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A THEORY OF REFINEMENT STRUCTURE OF HEDGE ALGEBRAS AND ITS APPLICATIONS TO FUZZY LOGIC

NGUYEN CAT HO
Institute of Information Technology, NCST
Hanoi, Vietnam
E-mail: ncho@ioit.ncst.ac.vn

HUYNH VAN NAM
Department of Mathematics, Quinhon University of Pedagogy
170 Nguyen Hue, Quinhon, Vietnam

Abstract. In [13], an algebraic approach to the natural structure of domains of linguistic variables was introduced. In this approach, every linguistic domain can be interpreted as an algebraic structure called a hedge algebra. In this paper, a refinement structure of hedge algebras based on free distributive lattices generated by linguistic hedge operations will be examined in order to model structure of linguistic domains more properly. In solving this question, we restrict our consideration to the specific hedge algebras called PN-homogeneous hedge algebras. It is shown that any PN-homogeneous hedge algebra can be refined to a refined hedge algebra (RHA, for short) and every RHA with a chain of the primary generators is a distributive lattice. Especially, we shall examine RHAs with exactly two distinct generators, which will be called symmetrical RHAs. Furthermore, in the symmetrical RHAs of the linguistic truth variable, we are able to define negation and implication operation, which, according to their properties, may be interpreted as logical negation and implication in a kind of fuzzy logic called linguistic-valued logic. Some elementary properties of these operations will be also examined. This yields a possibility to construct a method in linguistic reasoning, which is based on linguistic-valued fuzzy logic corresponding to the symmetrical RHAs of the linguistic truth variable.

1. Introduction. It is known that humans reason by means of their own language and they can choose and decide alternatives by evaluating semantics of linguistic terms. The fundamental elements in human reasoning are sentences normally containing vague

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concepts, and these sentences have implicitly or explicitly a truth degree, which is often expressed also by linguistic terms such as *true*, *very true*, *more or less true*, *approximately true*, *false*, *very false*, etc. In connection with this, Rinks wrote in [22] that "verbal coding is a human way of repackaging material into a few chunks of rich information. Natural language is rather unique in this characteristic. Until recently, a unified theory for manipulating in a strict mathematical sense non-numerical-valued variables, such as linguistic terms, did not exist."

Furthermore, it is well-known that Boolean algebras, Post algebras, 3-valued and multiple-valued Łukasiewicz algebras, etc. are algebraic foundations of classical or non-classical logics (see, e.g., [4,8,20,21,23]). In this direction, we want to look for an algebraic structure for fuzzy logic based on a suitable structure of truth vague concepts. It is known that L.A. Zadeh introduced and examined fuzzy logic based on the notion of linguistic variables. A linguistic variable is characterised by a quintuple $(X, T(X), U, G, M)$, where X is the name of the variable; $T(X)$ denotes the term-set of X, U is a universe of discourse of the base variable, G is a syntactic rule for generating linguistic terms of $T(X)$, and M is a semantic rule which is a mapping assigning to each linguistic term a fuzzy set on U. Recall that a fuzzy set of U is an element of the set $F(U, [0,1])$ of all functions from U to the unit interval $[0,1]$. In our approach, each term is associated with an element in an RHA, and its meaning is expressed through the structure of such an RHA.

In the papers initiated by Ho & Wechler [5,13], an algebraic approach to the natural structure of domains of linguistic variables was examined. As mentioned above, the main aim of our investigation is to find an appropriate algebraic structure for fuzzy logic and fuzzy reasoning, which could model human reasoning in an advantageous way. There are three main reasons for pursuing the research in this direction. The first one is that the domains of linguistic variables can be embedded into mathematical structure: the lattice structure, which is well-known in applications to logic. In such a structure, these domains can be ordered in a reasonable way, based on intuitive meanings of vague concepts. The second one is that there exists a natural demand to find a mathematical method for manipulating immediately linguistic terms as depicted above. The third one is that the way ones interpret the meaning of linguistic terms as fuzzy sets loses the natural ordering structure of linguistic domains.

In [13], an axiomatization for the so-called hedge algebras was introduced. Recall that the axiomatization is based on a detailed discussion about the general characteristics of linguistic hedges and vague concepts in natural language. The idea of this research was suggested from the research works of Zadeh and Lakoff [18,26,28], in which linguistic hedges and vague concepts are considered within the framework of fuzzy set theory.

In the algebraic approach, every linguistic domain can be interpreted as an algebra $AX = (X, G, H, \leq)$, where $(X, \leq)$ is a poset and G is a set of the primary generators and H is a set of unary operations representing linguistic hedges under consideration. In [14], hedge algebras were extended by introducing two additional operations corresponding to infimum and supremum of the so-called concept category of an element x, i.e. the set $H(x)$, which is generated from x by means of hedge operations.

It is shown that every extended hedge algebra (EHA, for short) with a lattice of the primary generators is a lattice and they can be used as an algebraic basis for a fuzzy logic called linguistic-valued logic (see, e.g., Ho [5-7]). However, many linguistic terms, which contain logical connectives disjunction “or” and/or conjunction “and” like *’Approximately True or Possibly True’*, cannot be reasonably expressed by elements of hedge algebras. The reason lies in the fact that although we can define in these algebras operations of join $\cup$ and meet $\cap$, which may be interpreted as disjunction “or” and conjunction “and”, but, in our opinion, these structures are rather rough. For example, let us consider the set of all possible truth values

$$\begin{aligned}T = \{&\text{true}, \text{false}, \text{very true}, \text{very false}, \text{approximately true}, \text{possibly true},\\ &\text{approximately true or possibly true}, \text{approximately true and possibly true}, \ldots\}\end{aligned}$$

We can see that the above linguistic value “approximately true or possibly true” will be expressed by “true” in the structure of EHA of the set of linguistic truth values, i.e. they define the same element in this algebra, which is clearly unsuitable in nature. Another disadvantage is that EHA, in general, are not distributive and hence we are not able to discuss the disjunction and conjunction normal forms.

In this paper we shall introduce some new axioms and obtain a class of algebras called refined hedge algebras (RHAs, for short), which have a finer structure than that of hedge algebras.

The paper is organised as follows: In Section 2 we shall present a way of constructing the distributive lattices of hedge operations. We shall introduce in Section 3 an axiomatization for RHA. A characterization to determine the relative position of elements in an RHA and some fundamental properties of this structure will be examined. The main property, which says that every RHA with a chain of the primary generators is a distributive lattice, will be studied in Section 4. In Section 5, RHAs with exactly two distinct generators called symmetrical RHAs will be examined. As a consequence, these RHAs are distributive lattices. Moreover, in Section 6 we shall point out that, in the finite symmetrical RHAs of the domains of the linguistic truth variable, we are able to define negation operation and implication operation, which may be interpreted as logical negation and implication. Some elementary properties of these operations will be also presented. Finally, some concluding remarks will be given in Section 7.

2. Distributive lattices of hedge operations. As mentioned in the previous section, the main aim of our investigation is to find a finer structure than that of hedge algebras. In order to construct this structure, we need some preparations.

First we shall recall some notions and notations introduced in [1]. Let P be a partial ordered set (poset, for short).

DEFINITION 2.1. An element a is said to *cover* an element b in a poset P, if $a > b$ and there is no $x \in P$ such that $a > x > b$.

By the *order* $o(P)$ of a poset P we mean the number of its elements, and if this number is finite, P is called a finite poset. Denote by $l(P)$ the length of a poset P.

In a poset P of finite length with the least element denoted by O, the *height* of an element $x \in P$ is, by definition, the least upper bound of the length of the chains $O = x_0 < x_1 < \ldots < x_n = x$ between O and x, and it is denoted by $h(x)$. If P has the greatest element, denoted by 1, then clearly $h(1) = l(P)$. Clearly also $h(x) = 1$ iff x covers O.

DEFINITION 2.2. A poset P is said to be *graded* if there exists a function from P to the set Z of all integers with the natural ordering, $g : P \to Z$, such that:

G1. $x > y$ implies $g(x) > g(y)$.
G2. If x covers y then $g(x) = g(y) + 1$.

Such a function g is called the graded function of P. It is known that any modular lattice of finite length is graded by its height function $h(x)$.

Let L be a modular lattice of finite length, we can define a relation R on L as follows:

$$\forall x, y \in L, (x, y) \in R \text{ iff } h(x) = h(y)$$

It is easily shown that R is an equivalence relation and then we have $L = \bigcup_{i=0}^{l(L)} L_i$ where $L_i = \{x \in L/h(x) = i\}, i = 0, \ldots, l(L)$, are the equivalence classes of the relation R. Clearly, $L_0 = \{0\}$ and $L_{l(L)} = \{1\}$.

In order to model the structure of sets of linguistic hedges, we need the following assumption, which describes the fact that any two hedges belonging to two different equivalence classes are always comparable:

(C0) Either $x > y$ or $x < y$ for any $x \in L_i$ and $y \in L_j$ and $i \neq j$.

To illustrate this, the reader can see the classes $L_1 = \{I\}$, $L_2 = \{A, P, ML\}$ and $L_3 = \{L\}$ as in Figure 3.

