = 3$, or $k = j = 2$, then $\mathrm{bad}_\Sigma(\sigma, u, v) = \infty$.
(b) if $k = 3$ and $j = 2$, then there exists a unique (up to multiplication by -1) primitive linear functional $\mathbf{f} \in \mathrm{Hom}(\mathbb{Z}^3, \mathbb{Z})$ such that $\sigma \cap Z\mathbf{f} = \sigma \cap Z\Sigma$. Set
$$\mathrm{bad}_\Sigma(\sigma, u, v) = \begin{cases} 0, & \text{if } \mathbf{f}(u) \cdot \mathbf{f}(v) \geq 0; \\ |\mathbf{f}(u)| + |\mathbf{f}(v)|, & \text{otherwise.} \end{cases}$$
(c) if $j = 1$, then there exists a unique primitive $w \in \mathbb{Z}^3$ such that $\sigma \cap Z\Sigma = \langle w \rangle$. $w \in \operatorname{rel\,int} \sigma$, and can be written uniquely as a linear combination of the vertices of σ with integer coefficients > 0. Set $\mathrm{bad}_\Sigma(\sigma, u, v) = a + b$, where a, b are the coefficients of u, v in the above expression for w.

We shall use induction on eight parameters, ordered lexicographically from left to right as follows:

$$s_0(3,3), s_1(3,2), s_0(3,2), s_1(3,1), s_0(3,1), s_0(2,2), s_1(2,1), s_0(2,1).$$

These parameters are defined by:

- $s_0(k,k)$ = number of cones of type (k,k);
- for $j<k$, $s_1(k,j)=\sup\{\mathrm{bad}_\Sigma(\sigma,u,v) : \sigma$ is of type $(k,j)\}$;
- for $j<k$, $s_0(k,j)$ = number of triples (σ,u,v) such that σ is of type (k,j) and $\mathrm{bad}_\Sigma(\sigma,u,v)=s_1(k,j)$ (triples are unordered, so $(\sigma,u,v)=(\sigma,v,u)$).

Note that, for $j<k$, we have $s_1(k,j)=0$ iff $s_0(k,j)=0$ iff Σ contains no cones of type (k,j).

LEMMA 4.2. *If, for every $1\le j<k\in\{2,3\}$, Σ contains no cones of type (k,j), then $Z\Sigma$ is a union of cones of Σ.*

PROOF. Let $u\in Z\Sigma$, and let σ be the cone of Σ to whose relative interior u belongs. If σ is k-dimensional, then by our assumption $\sigma\cap Z\Sigma$ must be k-dimensional, too. By Lemma 3.3, $\sigma\subseteq Z\Sigma$. ∎

We equip types with the following order:

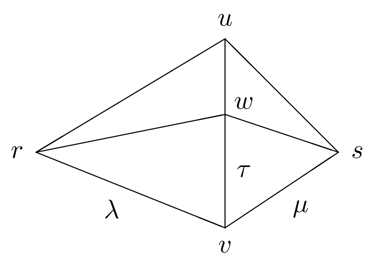
$$(3,3)\succ(3,2)\succ(3,1)\succ(2,2)\succ(2,1)\succ \text{no type}.$$

LEMMA 4.3. *Let Σ,Δ be complete unimodular fans, and assume that Δ is a subdivision of Σ. Let $\delta\in\Delta$, and let $\sigma=\bigcap\{\tau\in\Sigma : \delta\subseteq\tau\}$. Then the type of δ in Δ is less than or equal to the type of σ in Σ. If $\delta=\sigma$ and the two types are equal, then $bad_\Delta(\sigma)=bad_\Sigma(\sigma)$.*

PROOF. Clear from the definitions. ∎

LEMMA 4.4. *Suppose Σ contains a cone of type (k,j), with $1\le j<k\in\{2,3\}$; let $s_0(3,3),\dots,s_0(2,1)$ be the parameters associated to Σ. Then there exists a starring of Σ along a 2-dimensional cone such that—letting Σ' be the resulting fan and $s_0'(3,3),\dots,s_0'(2,1)$ its associated parameters—the 8-tuple $(s_0'(3,3),\dots,s_0'(2,1))$ is strictly smaller than $(s_0(3,3),\dots,s_0(2,1))$ in the lexicographic ordering.*

