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ON THE EXISTENCE OF PRIME IDEALS IN BOOLEAN ALGEBRAS

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Abstract. Rasiowa and Sikorski [5] showed that in any Boolean algebra there is an ultrafilter preserving countably many given infima. In [3] we proved an extension of this fact and gave some applications. Here, besides further remarks, we present some of these results in a more general setting.

1. Introduction. Let E be a subset and a an element of a Boolean algebra $\mathcal{B}$, $E \subseteq B$ and $a \in B$. Assume that a is the infimum of E, $a = \bigwedge E$. An ultrafilter U *preserves* $a = \bigwedge E$, if

$$a \notin U \text{ implies } e \notin U \text{ for some } e \in E.$$

In the section entitled "A theorem on the existence of prime ideals in Boolean algebras" of their paper "A proof of the completeness theorem of Gödel" (cf. [5]), Rasiowa and Sikorski prove the following theorem which is sometimes (cf. [4]) called *the* Lemma of Rasiowa and Sikorski.

Theorem 1.1. *Given infima $a_1 = \bigwedge E_1$, $a_2 = \bigwedge E_2, \ldots$ in a non-trivial (i.e., $0 \neq 1$) Boolean algebra there is an ultrafilter preserving all these infima.*

Since

$$a = \bigwedge E \quad \text{implies} \quad 0 = \bigwedge \{e \cap \sim a \mid e \in E\},$$

this result can be rephrased as:

Corollary 1.2. *Let $E_1, E_2, \ldots$ be subsets of a non-trivial Boolean algebra with $0 = \bigwedge E_1 = \bigwedge E_2 = \ldots$. Then*

$(*)$ *there is an ultrafilter U s.t. for all n there is $e \in E_n$ with $\sim e \in U$.*

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In [3] we gave necessary and sufficient "absolute" conditions for the existence of an ultrafilter as in this corollary in case we omit the hypothesis $0 = \bigwedge E_1 = \bigwedge E_2 = \ldots$. (As shown by $E_1 = \{a\}$ and $E_2 = \{\sim a\}$ with an arbitrary element a the hypothesis cannot simply be omitted.)

Our result and proof method were inspired by a corresponding characterization of the omissible types of (incomplete) first-order theories contained in [1], rediscovered and applied in [2]. It is well-known that one of the first important applications of the Lemma of Rasiowa and Sikorski is its use by Ryll-Nardzewski to characterize ω_0-categorical theories (cf. [6]). Implicitly, this characterization contains the so-called omitting types theorem.

In this paper we present our results extending the Lemma of Rasiowa and Sikorski in a more general setting.

2. Inflationary and monotone operations. Let B be a set and J an operation on ther power set of B,

$$J : \mathrm{Pow}(B) \to \mathrm{Pow}(B),$$

that is inflationary and monotone; here *inflationary* means that

$$X \subseteq J(X),$$

and *monotone* that

$$X \subseteq Y \quad \text{implies} \quad J(X) \subseteq J(Y).$$

By transfinite induction one defines the subsets J_α of B by

$$J_0 := \emptyset; \quad J_{\alpha+1} := J(J_\alpha); \quad J_\alpha := \bigcup_{\beta<\alpha} J_\beta.$$

Then,

$$J_\infty := \bigcup_\alpha J_\alpha$$

is the *least fixed-point* of J, i.e.,

$$J(J_\infty) = J_\infty \quad \text{and} \quad J(X) = X \text{ implies } J_\infty \subseteq X.$$

If κ is an infinite cardinal, we say that J is *κ-ary*, if

$$J(X) = \bigcup \{J(X_0) \mid X_0 \subseteq X \text{ and } |X_0| < \kappa\}$$

(here $|Y|$ denotes the cardinality of Y).