It is not difficult to see that the following holds:

PROPOSITION 2.1. *Let L be a modular lattice of finite length satisfying* (C0). *Then the following condition holds:*

If $o(L_i) > 1$ for an index $i \in \{1, \ldots, l(L) - 1\}$ then $o(L_{i-1}) = o(L_{i+1}) = 1$. Moreover, if we denote $e(L_{i+1})$ and $e(L_{i-1})$ the single element of L_{i+1} and L_{i-1}, respectively, then $e(L_{i+1}) = \vee_{x \in L_i} x$ and $e(L_{i-1}) = \wedge_{x \in L_i} x$, where $\vee$ and $\wedge$ are supremum and infimum in L, respectively.

We proceed now to consider a hedge algebra[1] $AX = (X, G, H, \leq)$, where $(X, \leq)$ is a poset, G is a set of the primary generators and H is a set of unary operations representing linguistic hedges under consideration. It is assumed that H can be decomposed into two disjoint subsets H^+ and H^- such that $H^+ + I$ and $H^- + I$ are finite modular lattices, where I is the identity, i.e. $Ix = x$ for every x in X, and considered as their zero-element. An example for this can be seen in Figure 3.

We will denote by N^+ and N^- the lengths of $H^+ + I$ and $H^- + I$, respectively. Suppose that g^+ and g^- are the graded functions of $H^+ + I$ and $H^- + I$, respectively.

Unless stated otherwise, in the sequel we shall always adopt the assumption that $H^+ + I$ and $H^- + I$ are finite modular lattices and satisfy the condition (C0). From now on, V and L stand for the unit-operations in $H^+ + I$ and $H^- + I$, respectively. Hence, we have $g^+(V) = N^+$, $g^-(L) = N^-$ and

$$H^+ + I = \bigcup_{i=0}^{N^+} H_i^+ \text{ where } H_i^+ = \{h \in H^+ + I/g^+(h) = i\},$$
$$H^- + I = \bigcup_{i=0}^{N^-} H_i^- \text{ where } H_i^- = \{h \in H^- + I/g^-(h) = i\}.$$

[1]See Ho & Wechler [14] for more details.

We shall now construct lattices, which are "freely" generated from $H^+ + I$ and $H^- + I$.

Let us consider $H^+ + I$. Assume that for some index $i \in 1, \dots, N^+$, $o(H_i^+) > 1$, and $H_i^+ = \{h_1^i, \dots, h_n^i\}$. By Proposition 2.1, the sets $H_{i+1}^+ = \{h^{i+1}\}$ and $H_{i-1}^+ = \{h^{i-1}\}$ are single-element sets. For such i, the ordering relationships between the elements of $H_{i-1}^+, H_i^+, H_{i+1}^+$, can be expressed as in Figure 1. Note that, as assumed above, there exists a natural ordering relation between classes H_i^+ and $H_i^+ \le H_j^+$ iff $i \le j$, where $H_i^+ \le H_j^+$ means that $x \le y$ for every $x \in H_i^+$ and $y \in H_j^+$.

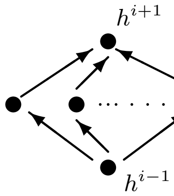


Fig. 1

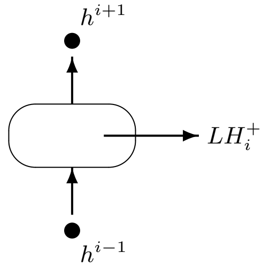


Fig. 2

Denote by $LH_i^+ = (L(H_i^+), \vee, \wedge)$ the free distributive lattice[2] generated from the incomparable elements $h_1^i, \dots, h_n^i$ of H_i^+. Particularly, for an index i such that $o(H_i^+) = 1$, we have $LH_i^+ = H_i^+$. Put $L_u^+ = \bigcup_{i=0}^{N^+} LH_i^+$ and $LH_i^+ + I = (L_u^+, H^+ + I, \vee, \wedge)$. Lu^+ becomes a distributive lattice under the ordering relation induced by the ordering relation of the lattices LH_i^+ and that defined between classes H_i^+, i.e. we have $LH_i^+ \le LH_j^+$, for any i, j such that $i \le j$. Figure 2 shows a picture of a segment of the constructed lattice $LH^+ + I$, where $o(H_i^+) > 1$.

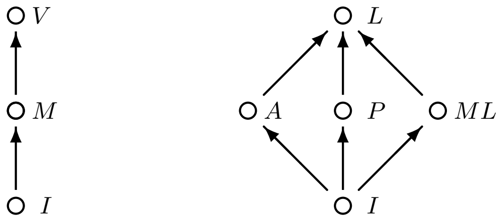
In an analogous way, we can construct the lattice $LH^- + I = (L_u^-, H^- + I, \vee, \wedge)$, generated from $H^- + I$. Here, there is no confusion, because H^+ and H^- are assumed to be disjoint and hence, so are LH^+ and LH^-, where $LH^+ = LH^+ + I \setminus \{I\}$ and $LH^- = LH^- + I \setminus \{I\}$. Thus, we have the following

Theorem 2.1. *$LH^+ + I = (L_u^+, \vee, \wedge, I, V, \le)$ and $LH^- + I = (L_u^-, \vee, \wedge, I, L, \le)$ are finite distributive lattices.*

Example 2.1. Let us consider the algebraic structure $AX = (X, G, H, \le)$, in which $G = \{\text{True}, \text{False}\}$ and $H^+ = \{V, M\}$ and $H^- = \{L, A, P, ML\}$. Here, for short, V, M, L, A, P, ML stand for Very, More, Little, Approximately, Possibly, More or Less, correspondingly, and $H^+ + I$ and $H^- + I$ are lattices depicted in Figure 3. Clearly, $H^+ + I$ and $H^- + I$ are finite modular lattices and satisfy condition (C0). By a construction as above, the distributive lattices $LH^+ + I$ and $LH^- + I$ generated from $H^+ + I$ and $H^- + I$, respectively, can be represented as in Figure 4, where

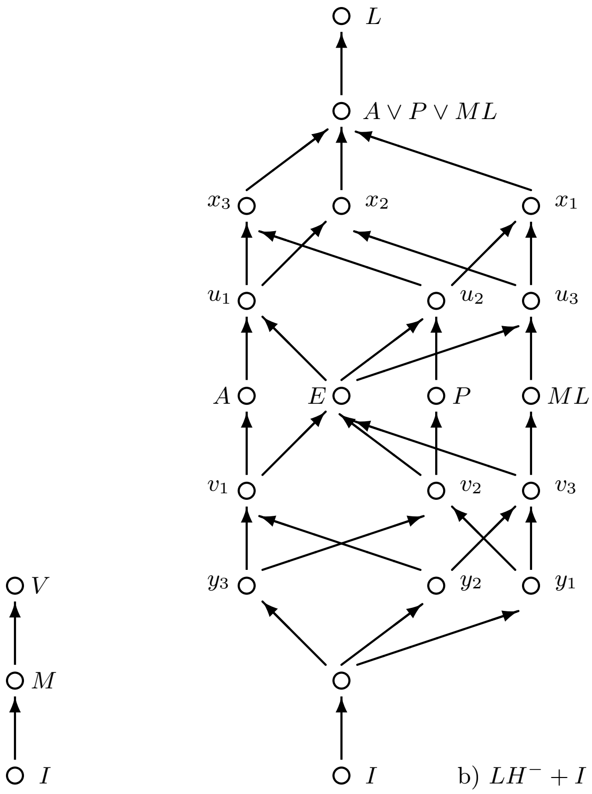
$x_1 = P \vee ML$, $x_2 = ML \vee A$, $x_3 = A \vee P$, $u_1 = x_2 \wedge x_3$, $u_2 = x_3 \wedge x_1$, $u_3 = x_1 \wedge x_2$,
$y_1 = P \wedge ML$, $y_2 = ML \wedge A$, $y_3 = A \wedge P$, $v_1 = y_2 \vee y_3$, $v_2 = y_3 \vee y_1$, $v_3 = y_1 \vee y_2$,
$E = (A \vee P) \wedge (P \vee ML) \wedge (ML \vee A) = (A \wedge P) \vee (P \wedge ML) \vee (ML \wedge A)$.

[2]See, e.g., Birkhoff [1].



a) H^+ b) H^-

Fig. 3



a) $LH^+ + I = H^+ + I$ b) $LH^- + I$

Fig. 4

3. An axiomatization for RHA and its elementary properties. Let us consider a hedge algebra $AX = (X, G, H, \leq)$, where $H^+ + I$ and $H^- + I$ are finite modular lattices satisfying condition (C0). Suppose that $LH^+ + I$ and $LH^- + I$ are distributive lattices, which are generated from $H^+ + I$ and $H^- + I$, respectively, as presented in the previous section. Let $I^+ = \{0, 1, \ldots, N^+\}$, $I^- = \{0, 1, \ldots, N^-\}$, and $SI^+ = \{i \in I^+/o(H_i^+) > 1\}$ and $SI^- = \{i \in I^-/o(H_i^-) > 1\}$. For simplifying the formulation of some statements, in the sequel by '^{c}' we mean either '$^+$' or '$^-$', and then for a statement containing, for instance, the notation LH_i^c for some $i \in SI^c$, we mean the statement presents two instances obtained by substituting "c" in turn by "+" and "−". For example, under such convention we can state the following: for any $i \in SI^c$, LH_i^c is the free distributive lattice generated from the incomparable elements of H_i^c and is a sublattice of $LH^c + I$; and for $i \in I^c \setminus SI^c$, LH_i^c is a single-element set, $LH_i^c = H_i^c$ and we have

$$LH^c + I = \bigcup_{i=0}^{N^c} LH_i^c.$$

Put $LH = LH^+ \cup LH^- \cup \{I\}$.

