ON THE EXISTENCE OF PRIME IDEALS IN BOOLEAN ALGEBRAS

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Abstract. Rasiowa and Sikorski [5] showed that in any Boolean algebra there is an ultrafilter preserving countably many given infima. In [3] we proved an extension of this fact and gave some applications. Here, besides further remarks, we present some of these results in a more general setting.

1. Introduction. Let $E$ be a subset and $a$ an element of a Boolean algebra $B$, $E \subseteq B$ and $a \in B$. Assume that $a$ is the infimum of $E$, $a = \bigwedge E$. An ultrafilter $U$ preserves $a = \bigwedge E$, if

$$a \not\in U \text{ implies } e \not\in U \text{ for some } e \in E.$$ 

In the section entitled “A theorem on the existence of prime ideals in Boolean algebras” of their paper “A proof of the completeness theorem of Gödel” (cf. [5]), Rasiowa and Sikorski prove the following theorem which is sometimes (cf. [4]) called the Lemma of Rasiowa and Sikorski.

**Theorem 1.1.** Given infima $a_1 = \bigwedge E_1$, $a_2 = \bigwedge E_2$,... in a non-trivial (i.e., $0 \neq 1$) Boolean algebra there is an ultrafilter preserving all these infima. Since

$$a = \bigwedge E \quad \text{implies} \quad 0 = \bigwedge \{e \cap \sim a \mid e \in E\},$$

this result can be rephrased as:

**Corollary 1.2.** Let $E_1, E_2, ...$ be subsets of a non-trivial Boolean algebra with $0 = \bigwedge E_1 = \bigwedge E_2 = ...$. Then

$$(*) \quad \text{there is an ultrafilter } U \text{ s.t. for all } n \text{ there is } e \in E_n \text{ with } \sim e \in U.$$

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[119]
In [3] we gave necessary and sufficient “absolute” conditions for the existence of an ultrafilter as in this corollary in case we omit the hypothesis $0 = \bigwedge E_1 = \bigwedge E_2 = \ldots$ (As shown by $E_1 = \{a\}$ and $E_2 = \{\sim a\}$ with an arbitrary element $a$ the hypothesis cannot simply be omitted.)

Our result and proof method were inspired by a corresponding characterization of the omissible types of (incomplete) first-order theories contained in [1], rediscovered and applied in [2]. It is well-known that one of the first important applications of the Lemma of Rasiowa and Sikorski is its use by Ryll-Nardzewski to characterize $\omega_0$-categorical theories (cf. [6]). Implicitly, this characterization contains the so-called omitting types theorem.

In this paper we present our results extending the Lemma of Rasiowa and Sikorski in a more general setting.

2. Inflationary and monotone operations. Let $B$ be a set and $J$ an operation on the power set of $B$,

$$J : \text{Pow}(B) \to \text{Pow}(B),$$

that is inflationary and monotone; here inflationary means that

$$X \subseteq J(X),$$

and monotone that

$$X \subseteq Y \text{ implies } J(X) \subseteq J(Y).$$

By transfinite induction one defines the subsets $J_\alpha$ of $B$ by

$$J_0 := \emptyset; \quad J_{\alpha+1} := J(J_\alpha); \quad J_\alpha := \bigcup_{\beta < \alpha} J_\beta.$$

Then,

$$J_\infty := \bigcup_{\alpha} J_\alpha$$

is the least fixed-point of $J$, i.e.,

$$J(J_\infty) = J_\infty \quad \text{and} \quad J(X) = X \text{ implies } J_\infty \subseteq X.$$

If $\kappa$ is an infinite cardinal, we say that $J$ is $\kappa$-ary, if

$$J(X) = \bigcup \{J(X_0) \mid X_0 \subseteq X \text{ and } |X_0| < \kappa\}$$

(Here $|Y|$ denotes the cardinality of $Y$).

Now let $I$ be a set and for $i \in I$ let $J^i$ be an inflationary and monotone operation on the power set of $B$. Define the union of $J^I$ of the $J^i$’s,

$$J^I : \text{Pow}(B) \to \text{Pow}(B),$$

by

$$J^I(X) := \bigcup \{J^i(X) \mid i \in I\}.$$

Clearly, $J^I$ is inflationary and monotone. Moreover,

1. Every fixed-point of $J^I$ is a fixed-point of each $J^i$; in particular, $J^I_\infty$ is a fixed-point of each $J^i$. 

As a corollary we get:

\[ B \]

Hence,

\[ I \]

(2) If each \( J^i \) is \( \kappa \)-ary, then so is \( J^I \) and, for any \( \alpha \),

\[
J^I_\alpha = \bigcup \{ J^I_\beta \mid I_0 \subseteq I \text{ and } |I_0| < \kappa \}
\]

(here, \( J^I_\beta \) is the union of the \( J^i \)'s with \( i \in I_0 \)).