PROOF. Let (k,j) be the largest type (with respect to $\succ$) such that $s_1(k,j)\neq 0$. Let $\sigma\in\Sigma$ be of type (k,j), and let u,v be vertices of σ with $\mathrm{bad}_\Sigma(\sigma,u,v)=s_1(k,j)$. We star Σ along $\tau=\langle u,v\rangle$, obtaining Σ'; let $w=u+v$. τ is the intersection of two uniquely determined 3-dimensional cones $\lambda,\mu\in\Sigma$. If $k=2$, then $\sigma=\tau$, while if $k=3$, we assume $\sigma=\lambda$. Upon taking a section of these cones by a suitable plane we obtain the following picture:



The starring replaces:

- λ with $\lambda'=\langle u,w,r\rangle$, $\lambda''=\langle w,v,r\rangle$, and $\langle w,r\rangle$;

- τ with $\tau' = \langle u, w\rangle$, $\tau'' = \langle w, v\rangle$, and $\langle w\rangle$;
- μ with $\mu' = \langle u, w, s\rangle$, $\mu'' = \langle w, v, s\rangle$, and $\langle w, s\rangle$.

We proceed arguing by cases:

Case 1. $k = 3$, $j = 2$, $\sigma = \lambda$. Then $\sigma \cap Z\Sigma = \sigma \cap Z\mathbf{f}$, for some primitive $\mathbf{f} \in \mathrm{Hom}(\mathbb{Z}^3, \mathbb{Z})$. Without loss of generality, $\mathbf{f}(u) < 0$. Also, $\tau \cap Z\Sigma = \tau \cap Z\mathbf{f} = \langle p\rangle$, with $p \in \mathrm{rel\,int}\,\tau$; by Lemmas 3.3 and 4.3, each of $\mu, \lambda', \lambda'', \mu', \mu''$ is of type $\preceq (3, 2)$. By Lemma 4.3, the parameter $s_0(3, 3)$ cannot increase. If it decreases, we are through. Otherwise, it suffices to prove that, for every $\rho \in \{\lambda', \lambda'', \mu', \mu''\}$, if ρ is of type $(3, 2)$, u', v' are vertices of ρ, and $\langle u', v'\rangle$ is not a cone of Σ, then $\mathrm{bad}_{\Sigma'}(\rho, u', v') < \mathrm{bad}_\Sigma(\sigma, u, v)$. To this purpose, we again distinguish:

Subcase 1.1. $\rho = \lambda'$, $u' = u$, $v' = w$. By Lemma 4.3, together with the assumption that λ' is of type $(3, 2)$, in Σ', we have $\lambda' \cap Z\Sigma' = \lambda' \cap Z\mathbf{f}$. If $p \notin \mathrm{rel\,int}\langle u, w\rangle$, then $\mathrm{bad}_{\Sigma'}(\rho, u', v') = 0$. Otherwise, $\mathbf{f}(u) < 0 < \mathbf{f}(w) < \mathbf{f}(v)$, and $\mathrm{bad}_{\Sigma'}(\rho, u', v') = |\mathbf{f}(u)| + |\mathbf{f}(w)| < |\mathbf{f}(u)| + |\mathbf{f}(v)| = \mathrm{bad}_\Sigma(\sigma, u, v)$.