Let us denote by UOS the set of two elements V and L, which are unit-operations in $LH^+ + I$ and in $LH^- + I$, respectively. Denote by *Nat* the set of all non-negative integers. We introduce the following notion which will be used as an assumption throughout the paper:

Definition 3.1. A hedge algebra $AX = (X, G, H, \leq)$ is said to be *PN-homogeneous*, where PN is an abbreviation of Positive and Negative, provided that for any set H_i^c if the unit operation V in $H^+ + I$ is positive[3] (negative, resp.) w.r.t. a certain operation h in H_i^c, then V is also positive (negative resp.) w.r.t. any other one in H_i^c.

For example, the hedge algebra $AX = (X, G, H, \leq)$ in Example 2.1 is a PN-homogeneous hedge algebra. Since every hedge h is a mapping from X into X, the image of an element x in X under h will be denoted by hx instead of $h(x)$, for convenience. Thus, we can write khx instead of $k(h(x))$ for any $h,\ k \in LH$ and $x \in X$. According to our convention, for simplicity in formulating several statements as mentioned in [13], for any $h \in LH$, we define $hIx = Ix = x$, i.e. when I occurs in an expression explicitly, any h applying to Ix will have no effect.

The following definition gives us the semantics of the inequality $h > k$, which describes a property in the natural language saying that a hedge is stronger than another one, e.g. Little is stronger than Possibly.

Definition 3.2. An algebra $AX = (X, G, LH, \leq)$ is said to be *semantically consistent* if for any $h,\ k \in LH^c + I$, $x \in X$ and $hx \neq kx$, h and k are comparable in $LH^c + I$ iff hx and kx are comparable and if $h > k$ then $hx > kx$, when $hx > x$, and $hx < kx$, when $hx < x$.

Throughout the paper we always assume the considered algebra AX satisfies the semantic consistency in Definition 3.2. For any two hedges $h,\ k$ in LH, if the statement $x \leq hx$ iff $kx \leq x$ holds, for every x in X, then h and k are said to be *converse*, or h is converse to k and vice-versa. If the statement $x \leq hx$ iff $x \leq kx$ holds, for every x in X, then h and k are said to be *compatible*.

Consider an algebra $AX = (X, G, LH, \leq)$, where G is a set of zero-argument operations, LH is a set of one-argument operations.

For every $x \in X$, $LH(x)$ denotes the set of all elements generated from x by means of operations in LH. More generally, for $Y \subset X$ and $H' \subset LH$, $H'(Y)$ denotes the subset of X generated from the elements in Y by means of the operations in H'. Particularly, $H'(Ix) = \{x\}$. As usual, LH^* denotes the set of all strings of hedges in LH.

[3]See Ho & Wechler [13].

REMARK 3.1. From the construction of the lattices $LH^+ + I$ and $LH^- + I$, it can be seen that the lattices $LH^+ + I$ and $LH^- + I$ also satisfy condition (C0), in which the notations L_i and L_j are replaced with LH_i^c and LH_j^c, respectively.

Now, we introduce an axiomatization for a refinement structure of hedge algebras.

DEFINITION 3.3. An algebra $AX = (X, G, LH, \leq)$ is said to be a *refined hedge algebra* (or, briefly, RHA), if $(H(G), G, H, \leq)$ is a PN-homogeneous hedge algebra and the following conditions hold:

(R1) Every operation in LH^+ is a converse operation of the operations in LH^-. In addition, the unit operation V in LH^+ is either positive or negative w.r.t. any operations in LH.

(R2) If u and v are independent, i.e. $u \notin LH(v)$ and $v \notin LH(u)$, then $x \notin LH(v)$ for any $x \in LH(u)$. For $x \neq hx$, $x \notin LH(hx)$. Especially, if $a, b \in G$ and $a < b$ then $LH(a) < LH(b)$.

(R3) If hx and kx are incomparable, then so are any elements $u \in LH(hx)$ and $v \in LH(kx)$. For any $h \neq k$ and $hx \leq kx$:

(i) If $h, k \in LH_i^c$, for $i \in SI^c$, and $hx \neq kx$ then $\delta hx < \delta kx$, for any string of hedges δ.

Furthermore, for any $y \in LH(kx)$ such that $y \not\geq \delta kx$, δhx and y are incomparable, and for any $z \in LH(hx)$ such that $z \not\leq \delta hx$, δkx and z are incomparable.

(ii) If both h and k are different from I and do not belong to the same sublattice LH_i^c or $hx = kx$, then $h'hx \leq k'kx$, for any $h', k' \in UOS$.

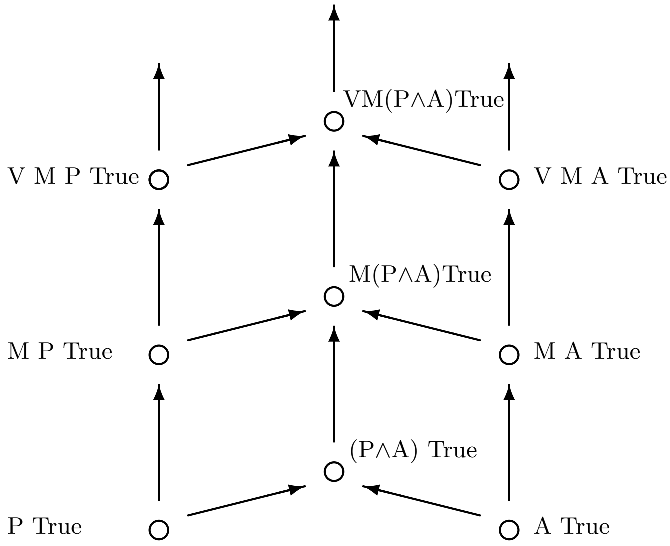
(iii) If $hx \neq kx$ then hx and kx are independent.

(R4) If $u \in LH(x)$ and suppose that $u \notin LH(hx)$, for any $h \in LH_i^c$, $i \in I^c$ then $u \geq v$ $(u \leq v)$ for $v \in LH(hx)$ implies $u \geq h'v$ $(u \leq h'v)$, for any $h' \in UOS$.

Now, we give an intuitive illustration of some axioms in Definition 3.3. (R2) describes a linguistic property saying that, for instance, if $u =$ Possibly true and $v =$ Approximately true, then u and v are independent and any term x generated from u, e.g. $x =$ Very Poss. true, must inherit the meaning of Possibly true and, hence, it cannot be generated from Approximately true. (R4) models the following semantic property of natural language: if $hx =$ Approximately True and u satisfies the condition in (R4) with $v =$ Very Approximately true $\geq u$ then u must be a term generated from Little true and hence $u \leq h'v$, where h' is either Very or Little. The statement (i) of (R3) is the basis to establish a partially ordering between the elements presented in Figure 5, that suits our intuition. The statement (ii) of (R3) guarantees that elements of L(A,P,ML)(True) in Figure 6 must be less than I.True = True and greater than L.True. The statement (iii) of (R3) states that a linguistic meaning generated from hx is not deduced from kx and vice-versa.

Note that the first part of (ii) in (R3) can be reformulated to include the case where one of h and k is to be the identity I, based on our convention upon I. But, then, it will be a consequence of (R4).

EXAMPLE 3.1. Let us consider an algebraic structure $AX = (X, G, LH, \leq)$, where $H = \{V, M, L, A, P, ML\}$ is the set of hedges considered in Example 2.1. For every hedge



P.True:= Possibly True
A.True:= Approx. True
M:= More; V:= Very

Fig. 5

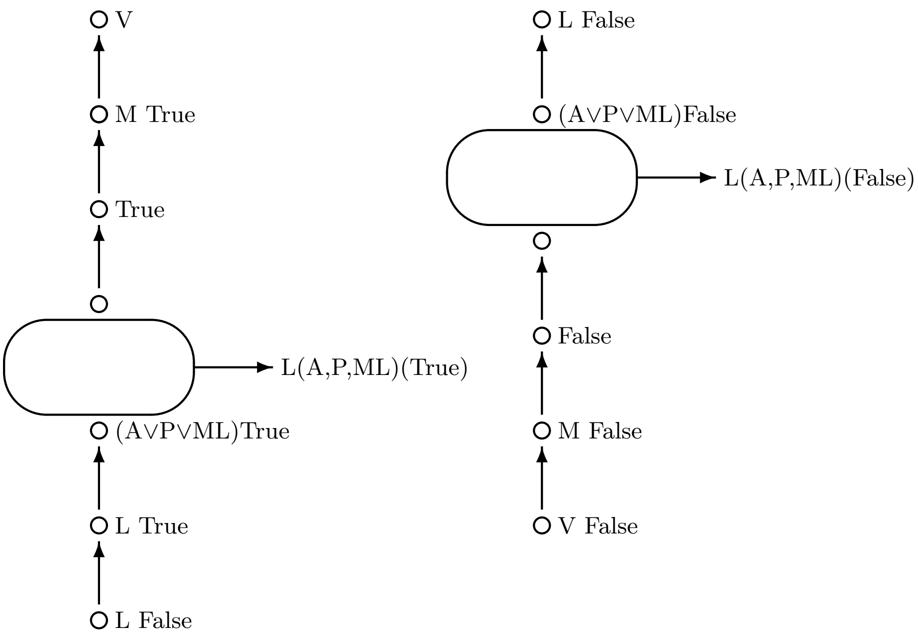


Fig. 6

operation h in LH, hTrue and hFalse are the elements represented in Figure 6. For $x \neq$ True and $x \neq$ False, we define $hx = x$. It can easily be seen that the operations are well defined and AX satisfies the conditions in Definition 3.3.