\textbf{Proof.} Clearly, the equality holds for \( \alpha = 0 \). For \( \alpha = \beta + 1 \) we have

\[
J^I_{\beta+1} := J^I(J^I_\beta) = \bigcup_{i \in I} J^I(J^I_\beta)
\]

\[
= \bigcup_{i \in I} \bigcup_{I_0 \subseteq I, |I_0| < \kappa} J^I(J^I_\beta)
\]

\[
= \bigcup_{I_0 \subseteq I, |I_0| < \kappa} J^I_\beta = \bigcup_{I_0 \subseteq I, |I_0| < \kappa} J^I_\alpha + 1
\]

(in deriving the first equality in the last line note that \( I_1 \subseteq I_2 \) implies \( J^I_1(X) \subseteq J^I_2(X) \)). If \( \alpha \) a limit ordinal then

\[
J^I_\alpha = \bigcup_{\beta < \alpha} J^I_\beta = \bigcup_{I_0 \subseteq I, |I_0| < \kappa} J^I_\beta = \bigcup_{I_0 \subseteq I, |I_0| < \kappa} J^I_\alpha.
\]

As a corollary we get:

(3) If each \( J^i \) is \( \kappa \)-ary, then \( J^I_\infty = \bigcup \{ J^I_\beta \mid I_0 \subseteq I \text{ and } |I_0| < \kappa \} \).

Hence,

(4) If each \( J^i \) is \( \kappa \)-ary, then for \( a \in B \),

\[
a \in J^I_\infty \quad \text{iff} \quad a \in J^I_\alpha \quad \text{for some } I_0 \subseteq I \text{ with } |I_0| < \kappa.
\]

3. The generalization of the Lemma of Rasiowa and Sikorski. Fix a Boolean algebra \( B \). For a subset \( X \) of \( B \) denote by \( F(X) \) the filter generated by \( X \),

\[
F(X) := \{ b \mid \text{there are } n \geq 0, b_0, \ldots, b_n \in X \text{ with } a_0 \cap \ldots \cap a_n \leq b \}.
\]

A filter \( F \) is proper, if \( 0 \notin F \). Henceforth, we shall use the letter \( U \) to denote ultrafilters, i.e., proper filters such that \( a \in U \) or \( \sim a \in U \) for all \( a \in B \).

An ultrafilter \( U \) omits \( E \), if there is \( e \in E \) such that \( \sim e \in U \) (cf. 1.2). Then, we say that \( E \) is omissible. Define \( J := J^E \), \( J : \text{Pow}(B) \to \text{Pow}(B) \), by

\[
J(X) := \{ \sim a \mid E \subseteq F(X \cup \{ a \}) \}.
\]

Clearly,

(5) \( J \) is inflationary and monotone; if \( |E| < \kappa \) then \( J \) is \( \kappa \)-ary.

(6) If \( X \subseteq U \) and \( U \) omits \( E \), then \( J(X) \subseteq U \).

\textbf{Proof.} Assume \( X \subseteq U \), \( U \) omits \( E \), and let \( \sim a \in J(X) \). Then, \( E \subseteq F(X \cup \{ a \}) \subseteq F(U \cup \{ a \}) \). Therefore, \( \sim a \in U \). 

A simple transfinite induction using (6) shows

(7) if \( U \) omits \( E \) then \( J_\infty \subseteq U \).
Moreover,

(8) $J(X) = X \iff X$ is a filter and in the quotient Boolean algebra $B/X$

we have $\bigwedge \overline{E} = 0$

(here $\overline{E} = \{ \bar{e} \mid e \in E \}$, where $\bar{e}$ denotes the equivalence class of $e$).

**Proof.** First, assume the right side of the equivalence. We only must show that $J(X) \subseteq X$. So assume $b \in J(X)$. Then $E \subseteq F(X \cup \{ \sim b \})$. Since $\bigwedge \overline{E} = 0$ in $B/X$, we have $\sim \bar{b} = 0$ in $B/X$, thus $b \in X$.

Now assume $J(X) = X$. Let $x \in X$. Then, $E \subseteq F(X \cup \{ \sim x \})$. If $y \in X$ then $F(X \cup \{ \sim x \}) = F(X \cup \{ \sim x \} \cup \sim y)$, hence, $E \subseteq F(X \cup \{ \sim x \} \cup \sim y)$, thus $x \cap y \in J(X) = X$. If $x \leq y$ then $F(X \cup \{ \sim y \}) \supseteq F(X \cup \{ \sim x \}) \supseteq E$ and therefore, $y \in J(X) = X$. Finally, let $a \in B$, and assume that in $B/X$,

$$\bar{a} \leq \bar{e} \text{ for all } e \in E.$$ 

Then $E \subseteq F(X \cup \{ a \})$, thus, $\sim a \in J(X) = X$, hence, $\bar{a} = 0$. □

Now let $E$ be a non-empty class of subsets of $B$. We say that $E$ is onissible, if there is an ultrafilter $U$ that omits $E$, i.e., that omits each $E$ in $E$. Let $J_E$ be the union of the $J^E$ is for $E \in E$, i.e.,

$$J^E(X) = \bigcup_{E \in E} J^E(X) = \{ \sim a \mid E \subseteq F(X \cup \{ a \}) \} \text{ for some } E \in E \}.$$ 

A transfinite induction, using (7), shows:

(9) If $U$ omits $E$ then $J^E_\infty \subseteq U$.