Subcase 1.2. $\rho = \lambda'$, $u' = r$, $v' = w$. Again, $\lambda' \cap Z\Sigma' = \lambda' \cap Z\mathbf{f}$. We may assume $\mathbf{f}(r) \cdot \mathbf{f}(w) < 0$ for, otherwise, $\mathrm{bad}_{\Sigma'}(\lambda', r, w) = 0$. Either $\mathbf{f}(u) < 0 < \mathbf{f}(w) < \mathbf{f}(v)$, or $\mathbf{f}(u) < \mathbf{f}(w) < 0 < \mathbf{f}(v)$. In the first case, $\mathrm{bad}_{\Sigma'}(\lambda', r, w) = |\mathbf{f}(r)| + |\mathbf{f}(w)| < |\mathbf{f}(r)| + |\mathbf{f}(v)| = \mathrm{bad}_\Sigma(\sigma, r, v) \le \mathrm{bad}_\Sigma(\sigma, u, v)$. In the second case, $\mathrm{bad}_{\Sigma'}(\lambda', r, w) = |\mathbf{f}(r)| + |\mathbf{f}(w)| < |\mathbf{f}(r)| + |\mathbf{f}(u)| = \mathrm{bad}_\Sigma(\sigma, r, u) \le \mathrm{bad}_\Sigma(\sigma, u, v)$.

Subcase 1.3. $\rho = \mu'$, $u' = u$, $v' = w$. Then μ is of type $(3, 2)$ in Σ. Let $\mathbf{g} \in \mathrm{Hom}(\mathbb{Z}^3, \mathbb{Z})$ be primitive satisfying $\mu \cap Z\mathbf{g} = \mu \cap Z\Sigma$; then $\mu' \cap Z\mathbf{g} = \mu' \cap Z\Sigma'$ and $\mathbf{g}(p) = 0$. If $p \notin \mathrm{rel\,int}\langle u, w\rangle$, then $\mathrm{bad}_{\Sigma'}(\mu', u, w) = 0$. Otherwise, without loss of generality, $\mathbf{g}(u) < 0 < \mathbf{g}(w) < \mathbf{g}(v)$, and $\mathrm{bad}_{\Sigma'}(\mu', u, w) = |\mathbf{g}(u)| + |\mathbf{g}(w)| < |\mathbf{g}(u)| + |\mathbf{g}(v)| = \mathrm{bad}_\Sigma(\mu, u, v) \le s_1(3, 2)$.

Subcase 1.4. $\rho = \mu'$, $u' = s$, $v' = w$. Then arguing as for Subcase 1.2 one gets the desired conclusion.

Subcase 1.5. $\rho = \lambda''$, or $\rho = \mu''$. The proof is the same as for the previous subcases.

Case 2. $k = 3$, $j = 1$, $\sigma = \lambda$. By our choice of (k, j), Σ contains no cones of type $(3, 2)$; by Lemmas 3.3 and 4.3, each of $\mu, \lambda', \lambda'', \mu', \mu''$ is either of type $(3, 1)$, or of no type. If either parameter $s_0(3, 3), s_1(3, 2), s_0(3, 2)$ happens to change, then, by Lemma 4.3, $s_0(3, 3)$ must decrease—and we are done. If, on the other hand, none of these parameters changes, then to obtain the desired conclusion it suffices to prove that, for every $\rho \in \{\lambda', \lambda'', \mu', \mu''\}$, if ρ is of type $(3, 1)$, u', v' are vertices of ρ, and $\langle u', v'\rangle \notin \Sigma$, then $\mathrm{bad}_{\Sigma'}(\rho, u', v') < \mathrm{bad}_\Sigma(\sigma, u, v)$. We only consider the case $\rho = \lambda'$, the other cases being similar. If $\rho = \lambda'$, then $\sigma \cap Z\Sigma = \lambda' \cap Z\Sigma' = \langle p\rangle$, for a uniquely determined primitive $p \in \mathbb{Z}^3$. We have $p = au + bv + cr$, for uniquely determined $a, b, c \in \mathbb{Z}^+ \setminus \{0\}$. Since $p \in \mathrm{rel\,int}\,\lambda'$, we have $a > b$. Also, $p = (a - b)u + bw + cr$, uniquely. Since $\langle u', v'\rangle \notin \Sigma$, we may assume $v' = w$. If $u' = u$, then $\mathrm{bad}_{\Sigma'}(\lambda', u', v') = (a - b) + b < a + b = \mathrm{bad}_\Sigma(\sigma, u, v)$. If $u' = r$, then $\mathrm{bad}_{\Sigma'}(\lambda', u', v') = b + c < a + c = \mathrm{bad}_\Sigma(\sigma, u, r) \le \mathrm{bad}_\Sigma(\sigma, u, v)$.