For the sake of convenience, we recall some definitions in [13].

DEFINITION 3.4. For any h, $k \in LH$, we shall write $hx <\leq kx$ ($hx <\leq Ix$) if for any h', k' in UOS and any $m, n \in Nat$, $V^n h'hx \leq V^m k'kx$ ($V^n h'hx \leq Ix$). If the last inequalities are always strict, then we shall write $hx << kx$ ($hx << Ix$).

As an example, the inequality V^nVery More true $\leq V^m$Little Very true holds intuitively, for all n and m and so we can write More true $<\leq$ Very true.

DEFINITION 3.5. Let x and u be two elements in an RHA $AX = (X, G, LH, \leq)$. The expression $h_n \dots h_1 u$ is said to be a *canonical representation* of x w.r.t. u in AX if

(i) $x = h_n \dots h_1 u$; (ii) $h_i \dots h_1 u \neq h_{i-1} \dots h_1 u$ for every $i \leq n$.

THEOREM 3.1. *Let $AX = (X, G, LH, \leq)$ be an RHA. Then, the following statements hold:*

(o) *If $hx <\leq kx$ then $hx \leq kx$.*

(i) *The operations in LH^c are compatible.*

(ii) *If $x \in X$ is a fixed point of an operation h in LH, i.e. $hx = x$, then it is a fixed point of the other ones.*

(iii) *If $x = h_n \dots h_1 u$, then there exists an index i such that the suffix $h_i \dots h_1 u$ of x is a canonical representation of x w.r.t. u and $h_j x = x$, for all $j > i$.*

(iv) *If $h \neq k$ and $hx = kx$ then x is a fixed point.*

(v) *For any h, $k \in LH$, if $x \leq hx$ ($x \geq hx$) then $Ix <\leq hx$ ($Ix \geq> hx$) and if $hx \leq kx, h \neq k$ and there is no i in SI^c such that both h and k belong to LH_i^c, then $hx <\leq kx$.*

PROOF. First, we observe that the condition (R1) is the same as the axioms (A1) and (A2) in Definition 3[13]. Therefore, the proofs of (o), (i), (ii), (iii) which are based on (R1) are similar as that in [13].

By (ii), we can use the terminology "a fixed point" instead of "a fixed point of an operation".

Now we prove (iv). Assume the contrary, that x is not a fixed point. Suppose that $hx > x$. If h and k are converse the $kx \leq x$. Hence, $hx > x \geq kx$, which contradicts the hypothesis. If h and k are compatible then on account of Definition 3.2 and $h \neq k$, it follows that $hx \neq kx$ if $hx \neq x$, which is again impossible. Thus $hx = x$. For the case where $hx < x$, the proof is similar. This concludes the proof of (iv).

To prove (v), suppose that $x \leq hx$. If $hx = x$ then x is a fixed point and so $x \leq V^n h'hx$, for each $h' \in UOS$. If $hx > x$, by virtue of (R2), we have $x \notin LH(hx)$ and $x \leq h'hx$, for $h' \in UOS$, by (R4). Again by (R4) we obtain $x \leq Vh'hx$. Since $x \in LH(x)$ and $x \notin LH(hx)$, applying (R4) repeatedly, we have $x \leq V^n h'hx$, i.e. $Ix <\leq hx$, by Definition 3.4.

For the case where $x \geq hx$, the proof is similar.

Now suppose that $hx \leq kx$, $h \neq k$ and h and k do not together belong to LH_i^c for any i. If $hx = kx$ then x is a fixed point, by (iv) of the theorem. Thus, $V^n h'hx = V^m k'kx$, for all h', $k' \in UOS$ and m, $n \in Nat$.

Assume that $hx < kx$ and $k \in LH_{i_0}^c$, and $h \in LH_{i_1}^c$, for $i_1, i_0 \in I^c$ and $i_1 \neq i_0$. Assuming that h and k are converse, we have $hx < x < kx$. As proved above, it follows that $hx << x$ and $x << kx$. Thus $hx << kx$. Now assume that h and k are compatible.

Put $u = hx$. Since $hx \neq kx$, it follows from (iv) of the theorem that $hx \neq k_1x$ for any $k_1 \in LH^c_{i_0}$. By (R3), hx and k_1x are independent and $u = hx \notin LH(k_1x)$, for any $k_1 \in LH^c_{i_0}$. Since $u \in LH(x)$, it follows from (R4) that $u < k'kx$, for any $k' \in UOS$. Applying (R4) again to the last inequality, we get $u < Vk'kx$. Repeating this argument, it implies that $u = hx < V^m k'kx$, for any $m \in Nat$ and $k' \in UOS$. It can be seen that h and k play a similar role and hence we can use the analogous argument as above, where $u = V^m k'kx$, and we obtain $V^m k'kx > V^n h'hx$, for any h', $k' \in UOS$ and $m, n \in Nat$, which means that $hx << kx$. So, we have proved that $hx <\leq kx$, which completes the proof of (v). ∎

The following theorem is a reformulation of Theorem 2 in [13] for RHAs.

THEOREM 3.2. *For any $h \in LH$, there exist two unit operations h^- and h^+ such that h^- is negative and h^+ is positive w.r.t. h and for any $h_1, \ldots, h_n \in LH$, $x \in X$,*

$$V^n h^- hx \leq h_n \ldots h_1 hx \leq V^n h^+ hx \quad \textit{if} \ \ hx \geq x,$$
$$V^n h^- hx \geq h_n \ldots h_1 hx \geq V^n h^+ hx \quad \textit{if} \ \ hx \leq x.$$

PROOF. We shall prove the theorem by induction on the number n of hedge operations. Assume $n = 1$ and $hx \geq x$. If h_1 is positive w.r.t. h then we have $h_1hx \geq hx$. From the assumptions of the operations h^+ and h^-, we have $h^+hx \geq h_1hx \geq hx \geq h^-hx$. Since V is positive w.r.t. h^+ and h^- (see [13]), we obtain $Vh^+hx \geq h^+hx \geq h_1hx \geq hx \geq h^-hx \geq Vh^-hx$, which are the required inequalities.

If h_1 is negative w.r.t. h, we have $h_1hx \leq hx$. By the assumption on the hedge h^-, it follows that h^- and h_1 are compatible and $h^- \geq h_1$. So $h^-hx \leq h_1hx \leq hx$. By the same argument as above, we obtain again $Vh^+hx \geq h^+hx \geq hx \geq h_1hx \geq h^-hx \geq Vh^-hx$.

For the case $hx \leq x$, the proof is similar. Consequently, it has been proved that the inequalities in the theorem hold for $n = 1$.

Assume that the theorem holds for $n = i$, i.e. if $h_1hx \leq hx$, then $V^i h_1^+ h_1hx \leq h_{i+1}h_i \ldots h_1hx \leq V^i h_1^- h_1x$ and if $h_1hx \geq hx$ then $V^i h_1^+ h_1hx \geq h_{i+1}h_i \ldots h_1hx \geq V^i h_1^- h_1x$, where h_1^+, h_1^- and h_1 satisfy the assumption like that made on h^+, h^- and h.

Now we shall prove the induction conclusion for the case $hx \leq x$. For the opposite case, the proof is similar.

Suppose first that h_1 is positive w.r.t. h, and so $h_1hx \leq hx$. From the induction hypothesis it follows that $h_{i+1}h_i \ldots h_1hx \leq V^i h_1^- h_1hx$, and by (v) of Theorem 3.1, it implies that $V^i h_1^- h_1hx \leq hx \leq h^-hx \leq V^{i+1}h^-hx$, with a notice that h^- is negative w.r.t. h and V is positive w.r.t. h^- and V. So, one of the two required inequalities is true.

Since both h_1 and h^+ are positive w.r.t. h, it follows that they together belong to either LH^+ or LH^-. So, $h^+ \geq h_1$ and hence $h^+hx \leq h_1hx$. In addition, if $h^+ \neq h_1$, then by (v) of Theorem 3.1, it follows that $h^+hx <\leq h_1hx$. According to the induction hypothesis and Definition 3.4, we have $h_{i+1}h_i \ldots h_1hx \geq V^i h_1^+ h_1hx \geq V^i Vh^+hx$. If $h^+ = h_1$, then h_1 is either V or L. In both cases $h^+ = V$ and, hence, from the induction hypothesis it follows that $h_{i+1}h_i \ldots h_1hx \geq V^i Vh_1hx = V^{i+1}h^+hx$. Thus, for the case where h_1 is positive w.r.t. h, the induction conclusion follows. Since the proof for the case where h_1 is negative w.r.t. h is similar, the theorem is completely proved. ∎

COROLLARY 3.1. (i) *For any $x \in X$, if $hx < kx$ and there is no $i \in SI^c$ such that both h and k belong to LH_i^c, then for any two strings of hedges δ and δ', the inequality $\delta hx < \delta' kx$ holds.*

(ii) *Let u be an arbitrary element in X and $x \in LH(u)$. Then, there exist always elements y, $z \in UOS(u)$, i.e. z and y are generated from u by means of the unit operations, such that $y \geq x \geq z$. Furthermore, either one of the equalities $u \leq x \leq V^n hu$ and $u \geq x \geq V^n hu$ holds, for a suitably chosen $h \in LH$ and for sufficiently great number $n \in Nat$.*

PROOF. For the proof of this corollary, we refer the reader to [13]. ∎

Now, the following theorem gives us a characterisation to determine the relative position of elements in an RHA. Here, the notation x_j is defined as follows: if $x = h_n \dots h_1 u$, then x_j denotes the expression $h_{j-1} \dots h_1 u$, for $1 \leq j \leq n$.