Now let I be a set and for $i \in I$ let J^i be an inflationary and monotone operation on the power set of B. Define the *union* of J^I of the J^i's,

$$J^I : \mathrm{Pow}(B) \to \mathrm{Pow}(B),$$

by

$$J^I(X) \quad := \quad \bigcup \{J^i(X) \mid i \in I\}.$$

Clearly, J^I is inflationary and monotone. Moreover,

(1) Every fixed-point of J^I is a fixed-point of each J^i; in particular, J^I_∞ is a fixed-point of each J^i.

Proof. Assume that $J^I(X) = X$. Since J^i is inflationary, we have $X \subseteq J^i(X) \subseteq J^I(X) = X$. ■

(2) If each J^i is κ-ary, then so is J^I and, for any α,

$$J^I_\alpha = \bigcup\{J^{I_0}_\alpha \mid I_0 \subseteq I \text{ and } |I_0| < \kappa\}$$

(here, J^{I_0} is the *union* of the J^i's with $i \in I_0$).

Proof. Clearly, the equality holds for $\alpha = 0$. For $\alpha = \beta + 1$ we have

$$\begin{aligned} J^I_{\beta+1} &:= J^I(J^I_\beta) = \bigcup_{i\in I} J^i(J^I_\beta) \\ &= \bigcup_{i\in I} \bigcup_{I_0\subseteq I,\, |I_0|<\kappa} J^i(J^{I_0}_\beta) \\ &= \bigcup_{I_0\subseteq I,\, |I_0|<\kappa} J^{I_0}(J^{I_0}_\beta) = \bigcup_{I_0\subseteq I,\, |I_0|<\kappa} J^{I_0}_{\beta+1} \end{aligned}$$

(in deriving the first equality in the last line note that $I_1 \subseteq I_2$ implies $J^{I_1}(X) \subseteq J^{I_2}(X)$). If α a limit ordinal then

$$J^I_\alpha = \bigcup_{\beta<\alpha} J^I_\beta = \bigcup_{I_0\subseteq I,\, |I_0|<\kappa} \bigcup_{\beta<\alpha} J^{I_0}_\beta = \bigcup_{I_0\subseteq I,\, |I_0|<\kappa} J^{I_0}_\alpha. \ ■$$

As a corollary we get:

(3) If each J^i is κ-ary, then $J^I_\infty = \bigcup\{J^{I_0}_\infty \mid I_0 \subseteq I \text{ and } |I_0| < \kappa\}$.

Hence,

(4) If each J^i is κ-ary, then for $a \in B$,

$$a \in J^I_\infty \quad \text{iff} \quad a \in J^{I_0}_\infty \text{ for some } I_0 \subseteq I \text{ with } |I_0| < \kappa.$$

3. The generalization of the Lemma of Rasiowa and Sikorski. Fix a Boolean algebra $\mathcal{B}$. For a subset X of B denote by $F(X)$ the filter generated by X,

$$F(X) \quad := \quad \{b \mid \text{there are } n \geq 0,\, a_0, \dots, a_n \in X \text{ with } a_0 \cap \ldots \cap a_n \leq b\}.$$

A filter F is *proper*, if $0 \notin F$. Henceforth, we shall use the letter U to denote ultrafilters, i.e., proper filters such that $a \in U$ or $\sim a \in U$ for all $a \in B$.

An ultrafilter U *omits* E, if there is $e \in E$ such that $\sim e \in U$ (cf. 1.2). Then, we say that E is *omissible*. Define $J\,(= J^E)$, $J : \mathrm{Pow}(B) \to \mathrm{Pow}(B)$, by

$$J(X) \quad := \quad \{\sim a \mid E \subseteq F(X \cup \{a\})\}.$$

Clearly,

(5) J is inflationary and monotone; if $|E| < \kappa$ then J is κ-ary.
(6) If $X \subseteq U$ and U omits E, then $J(X) \subseteq U$.

Proof. Assume $X \subseteq U$, U omits E, and let $\sim a \in J(X)$. Then, $E \subseteq F(X \cup \{a\}) \subseteq F(U \cup \{a\})$. Therefore, $\sim a \in U$. ■

A simple transfinite induction using (6) shows

(7) if U omits E then $J_\infty \subseteq U$.