Case 3. $k = 2$, $j = 1$, $\sigma = \tau$. This is even simpler than Case 2. It suffices to observe that, by our choice of (k, j), Σ contains no cones of either type $(3, 2)$ or $(3, 1)$. Also, each of $\lambda, \mu, \lambda', \lambda'', \mu', \mu''$ is of no type. In case one of the parameters $\succ s_1(2, 1)$ happens to change, then, by Lemma 4.3, the greatest such changing parameter must actually decrease. If, on the other hand, no parameter $\succ s_1(2,1)$ does change, then the same argument as in Case 2 yields the desired conclusion. ∎

The proof of Theorem 3.6 is now complete: as a matter of fact, let a complete unimodular fan Σ be given, $|\Sigma| = \mathbb{R}^3$. If $Z\Sigma$ is not a union of cones of Σ, then, by Lemma 4.2, Σ contains a cone of type (k,j), for some $j < k$. Using Lemma 4.4, we have a chain of starrings $\Sigma > \Sigma' > \Sigma'' > \cdots$, which must terminate, since the set of 8-tuples of parameters is well ordered; say it stops at Δ. Then Δ contains no cones of type (k,j) for $k > j$, and Lemma 4.2 yields the desired conclusion.

5. Conclusion

THEOREM 5.1. *Every ℓ-group G with three generators is ultrasimplicial.*

PROOF. We can safely identify G with the quotient ℓ-group $\mathrm{F}\ell(3)/J$, for some ℓ-ideal J. Let $\mathbf{f}_1/J, \ldots, \mathbf{f}_r/J \in (\mathrm{F}\ell(3)/J)^+$. Replacing each $\mathbf{f}_i$ by $\mathbf{f}_i \vee \mathbf{0}$, and deleting all elements annihilated by the quotient map, we may assume $\mathbf{f}_1, \ldots, \mathbf{f}_r \in \mathrm{F}\ell(3)^+ \setminus J$. Each $\mathbf{f}_i$ is of the form

$$\mathbf{f}_i = \bigvee_{s \in S} \bigwedge_{t \in T} \mathbf{g}^i_{st}$$

where S, T are finite index sets, and each $\mathbf{g}^i_{st}$ is in $\mathrm{Hom}(\mathbb{Z}^3, \mathbb{Z})$. Let $\mathbf{g}_1, \ldots, \mathbf{g}_k$ display all $\mathbf{g}^i_{st}$, for $1 \le i \le r$. For every permutation φ of $\{1, \ldots, k\}$, let

$$\sigma_\varphi = \{x \in \mathbb{R}^3 : \mathbf{g}_{\varphi(1)}(x) \le \mathbf{g}_{\varphi(2)}(x) \le \cdots \le \mathbf{g}_{\varphi(k)}(x)\}.$$

Then a routine argument shows that each σ_φ is a polyhedral cone and that the set Γ of all faces of all σ_φ's is a complete fan; moreover, every $\mathbf{f}_i$ is linear on every cone of Γ.

We make three successive refinements. The first two are standard constructions:

- firstly, we refine Γ to a fan Π whose cones are all simplicial. This can be accomplished without introducing new vertices, following [Ewa96, Theorem 4.2];
- secondly, we refine Π to a complete unimodular fan Σ, as in the proof of [Ewa96, Theorem 8.5].

Thirdly, in the light of Theorem 3.6,

- we refine Σ to a fan Δ such that $Z\Delta$ (relative to the ideal J) is a union of cones of Δ.