THEOREM 3.3. *Let $x = h_n \dots h_1 u$ and $y = k_m \dots k_1 u$ be two arbitrary canonical representations of x and y w.r.t. u, respectively. Then there exists an index $j \leq \min(m, n) + 1$ such that $h_{j'} = k_{j'}$, for all $j' < j$ and*

(1) *$x < y$ iff one of the following conditions holds*

(i) *$h_j x_j < k_j x_j$ and $\delta k_j x_j \leq \delta' k_j x_j$ or $\delta h_j x_j \leq \delta' h_j x_j$ if h_j and k_j together belong to LH_i^c for some $i \in SI^c$, where $x_j = h_{j-1} \dots h_1 u$, $\delta = h_n \dots h_{j+1}$, $\delta' = k_m \dots k_{j+1}$;*

(ii) *$h_j x_j < k_j x_j$, otherwise;*

(2) *$x = y$ iff $m = n$ and $h_j = k_j$ for all $j \leq n$;*

(3) *x and y are incomparable iff there exists $i \in SI^c$ such that both h_j and k_j belong to LH_i^c and one of the following conditions holds:*

(i) *$h_j x_j$ and $k_j x_j$ are incomparable,*

(ii) *$h_j x_j < k_j x_j$ and $\delta k_j x_j \not\leq \delta' k_j x_j$,*

(iii) *$h_j x_j > k_j x_j$ and $\delta' h_j x_j \not\leq \delta h_j x_j$.*

PROOF. Let j be the least index such that $h_j \neq k_j$. It can be seen that $j \leq \min(m, n) + 1$, since $I \neq h$ for every $h \in LH$.

Sufficiency: To prove the sufficiency of (1), suppose first that $h_j x_j < k_j x_j$ and there is no index i_0 in SI^c such that both h_j and k_j belong to $LH_{i_0}^c$. From (v) of Theorem 3.1, we obtain $h_j x_j << k_j x_j$ and $V^p h h_j x_j < V^q k k_j x_j$, for any $h, k \in UOS$ and $p, q \in Nat$. By Theorem 3.2, there exist $h', k' \in UOS$ such that $h_n \dots h_j x_j \leq V^{n-j-1} h' h_j x_j$ and $k_m \dots k_j x_j \geq V^{m-j-1} k' k_j x_j$, which imply that $x < y$.

If there exists an index i_0 in SI^c, such that both h_j and k_j belong to $LH_{i_0}^c$, and $h_j x_j < k_j x_j$ and $\delta k_j x_j \leq \delta' k_j x_j$, then by (R3), we have $\delta h_j x_j < \delta k_j x_j$. Hence, $x = \delta h_j x_j < \delta k_j x_j \leq \delta' k_j x_j = y$.

Since the sufficiency of (2) is evident, we prove the sufficiency of (3). Suppose that there exists an index i_0 in SI^c such that both h_j and k_j belong to $LH_{i_0}^c$. If (i) holds, it follows from (R3) that x and y are incomparable. If (ii) holds, i.e. $h_j x_j < k_j x_j$ and $\delta k_j x_j \not\leq \delta' k_j x_j$, then by (R3), we infer $\delta h_j x_j < \delta k_j x_j$. Moreover, it follows from (R3) that $\delta h_j x_j$ and z are incomparable, for any $z \in LH(k_j x_j)$ such that $\delta k_j x_j \not\leq z$. Thus, $x = \delta h_j x_j$ and $y = \delta' k_j x_j$ are incomparable. In the case (iii) holds, the proof is similar.

Necessity: Suppose that there is no index j such that $h_j \neq k_j$. Note that one of h_j and k_j may be the operation I. Then, it is evident that the two canonical representations of x and y are identical and hence $x = y$. Therefore, assuming that these two canonical representations are different, there exists the least index j such that $h_j \neq k_j$. Obviously, $j \leq \min(m, n) + 1$. Between $h_j x_j$ and $k_j x_j$ there are the following ordering relationships: $h_j x_j = k_j x_j$, $h_j x_j < k_j x_j$, $h_j x_j > k_j x_j$ and $h_j x_j$ and $k_j x_j$ are incomparable. From the proof of the sufficiency, we have the following:

(1) If $x < y$ then $h_j x_j < k_j x_j$. Furthermore, if there exists i_0 in SI^c such that both h_j and k_j belong to $LH^c_{i_0}$, then, by (R3), from $\delta k_j x_j \not\leq \delta' k_j x_j$ it follows that x and y are incomparable. This contradicts the hypothesis and hence, $\delta k_j x_j \leq \delta' k_j x_j$. Likewise, it can be proved that $\delta h_j x_j \leq \delta' h_j x_j$.

(2) If $x = y$ then $h_j x_j = k_j x_j$. It remains to prove that if $h_j x_j = k_j x_j$ then $m = n$ and $h_j = k_j$. In fact, if $h_j \neq k_j$, it follows by (iv) of Theorem 3.1 that x_j is a fixed point. Thus, from the definition of the canonical representations it follows that $m = n$ and $h_j = k_j$ for all $j \leq n$.

(3) Suppose that x and y are incomparable. Then, there are only three possibilities: $h_j x_j < k_j x_j$, $h_j x_j > k_j x_j$ and $h_j x_j$ and $k_j x_j$ are incomparable. If h_j and k_j are converse then it is easy to infer that x and y are comparable, a contradiction. If h_j and k_j are compatible and there is no i_0 in SI^c such that both h_j and k_j belong to $LH^c_{i_0}$, then from Remark 3.1, it follows that h_j and k_j are comparable. So, $h_j x_j$ and $k_j x_j$ are also comparable. Furthermore, from (v) of Theorem 3.1 and (ii) of Corollary 3.1, it is easy to check that x and y are comparable, as well. This contradicts the assumption. Thus, we have proved that if x and y are incomparable, then there exists an i_0 in SI^c such that both h_j and k_j belong to $LH^c_{i_0}$. Assume now that $h_j x_j < k_j x_j$. We have to prove that $\delta k_j x_j \not\leq \delta' k_j x_j$. In fact, by (R3), we have $\delta h_j x_j < \delta k_j x_j$. If $\delta k_j x_j \leq \delta' k_j x_j$ then $x = \delta h_j x_j < \delta k_j x_j \leq \delta' k_j x_j = y$, which contradicts the hypothesis. Thus, we have $\delta k_j x_j \not\leq \delta' k_j x_j$. In the case $h_j x_j > k_j x_j$, by an analogous argument we have $\delta' h_j x_j \not\leq \delta h_j x_j$. This concludes the proof. ■

Remark 3.2. At first glance, one may think that the theorem is meaningless, because it replaces the comparison of two elements by the comparison of two others: The comparison between $x = \delta h_j x_j$ and $y = \delta' k_j x_j$ is changed to that between $x' = \delta k_j x_j$ and $y' = \delta' k_j x_j$ or between $x' = \delta h_j x_j$ and $y' = \delta' h_j x_j$. But, notice that the length of the common suffix of x' and y' is greater than that of x and y. It leads to a procedure that with a finite number of steps one can decide whether the given elements x and y are comparable and which one is greater than the other.

Corollary 3.2. *If x is not a fixed point and u is any element in X, then the canonical representation of x w.r.t. u, if it exists, is unique, i.e. if $h_n \ldots h_1 u$ and $k_m \ldots k_1 u$ are two canonical representations of x w.r. t. u, then $m = n$ and $h_i = k_i$, for all $i \leq n$.*

The following proposition shows that if both h and k belong to LH^c_i, for $i \in SI^c$, then from the property hx is a fixed point we can deduce that kx is also fixed point and vice-versa. Intuitively, it means that, for such h and k, h can generate a proper meaning from an element x (i.e. $hx \neq x$) iff k does so.

PROPOSITION 3.1. *For any $x \in X$ and $i \in SI^c$. If there exists a hedge $h \in LH_i^c$ such that hx is a fixed point, then so is kx, for any $k \in LH_i^c$.*

PROOF. If $hx = kx$, then the assertion is evident. Assume that $hx \neq kx$. We shall prove by cases as follows:

(i) Assume that $hx < kx$. It follows that $Vhx < Vkx$ and if $Vkx > kx$, then kx and Vhx are incomparable, by (i) of (R3). It contradicts the assumption that $Vhx = hx < kx$. If $Vkx < kx$, then Vkx and hx are incomparable, by (i) of (R3) and this again contradicts the fact that $hx = Vhx < Vkx$. Since, by (R1), Vkx and kx must be comparable, it follows that $Vkx = kx$, i.e. kx is a fixed point.

(ii) For the case where $hx > kx$, the proof is similar.

(iii) Suppose that hx and kx are incomparable. Since LH_i^c, for $i \in SI^c$, is a sublattice of LH^c+I, it follows that $(h \vee k)$ belongs to LH_i^c. Clearly, hx and $(h \vee k)x$ are comparable. By the cases proved above, it follows that $(h \vee k)x$ is a fixed point and, hence, by the same reason, kx is a fixed point. The proof is completed. ■

The following proposition can be considered as a generalization of Proposition 3.1.