By (1) and (8) we get

(10) $J^E_\infty$ is a filter and in the quotient Boolean algebra $B/J^E_\infty$ we have $\bigwedge \overline{E} = 0$ for every $E \in E$.

Let $C$ be a class of Boolean algebras and $\lambda$ a cardinal. We say that $C$ is R(asiowa) S(ikorski)(\lambda)-good, if for any non-trivial Boolean algebra $B$ in $C$ and any set $E$, $|E| < \lambda$, of non-empty subsets $E$ of $B$ with $\bigwedge E = 0$, there is an ultrafilter $U$ that omits $E$. The classical Lemma of Rasiowa and Sikorski (cf. 1.2) tells us that the class of all Boolean algebras is RS($\omega_1$)-good. Martin’s axiom is (equivalent to) the statement that the class of all Boolean algebras with the countable chain condition is RS($2^{\omega_1}$)-good (a Boolean algebra satisfies the countable chain condition, if every subset of pairwise disjoint elements is countable). The class of all Boolean algebras is not RS($\omega_1^+$)-good; a counterexample is obtained by choosing an appropriate set $E$ in the Boolean algebra of regular open subsets of the partial order given by the set of partial functions from $\omega$ to $\omega_1$ with finite support (cf. [4]).

**Theorem 3.1.** Let $C$ be a RS($\lambda$)-good class of Boolean algebras closed under quotients. Then, for any Boolean algebra $B$ in $C$ and any family $E$, $|E| < \lambda$, of subsets of $B$,

$E$ is onissible $\iff 0 \notin J^E_\infty$.

**Proof.** If $U$ omits $E$, then $J^E_\infty \subseteq U$ by (9); hence, $0 \notin J^E_\infty$. Otherwise, if $0 \notin J^E_\infty$ then, by (8), $J^E_\infty$ is a proper filter, $B/J^E_\infty$ is a non-trivial Boolean algebra, and, in $B/J^E_\infty$,
we have $\bigwedge E = 0$ for all $E \in \mathcal{E}$. Hence, by the assumption of RS($\lambda$)-goodness there is an ultrafilter $U$ in $B/J^E_\infty$ that omits $\{E \mid E \in \mathcal{E}\}$. Therefore, $U^{-1} := \{b \in B \mid b \in U\}$ is an ultrafilter omitting $\mathcal{E}$.

Recall that a Boolean algebra $B$ is retractive, if for every proper filter $F$ in $B$ there is a homomorphism $f$ from $B/F$ to $B$ such that $\pi \circ f$ is the identity on $B/F$ (here, $\pi$ denotes the canonical homomorphism from $B$ onto $B/F$). Clearly,

if $B$ is retractive and has the ccc, then every quotient of $B$ has the ccc.

Every interval algebra and every tree algebra is retractive (see [4]). Hence, we obtain from the preceding theorem (taking as $C$ the class of interval algebras (or, the class of tree algebras) with ccc):

**Corollary 3.2.** Assume Martin’s axiom and let $B$ be an interval algebra or a tree algebra with the countable chain condition. Furthermore, let $\mathcal{E}, |\mathcal{E}| < 2^\omega$, be a family of subsets of $B$. Then $\mathcal{E}$ is omissible iff $0 \notin J^E_\infty$.

**Theorem 3.3.** Let $C$ be a RS($\lambda$)-good class of Boolean algebras closed under quotients. For $B$ in $C$ and any family $\mathcal{E}, |\mathcal{E}| < \lambda$, of subsets $E$ of $B$ with $|E| < \kappa$ the following holds: if every subfamily of $\mathcal{E}$ of cardinality less than $\kappa$ is omissible, then $\mathcal{E}$ is omissible.

**Proof.** Let $\mathcal{E}_0$ be an arbitrary subfamily of $\mathcal{E}$ of cardinality less than $\kappa$. Since $\mathcal{E}_0$ is omissible, $0 \notin J^E_\infty$ by (9). As $J^E$ is $\kappa$-ary (cf. (5) and (2)), we have by (3), $0 \notin J^E_\infty$. Hence, by the previous theorem, $\mathcal{E}$ is omissible.

An instance of this theorem is:

**Corollary 3.4.** Assume Martin’s axiom and let $\mathcal{E}, |\mathcal{E}| < 2^\omega$, be a family of countable subsets of an interval algebra or of a tree algebra with the countable chain condition. If every countable subfamily of $\mathcal{E}$ is omissible, then $\mathcal{E}$ is omissible.

**References**