Let $u_1, \ldots, u_t$ be the vertices of Δ, and $\mathbf{u}_1, \ldots, \mathbf{u}_t$ their associated Schauder hats. Since, for every i,

$$\mathbf{f}_i = \mathbf{f}_i(u_1) \cdot \mathbf{u}_1 + \cdots + \mathbf{f}_i(u_t) \cdot \mathbf{u}_t,$$

it follows that the set $\{\mathbf{u}_1/J, \ldots, \mathbf{u}_t/J\} \setminus \{\mathbf{0}/J\}$ spans $\mathbf{f}_1/J, \ldots, \mathbf{f}_r/J$ positively and, by Lemma 3.5, is linearly independent over $\mathbb{Z}$ in $\mathrm{F}\ell(3)/J$. Hence condition (U) is satisfied, and the proof is complete. ∎

Final remarks. A moment's reflection shows that a (constructive) proof of the ultrasimplicial property of all n-generated ℓ-groups immediately extends to all ℓ-groups.

Thus it is natural to consider the following question: can the techniques of this paper be extended to n-generated ℓ-groups ?

Theorem 3.6 is the only step in our proof that cannot be immediately generalized to higher dimensions. One can reasonably expect that the n-dimensional generalization of this theorem requires an induction argument over more complicated parameters. As a working hypothesis, for any j-dimensional cone $\sigma \cap Z\Sigma$ lying inside a k-dimensional cone $\sigma \in \Sigma$ natural badness parameters are provided by the Plücker coordinates of $\sigma \cap Z\Sigma$ with respect to the basis given by the j-dimensional faces of σ (see, e.g., [BML67, Chapter XVI] for background). The attentive reader may have noticed that also the parameters used in this paper are based on Plücker coordinates, although in a slightly disguised form.

For an instructive example, let $\sigma = \langle v_1, \ldots, v_4 \rangle$ be a 4-dimensional cone of a fan Σ, and suppose the 2-dimensional cone $\sigma \cap Z\Sigma$ to be positively spanned by the two linearly independent vectors $p, q \in \mathbb{Z}^n \cap \sigma$. Let us construct the exterior algebra $\bigwedge^2 \mathbb{Z}^4$, with basis $v_1 \wedge v_2, \ldots, v_3 \wedge v_4$, and assume that $p \wedge q$ has coordinates $\xi_{12}, \ldots, \xi_{34} \in \mathbb{Z}$ in $\bigwedge^2 \mathbb{Z}^4$, with $\xi_{12}, \ldots, \xi_{34}$ relatively prime. It follows that

$$s = |\xi_{12}| + \cdots + |\xi_{34}|$$

is a natural badness parameter for $\sigma \cap Z\Sigma$ in σ. Indeed, $\sigma \cap Z\Sigma$ coincides with a face of σ exactly when $s = 1$. We must star σ in such a way that s decreases. Starring σ along one of its faces corresponds to a base change in $\bigwedge^2 \mathbb{Z}^4$. Now, while in dimension ≤ 3 every base change corresponds to a starring, this does not hold in higher dimensions; for example, the base change arising from the substitution of $v_1 \wedge v_2 + v_3 \wedge v_4$ for $v_1 \wedge v_2$ does not correspond to any starring, because $v_1 \wedge v_2 + v_3 \wedge v_4$ is not reducible in $\bigwedge^2 \mathbb{Z}^4$. In algebraic-geometric terms, one has to figure out a path of starrings, leading from the base points $v_1 \wedge v_2, \ldots, v_3 \wedge v_4$ to the point $p \wedge q$, and never leaving the Grassmannian of lines in $\mathbb{P}^3$.

Further complications arise from the requirement that the starrings employed to decrease the badness of $\sigma \cap Z\Sigma$ in σ should not result in increasing the badness of $\sigma' \cap Z\Sigma$ in σ', for any $\sigma' \in \Sigma$.

Closing a circle of ideas, as a final source of complication it might well be the case that the n-dimensional generalization of the results of this paper requires that the starring operation should be performed along arbitrary cones of Σ (see [Oda88] or [Ewa96]), rather than only 2-dimensional cones.

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