PROPOSITION 3.2. *For any $x \in X$ and $h, k \in LH_i^c$, for some $i \in SI^c$ and for any string of hedges δ, δhx is a fixed point iff δkx is a fixed point.*

PROOF. Suppose that δhx is a fixed point. There are two cases:

Case (i): hx and kx are comparable. Without loss of generality, suppose that $hx \leq kx$. If $hx = kx$ then, by (iv) of Theorem 3.1, x is a fixed point and, hence, so is $\delta kx = x$. Now, assume that $hx < kx$. By (i) of (R3), it follows that $\delta hx < \delta kx$.

Suppose the contrary that δkx is not a fixed point. Take a suitable h' so that $h'\delta kx < \delta kx$. By (i) of (R3), δhx and $h'\delta kx$ are incomparable. Again by (i)(R3), from $hx < kx$ it follows that $\delta hx = h'\delta hx < h'\delta kx$. We have a contradiction. Therefore, δkx is a fixed point.

Case (ii): hx and kx are incomparable. By the same argument as in Case (iii) of the proof of Proposition 3.1, we can prove that δkx is a fixed point.

Since h and k play symmetrical roles, the proof is completed. ■

Since the RHA is constructed from a given PN-homogeneous hedge algebra, a natural question arises whether the PN-homogeneous property for the unit-operation V in $LH^+ + I$, but not in $H^+ + I$, still holds if we replace H_i^c in Definition 3.1 with LH_i^c. The following proposition answers this question.

PROPOSITION 3.3. *If the unit operation V in $LH^+ + I$ is positive (negative, resp.) w.r.t. a certain h in H_i^c, for i in SI^c, then V is also positive (negative, resp.) w.r.t. any operation in LH_i^c.*

PROOF. We shall prove the assertion for the case of "positive". The proof for the case of "negative" is similar.

Assume that V is positive w.r.t. $h \in H_i^c$, for some $i \in SI^c$. Since the hedge algebra $(H(G), G, H, \leq)$ is PN-homogeneous, it follows that V is also positive w.r.t. any operation in H_i^c. Since LH_i^c is a free distributive lattice generated from incomparable elements in H_i^c, for every $k \in LH_i^c$ there exists $h' \in H_i^c$ such that either $k \geq h'$ or $k \leq h'$. If $k = h'$, then V is already positive w.r.t. k. Consider the case $h' \neq k$ and suppose that $h'x \leq kx$. In the case $h'x = kx$, by (iv) of Theorem 3.1, x is a fixed point and so $Vkx = kx$. Assume that $h'x < kx$, we have $Vh'x < Vkx$, by (i)(R3). If $h'x$ is a fixed point, then so is kx, by Proposition 3.1, i.e. $Vkx = kx$.

If $Vh'x > h'x$, then $h'x > x$, since V is positive w.r.t. h'. Hence, $kx \geq x$ and, again by (i)(R3) with δ to be empty, kx and Vhx are incomparable. From this fact and the inequality $Vh'x < Vkx$ it follows from (i)(R3) that $Vkx \geq kx \geq x$.

If $Vh'x < h'x$, then $h'x < x$, since V is positive w.r.t. h'. From (i)(R3), it follows that $h'x$ and Vkx are incomparable and, similarly as above, it can be derived that $Vkx \leq kx \leq x$.

Since an analogous argument can be used for the case $h'x \geq kx$, we have proved that, in any case, either $Vkx \leq kx \leq x$ or $Vkx \geq kx \geq x$, i.e. V is positive w.r.t. k. This concludes the proof. ∎

The following proposition states that hedge operations in the same sublattice have analogous semantic properties in terms of $\leq$.

PROPOSITION 3.4. *For any $h, k \in LH_i^c$, for some $i \in SI^c$, and for any $x \in X$, we have the following assertions:*

(i) *$\delta hx > x$ ($\delta hx < x$) iff $\delta kx > x$ ($\delta kx < x$), for any $\delta \in LH^*$.*

(ii) *If $hx \neq kx$, then δhx and $\delta' hx$ are incomparable iff δkx and $\delta' kx$ are incomparable, for any $\delta, \delta' \in LH^*$.*

(iii) *$\delta hx > \delta' hx$ iff $\delta kx > \delta' kx$, for any $\delta, \delta' \in LH^*$.*

PROOF. The assertion (i) can easily be proved from the given assumption on h and k.

(ii) It is sufficient to prove the statement for the case where h and k are comparable, since if h and k are incomparable in LH_i^c, then there exists h' in LH_i^c such that $h' \geq k$ and $h' \geq h$. Hence, it can easily be seen that the assertion can be deduced from the case being proved now. Moreover, without loss of generality, we can assume that $hx > kx$. For any two strings of hedges δ and δ', it follows from (i)(R3) that $\delta hx > \delta kx$ and $\delta' hx > \delta' kx$, and that the incomparability of δhx and $\delta' hx$ implies the incomparability of two elements $\delta' hx$ and δkx and that of two elements δhx and $\delta' kx$. Now, it can be verified that the comparability of δkx and $\delta' kx$ leads to a contradiction. Similarly, we can prove that the incomparability of δkx and $\delta' kx$ implies the incomparability of δhx and δhx.

(iii) Similar as in the proof of (ii), we can assume without loss of generality that $hx > kx$. It follows from (R3) that $\delta hx > \delta kx$ and $\delta' hx > \delta' kx$, for any $\delta, \delta' \in LH^*$. Suppose now that $\delta hx > \delta' hx$. It implies from (ii) that δkx and $\delta' kx$ are comparable. Further, by (R3), we infer that δkx and $\delta' hx$ are incomparable and so, if $\delta kx \leq \delta' kx$ then $\delta kx < \delta' hx$, we have a contradiction. Hence, $\delta kx > \delta' kx$. Since the sufficiency is evident, the proof is completed. ∎

4. Lattice characteristic and distributivity of RHAs. In this section, we shall study the main property of RHAs. It will be shown that RHA is a distributive lattice if the set of the primary generators is a chain. Firstly, we shall prove the following theorem saying that RHA with a chain of the primary generators is a lattice. It also gives us recursive formulas for computing infimum and supremum of elements in RHA.

THEOREM 4.1. *Let $AX = (X, G, LH, \leq)$ be an RHA and G be a chain of generators. Then AX is a lattice. Moreover, for any two incomparable elements x and y in X, if $x = \delta hw$, and $y = \delta' kw$, where $\delta, \delta' \in LH^*$ and $w \in LH(a)$ for some $a \in G$, are canonical representations of x and y, respectively, then both h and k belong to LH_i^c, for some $i \in SI^c$ and*

$$x \cup y = \begin{cases} \delta(h \vee k)w \cup \delta(h \vee k)w \text{ if } hw > w \\ \delta(h \wedge k)w \cup \delta(h \wedge k)w \text{ if } hw < w \end{cases}$$

$$x \cap y = \begin{cases} \delta(h \wedge k)w \cup \delta(h \wedge k)w \text{ if } hw > w \\ \delta(h \vee k)w \cup \delta(h \vee k)w \text{ if } hw < w \end{cases}$$

where $\cup$, $\cap$ stand for join, meet in AX, while $\vee$, $\wedge$ stand for join and meet in $LH^c + I$.

PROOF. From (R2) it follows that if x and y are incomparable in X, then there exists an element $a \in G$ such that $x, y \in LH(a)$, since G is a chain. Thus, there exist two canonical representations of x and y w.r.t. a, say $x = h_n \dots h_1 a$ and $y = k_m \dots k_1 a$. On account of Theorem 3.3, there exists an index $j \leq \min(m, n) + 1$ such that $h_i = k_i$, for any $i \leq j$. Furthermore, there exists $i_0 \in SI^c$ such that $h_j, k_j \in LH_{i_0}^c$. Let $\delta = h_n \dots h_{j+1}$, $\delta' = k_m \dots k_{j+1}$, $h = h_j$, $k = k_j$. With this notation we have $x = \delta hw$ and $y = \delta' kw$, where $w = h_{j-1} \dots h_1 a$.

We shall prove the theorem for the supremum. The proof for the infimum can be obtained by duality.

Let us first consider the case where $hw > w$. Then, we also have $kw > w$. It implies that $(h \vee k)w > w$ and $h \vee k \in LH_{i_0}^c$, since $LH_{i_0}^c$ is a sublattice of $LH^c + I$. By Definition 3.2 and (i)(R3), we have $\delta(h \vee k)w \geq \delta hw$, $\delta'(h \vee k)w \geq \delta' kw$.

We shall prove that $t \geq \{\delta(h \vee k)w, \delta'(h \vee k)w\}$, for any $t \in LH(a)$, $t > \{x, y\}$.