Moreover,

(8) $J(X) = X$ iff X is a filter and in the quotient Boolean algebra $\mathcal{B}/X$ we have $\bigwedge \overline{E} = 0$

(here $\overline{E} = \{\bar{e} \mid e \in E\}$, where $\bar{e}$ denotes the equivalence class of e).

PROOF. First, assume the right side of the equivalence. We only must show that $J(X) \subseteq X$. So assume $b \in J(X)$. Then $E \subseteq F(X \cup \{\sim b\})$. Since $\bigwedge \overline{E} = 0$ in $\mathcal{B}/X$, we have $\overline{\sim b} = 0$ in $\mathcal{B}/X$, thus $b \in X$.

Now assume $J(X) = X$. Let $x \in X$. Then, $E \subseteq F(X \cup \{\sim x\})$. If $y \in X$ then $F(X \cup \{\sim x\}) = F(X \cup \{\sim x \cup \sim y\})$, hence, $E \subseteq F(X \cup \{\sim (x \cap y)\})$, thus $x \cap y \in J(X) = X$. If $x \leq y$ then $F(X \cup \{\sim y\}) \supseteq F(X \cup \{\sim x\}) \supseteq E$ and therefore, $y \in J(X) = X$. Finally, let $a \in B$, and assume that in $\mathcal{B}/X$,

$$\bar{a} \leq \bar{e} \quad \text{for all } e \in E.$$

Then $E \subseteq F(X \cup \{a\})$, thus, $\sim a \in J(X) = X$, hence, $\bar{a} = 0$. ■

Now let $\mathcal{E}$ be a non-empty class of subsets of B. We say that $\mathcal{E}$ is *onissible*, if there is an ultrafilter U that *omits* $\mathcal{E}$, i.e., that omits each E in $\mathcal{E}$. Let $J^{\mathcal{E}}$ be the union of the J^E's for $E \in \mathcal{E}$, i.e.,

$$J^{\mathcal{E}}(X) = \bigcup_{E \in \mathcal{E}} J^E(X) = \{\sim a \mid E \subseteq F(X \cup \{a\}) \text{ for some } E \in \mathcal{E}\}.$$

A transfinite induction, using (7), shows:

(9) If U omits $\mathcal{E}$ then $J^{\mathcal{E}}_\infty \subseteq U$.

By (1) and (8) we get

(10) $J^{\mathcal{E}}_\infty$ is a filter and in the quotient Boolean algebra $\mathcal{B}/J^{\mathcal{E}}_\infty$ we have $\bigwedge \overline{E} = 0$ for every $E \in \mathcal{E}$.

Let $\mathcal{C}$ be a class of Boolean algebras and λ a cardinal. We say that $\mathcal{C}$ is R(asiowa) S(ikorski)(λ)-*good*, if for any non-trivial Boolean algebra $\mathcal{B}$ in $\mathcal{C}$ and any set $\mathcal{E}$, $|\mathcal{E}| < \lambda$, of non-empty subsets E of B with $\bigwedge E = 0$, there is an ultrafilter U that omits $\mathcal{E}$. The classical Lemma of Rasiowa and Sikorski (cf. 1.2) tells us that the class of all Boolean algebras is RS(ω_1)-good. Martin's axiom is (equivalent to) the statement that the class of all Boolean algebras with the countable chain condition is RS(2^ω)-good (a Boolean algebra satisfies the *countable chain condition*, if every subset of pairwise disjoint elements is countable). The class of all Boolean algebras is not RS(ω_1^+)-good; a counterexample is obtained by choosing an appropriate set $\mathcal{E}$ in the Boolean algebra of regular open subsets of the partial order given by the set of partial functions from ω to ω_1 with finite support (cf. [4]).