Suppose that $t = l_p \dots l_1 a$ is the canonical representation of t w.r.t. a. Consider the case that $t \in LH(w)$ and so we have $t = l_p \dots l_{j+1} l_j w$. Since $t > \{x, y\}$ it follows from Theorem 3.3 that $l_j w > \{hw, kw\}$. Remember that $hw > w$ and, hence, $l_j \geq h \vee k$ and $l_j w \geq (h \vee k)w$. If $l_i \notin LH_{i_0}^c$, by (v) Theorem 3.1 we obtain $l_j w >> (h \vee k)w$ and so, we can infer that

$$t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}.$$

Let $l_j \in LH_{i_0}^c$ and assume that $l_j w = (h \vee k)w$. If $l_j \neq (h \vee k)$, then by (iv) Theorem 3.1, w is a fixed point and hence $w = hw$, contrary to assumption. Thus, $l_j = (h \vee k)$ and, since $l_j w > hw$, $l_j w > kw$ and $t > \{x, y\}$, Theorem 3.3 yields

$$t \geq \{\delta(h \vee k)w, \delta'(h \vee k)w\}.$$

Now, assume that $l_j w > (h \vee k)w$. Thus, $t = l_p \dots l_j w > l_p \dots l_{j+1}(h \vee k)w$, by (i)(R3). Since $t > \{x, y\}$ and it is easily seen that $x = \delta hw < \delta l_j w$, $y = \delta' kw < \delta' l_j w$, we infer again by (i)(R3) that $t \geq \{\delta l_j w, \delta' l_j w\}$. Applying Proposition 3.4 to the last inequalities, we get

$$l_p \dots l_{j+1}(h \vee k)w \geq \{\delta(h \vee k)w, \delta'(h \vee k)w\}.$$

Hence, $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$, which is the desired inequality.

Now, consider the case that $t \notin LH(w)$. Then, there exists an index $j' \leq j-1$ such that $h_i = l_i$ for any $i < j'$ and $l_{j'}u > h_{j'}u$, where $u = h_{j'-1} \dots h_1 a$. If there is no $i_1 \in SI^c$ such that $h_{j'}$, $l_{j'} \in LH^c_{i_1}$ then it follows from (v) Theorem 3.1 that $l_{j'}u >> h_{j'}u$ and, hence, by Theorem 3.2, it can easily be verified that $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$. If there exists $i_1 \in SI^c$ such that $h_{j'}, l_{j'} \in LH^c_{i_1}$, we set $s = j - j' - 1$ and prove the assertion by induction on the number s of hedge operations.

For $s = 0$, i.e. $j' = j-1$, and $w = h_{j'}u$, we can write $t \geq \{x = \delta h h_{j'}u, y = \delta' k h_{j'}u\}$, and applying (i)(R3) to $l_{j'}u > h_{j'}u$, it follows that $t \geq \{\delta h l_{j'}u, \delta' k l_{j'}u\}$. Since $hh_{j'}u > h_{j'}u$ and the elements $x = \delta h h_{j'}u$ and $y = \delta' k h_{j'}u$ are incomparable, it follows also from Proposition 3.4 that $hl_{j'}u > l_{j'}u$ and that $\delta h l_{j'}u$ and $\delta' k l_{j'}u$ are incomparable. Clearly, $t \in LH(l_{j'}u)$ and analogously to the case where $t \in LH(w)$, with $w = l_{j'}u$, we can prove that $t \geq \{\delta(h \vee k)l_{j'}u, \delta'(h \vee k)l_{j'}u\}$. Moreover, it follows from (i)(R3) that $\delta(h \vee k)l_{j'}u > \delta(h \vee k)h_{j'}u = \delta(h \vee k)w$ and $\delta'(h \vee k)l_{j'}u > \delta'(h \vee k)h_{j'}u = \delta'(h \vee k)w$ and, hence, $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$.

Assume the induction hypothesis, that the inequality holds for every $s \leq i$. For $s = i+1$, we have $j' + i + 1 = j - 1$ and $w = h_{j-1} \dots h_{j'+1} h_{j'} u$. Set $w' = h_{j-1} \dots h_{j'+1} l_{j'} u$. It follows from Proposition 3.4 that $hw' > w'$, since $hw > w$ and $h_{j'}, l_{j'} \in LH^c_{i_1}$. Using again (i)(R3) as above, we get

$$t \geq \{\delta h h_{j-1} \dots h_{j'+1} l_{j'} u, \delta' k h_{j-1} \dots h_{j'+1} l_{j'} u\}$$

and by Proposition 3.4 we see that $\delta h h_{j-1} \dots h_{j'+1} l_{j'} u$ and $\delta' k h_{j-1} \dots h_{j'+1} l_{j'} u$ are incomparable. If $t \in LH(w')$ then, by the same argument as for the case $t \in LH(w)$, we obtain

$$t \geq \{\delta(h \vee k) h_{j-1} \dots h_{j'+1} l_{j'} u, \delta'(h \vee k) h_{j-1} \dots h_{j'+1} l_{j'} u\}$$

and, hence, $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$, on account of (i)(R3) applied to $h_{j'}u < l_{j'}u$.

If $t \notin LH(w')$ then there exists an index j'', which satisfies $j' + 1 \leq j'' \leq j - 1$, such that $h_{i'} = l_{i'}$ for any i' satisfying $j'' > i' \geq j' + 1$ and $l_{j''}u' > h_{j''}u'$, where $u' = l_{j''-1} \dots l_{j'} u$. By also an analogous argument as in the case where $t \notin LH(w)$, if there is no $i_2 \in SI^c$ such that $h_{j''}, l_{j''} \in LH^c_{i_2}$ then, by the same argument as for $t \notin LH(w)$ we obtain $t > \{\delta(h \vee k)w', \delta'(h \vee k)w'\}$ and hence, $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$.

If there exists $i_2 \in SI^c$ such that $h_{j''}, l_{j''} \in LH^c_{i_2}$ then by the induction hypothesis we have

$$t \geq \{\delta(h \vee k) h_{j-1} \dots h_{j''+1} l_{j''} u', \delta'(h \vee k) h_{j-1} \dots h_{j''+1} l_{j''} u'\}.$$

Since $h_{j''}u' < l_{j''}u'$, we have $\delta(h \vee k) h_{j-1} \dots h_{j''+1} l_{j''} u' > \delta(h \vee k) h_{j-1} \dots h_{j''+1} h_{j''} u'$ $> \delta(h \vee k)w$ and $\delta'(h \vee k) h_{j-1} \dots h_{j''+1} l_{j''} u' > \delta'(h \vee k) h_{j-1} \dots h_{j''+1} h_{j''} u' > \delta'(h \vee k)w$, and so $t > \{\delta(h \vee k)w, \delta'(h \vee k)w\}$, which is what we desire.

By the proved inequality, we can see that if supremum of two elements x and y exists then $\sup\{x, y\} = \sup\{\delta(h \vee k)w, \delta'(h \vee k)w\}$.

So, it remains to prove that $\sup\{\delta(h \vee k)w, \delta'(h \vee k)w\}$ always exists. Indeed, we shall argue by induction on the length of string δ of hedges. If $|\delta| = 0$ then the assertion is evident, since $(h \vee k)w$ and $\delta'(h \vee k)w$ are comparable. Assume that the assertion holds for $|\delta| \leq i$. For the case $|\delta| = i + 1$, if $\delta(h \vee k)w$ and $\delta'(h \vee k)w$ are comparable then the assertion is clearly true. If $x' = \delta(h \vee k)w$ and $y' = \delta'(h \vee k)w$ are incomparable, then we can use the same argument as for x and y to prove that for any $t \in X$, if $t > \{x', y'\}$ then $t \geq \{\delta_1(h' \vee k')w', \delta_1'(h' \vee k')w'\}$, where h', k' satisfy the same assumption like that on h and k. Since $|\delta_1| < i$, $\sup\{\delta_1(h' \vee k')w', \delta_1'(h' \vee k')w'\}$ exists by the induction hypothesis. Consequently,

$$\sup\{\delta(h \vee k)w, \delta'(h \vee k)w\} = \sup\{\delta_1(h' \vee k')w', \delta_1'(h' \vee k')w'\} = \sup\{x, y\}.$$

Since the proof for the case where $hw < w$ is similar, the theorem is completely proved. ■

For any $x \in X$, let us denote $LH[x] = \{hx/h \in LH\}$. Theorem 2.1 and Theorem 4.1 yield

COROLLARY 4.1. *Let $AX = (X, G, LH, \leq)$ be an RHA and G is a chain. The following statements hold*

(i) *$LH(x)$ is a sublattice of AX.*

(ii) *$LH[x]$ is a distributive sublattice of AX.*

PROPOSITION 4.1. *Let $AX = (X, G, LH, \leq)$ be an RHA and G be a chain. Then, for any $h, k \in LH_i^c$, where $i \in SI^c$, and for any $x \in X$ such that $hx \neq kx$, there exists a lattice isomorphism f from $LH(hx)$ onto $LH(kx)$ defined as follows: $f(\delta hx) = \delta kx$.*

PROOF. By Proposition 3.4. ■

Before proving the distributivity of RHA, we need the following

THEOREM 4.2 [2]. *Let L be a lattice. L is a non-distributive lattice iff M_5 or N_5 can be embedded into L, where M_5 or N_5 are two five-element lattices depicted in Figure 7.*

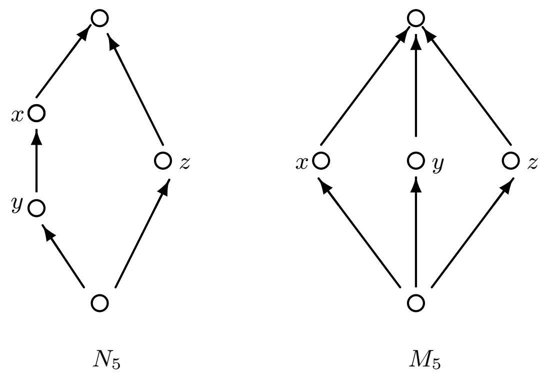


Fig. 7

Now, we shall prove the following theorem.