THEOREM 3.1. *Let $\mathcal{C}$ be a* RS(λ)*-good class of Boolean algebras closed under quotients. Then, for any Boolean algebra $\mathcal{B}$ in $\mathcal{C}$ and any family $\mathcal{E}$, $|\mathcal{E}| < \lambda$, of subsets of B,*

$$\mathcal{E} \text{ is omissible} \quad \text{iff} \quad 0 \notin J^{\mathcal{E}}_\infty.$$

PROOF. If U omits $\mathcal{E}$, then $J^{\mathcal{E}}_\infty \subseteq U$ by (9); hence, $0 \notin J^{\mathcal{E}}_\infty$. Otherwise, if $0 \notin J^{\mathcal{E}}_\infty$ then, by (8), $J^{\mathcal{E}}_\infty$ is a proper filter, $\mathcal{B}/J^{\mathcal{E}}_\infty$ is a non-trivial Boolean algebra, and, in $\mathcal{B}/J^{\mathcal{E}}_\infty$, we have $\bigwedge \overline{E} = 0$ for all $E \in \mathcal{E}$. Hence, by the assumption of RS(λ)-goodness there is an ultrafilter U in $\mathcal{B}/J^{\mathcal{E}}_{\infty}$ that omits $\{\overline{E} \mid E \in \mathcal{E}\}$. Therefore, $U^{-1} := \{b \in B \mid \bar{b} \in U\}$ is an ultrafilter omitting $\mathcal{E}$. ∎

Recall that a Boolean algebra $\mathcal{B}$ is *retractive*, if for every proper filter F in $\mathcal{B}$ there is a homomorphism f from $\mathcal{B}/F$ to $\mathcal{B}$ such that $\pi \circ f$ is the identity on $\mathcal{B}/F$ (here, π denotes the canonical homomorphism from $\mathcal{B}$ onto $\mathcal{B}/F$). Clearly,

if $\mathcal{B}$ is retractive and has the ccc, then every quotient of $\mathcal{B}$ has the ccc.

Every interval algebra and every tree algebra is retractive (see [4]). Hence, we obtain from the preceding theorem (taking as $\mathcal{C}$ the class of interval algebras (or, the class of tree algebras) with ccc):

COROLLARY 3.2. *Assume Martin's axiom and let $\mathcal{B}$ be an interval algebra or a tree algebra with the countable chain condition. Furthermore, let $\mathcal{E}$, $|\mathcal{E}| < 2^{\omega}$, be a family of subsets of $\mathcal{B}$. Then $\mathcal{E}$ is omissible iff $0 \notin J^{\mathcal{E}}_{\infty}$.*

THEOREM 3.3. *Let $\mathcal{C}$ be a* RS*(λ)-good class of Boolean algebras closed under quotients. For $\mathcal{B}$ in $\mathcal{C}$ and any family $\mathcal{E}$, $|\mathcal{E}| < \lambda$, of subsets E of $\mathcal{B}$ with $|E| < \kappa$ the following holds: if every subfamily of $\mathcal{E}$ of cardinality less than κ is omissible, then $\mathcal{E}$ is omissible.*

PROOF. Let $\mathcal{E}_0$ be an arbitrary subfamily of $\mathcal{E}$ of cardinality less than κ. Since $\mathcal{E}_0$ is omissible, $0 \notin J^{\mathcal{E}_0}_{\infty}$ by (9). As $J^{\mathcal{E}}$ is κ-ary (cf. (5) and (2)), we have by (3), $0 \notin J^{\mathcal{E}}_{\infty}$. Hence, by the previous theorem, $\mathcal{E}$ is omissible. ∎

An instance of this theorem is:

COROLLARY 3.4. *Assume Martin's axiom and let $\mathcal{E}$, $|\mathcal{E}| < 2^{\omega}$, be a family of countable subsets of an interval algebra or of a tree algebra with the countable chain condition. If every countable subfamily of $\mathcal{E}$ is omissible, then $\mathcal{E}$ is omissible.*

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