THEOREM 4.3. *Let $AX = (X, G, LH, \leq)$ be an RHA. If G is a chain then AX is a distributive lattice.*

PROOF. On account of Theorem 4.2, we suppose the contrary that N_5 can be embedded into AX as its sublattice, i.e. there exist elements $x, y, z \in X$ such that x and y are comparable, say $x > y$, and the pairs x,z and y, z are incomparable. In addition, the following equalities hold: $x \cap z = y \cap z$ and $x \cup z = y \cup z$. It can be seen that there exists $a \in G$ such that all elements x, y, z, $x \cap z, x \cup z \in LH(a)$.

Suppose that $x = h_n \dots h_1 a$, $y = k_m \dots k_1 a$, $z = l_p \dots l_1 a$ are canonical representations of x, y, z w.r.t. a, respectively. By Theorem 3.3, there exists an index $j \leq \min(n, m, p) + 1$ such that $h_{j'} = k_{j'} = l_{j'}$ for any $j' < j$, and at least one of the two operations h_j and k_j is different from l_j, say $h_j \neq l_j$. Since x and z are incomparable, by Theorem 3.3, h_j and l_j must belong to the same LH_c^i, for some $i \in SI^c$. Set $w = h_{j-1} \dots h_1 a$, $\delta_x = h_n \dots h_{j+1}$, $\delta_y = k_m \dots k_{j+1}$, $\delta_z = l_p \dots l_{j+1}$. If $k_j = l_j$ then $l_j w < h_j w$, by Theorem 3.3. It follows from Theorem 4.1 that $x \cup z \in LH(h_j w)$ and $y \cup z \in LH(l_j w)$, which contradicts the fact that $x \cup z = y \cup z$. Thus, $k_j \neq l_j$. If $k_j \notin LH_i^c$ then $k_j w < h_j w$ and, by Remark 3.1, we also have $k_j w < h_j w$, and hence, by Theorem 3.3, we obtain $y < z$, a contradiction. Thus, $k_j \in LH_i^c$. According to Theorem 4.1, it follows that

$$\left.\begin{array}{l} x \cup z \in LH((h_j \vee l_j)w), y \cup z \in LH((k_j \vee l_j)w) \\ x \cap z \in LH((h_j \wedge l_j)w), y \cap z \in LH((k_j \wedge l_j)w) \end{array}\right. \text{if } h_j w > w\,.$$

$$\left.\begin{array}{l} x \cup z \in LH((h_j \wedge l_j)w), y \cup z \in LH((k_j \wedge l_j)w) \\ x \cap z \in LH((h_j \vee l_j)w), y \cap z \in LH((k_j \vee l_j)w) \end{array}\right. \text{if } h_j w < w\,.$$

By virtue of axiom (i)(R3), it can easily be seen that $(h_j \vee l_j)w = (k_j \vee l_j)w$ and $(h_j \wedge l_j)w = (k_j \wedge l_j)w$, since $x \cup z = y \cup z$ and $x \cap z = y \cap z$. Consequently, it follows from (ii) of Corollary 4.1 that $h_j = k_j$.

Now, we shall show that the assumption concerning the sublattice N_5 will lead to a contradiction by induction on the length $|\delta_x|$ of the string δ_x mentioned above.

We shall only prove the case $h_j w > w$, since the argument for the other case is similar. Assume that $|\delta_x| = 0$. Then, it follows from Theorem 4.1 that

$$x \cup z = (h_j \vee l_j)w \cup \delta_z(h_j \vee l_j)w, y \cup z = \delta_y(h_j \vee l_j)w \cup \delta_z(h_j \vee l_j)w$$

and

$$x \cap z = (h_j \wedge l_j)w \cap \delta_z(h_j \wedge l_j)w, y \cap z = \delta_y(h_j \wedge l_j)w \cap \delta_z(h_j \wedge l_j)w.$$

Suppose that $(h_j \vee l_j)w$ is a fixed point. By Proposition 3.2, hw is a fixed point, for every $h \in LH_i^c$. By virtue of Theorem 3.3, it follows that $\{x, y, z, x \cup z, y \cap x\}$ is isomorphic to $\{h_j w, k_j w, l_j w, (h_j \vee l_j)w, (k_j \vee l_j)w\}$, which contradicts the fact that $LH[w]$ is distributive by (ii) of Corollary 4.1. Now suppose that $(h_j \vee l_j)w$, $(h_j \wedge l_j)w$ are not fixed points. So, if $(h_j \vee l_j)w = \delta_z(h_j \vee l_j)w$ then $|\delta_z| = 0$, and hence, $x \cup z = (h_j \vee l_j)w$, $x \cap z = (h_j \wedge l_j)w$. Since $x > y$ and $h_j = k_j$, it follows from (iii), Proposition 3.4 that $(h_j \vee l_j)w > \delta_y(h_j \vee l_j)w$ and $(h_j \wedge l_j)w > \delta_y(h_j \wedge l_j)w$. Thus, by Theorem 4.1, $y \cap z = \delta_y(h_j \wedge l_j)w$ and, hence, $x \cap z = (h_j \wedge l_j)w > y \cap z$, contrary to assumption.

If $(h_j \vee l_j)w > \delta_z(h_j \vee l_j)w$ then, by (iii) Proposition 3.4, $(h_j \wedge l_j)w > \delta_y(h_j \wedge l_j)w$, and again by Theorem 4.1, $x \cup z = (h_j \vee l_j)w$ and $x \cap z = \delta_z(h_j \wedge l_j)w$. On the other hand, since $y \cap z = \delta_y(h_j \wedge l_j)w \cap \delta_z(h_j \wedge l_j)w = x \cap z$, it follows that $\delta_y(h_j \wedge l_j)w \geq \delta_z(h_j \wedge l_j)w$. Also, by Proposition 3.4, it implies that $\delta_y(h_j \vee l_j)w \geq \delta_z(h_j \vee l_j)w$, which yields $y \cup z = \delta_y(h_j \vee l_j)w < (h_j \vee l_j)w = x \cup z$, contrary to assumption.

By an analogous argument, the assumption $(h_j \vee l_j)w < \delta_z(h_j \vee l_j)w$ also leads to a contradiction. This concludes the proof of the case where $|\delta_x| = 0$.

Now suppose that a contradiction will follow for all elements x, y and z satisfying the mentioned assumption and, as well, the condition $|\delta_x| < i$. Let us consider x, y and z, which satisfy this assumption as well as the equality $|\delta_x| = i$. It follows from Theorem 4.1 that

$$x \cup z = \delta_x(h_j \vee l_j)w \cup \delta_z(h_j \vee l_j)w, y \cup z = \delta_y(h_j \vee l_j)w \cup \delta_z(h_j \vee l_j)w$$

and

$$x \cap z = \delta_x(h_j \wedge l_j)w \cap \delta_z(h_j \wedge l_j)w, y \cap z = \delta_y(h_j \wedge l_j)w \cap \delta_z(h_j \wedge l_j)w$$

since $h_j w > w$.

Let $x' = \delta_x(h_j \vee l_j)w$, $y' = \delta_y(h_j \vee l_j)w$ and $z' = \delta_z(h_j \vee l_j)w$. By the assumption made on x, y and z, and by Proposition 4.1, it can be seen that x', y' and z' also satisfy the assumption like that made on x, y and z. Then, by an analogous argument as at the beginning of the proof, it follows that there exists an index j' satisfying $j < j' \leq \min(n, m, p) + 1$ such that $h_{j''} = k_{j''} = l_{j''}$ for any $j'' < j'$, and $k_{j'} = h_{j'} \neq l_{j'}$ and, moreover, $h_{j'}, l_{j'} \in LH^c_{i'}$, for some $i' \in SI^c$.

Set $w' = h_{j'-1} \dots h_{j+1}(h_j \vee l_j)w$, $\delta'_x = h_n \dots h_{j'+1}$, $\delta'_y = k_m \dots k_{j'+1}$, $\delta'_z = l_p \dots l_{j'+1}$.

Note that $|\delta'_x| < i$, and, hence, according to the induction hypothesis, it leads to a contradiction. This shows that N_5 cannot be embedded into AX as its sublattice.

Similarly, we can prove that M_5 cannot be embedded into AX as its sublattice, as well. This concludes the proof. ∎

5. Symmetrical RHA. In this section we prepare an algebraic foundation to investigate fuzzy logic, based on an algebraic point of view. As we know, L.A. Zadeh introduced and examined fuzzy logic based on the notion of linguistic variables. A linguistic variable of Truth is characterised by a quintuple $(\text{Truth}, T(\text{Truth}), U, G, M)$, where Truth is the name of the variable; $T(\text{Truth})$ denotes the term-set of Truth, U is a universe of discourse of the base variable, i.e. the unit interval [0,1], G is a syntactic rule for generating linguistic terms of T(Truth), and M is a semantic rule which is a mapping assigning to each linguistic term a fuzzy set on U. In our approach, each term is associated with an element in an RHA, and its meaning is expressed through the structure of such an RHA. Intuitively, we can recognise some what of symmetricity of the set T(Truth) and therefore, we have to examine the so-called symmetrical RHAs.

In natural languages there are many linguistic variables, which have only two distinct primary terms. These terms have intuitive contradictory meaning such as 'true' and 'false', 'old' and 'young', 'large' and 'small', 'tall' and 'short', etc. This suggested Ho and Wechler to investigate in [14] extended hedge algebras (EHAs) with exactly two generators, one of which is called positive generator, denoted by t, and the other is called negative generator, denoted by f. The positive and negative generators are characterised by $Vt \geq t$, $Vf \leq f$ and $t > f$. Under such a normalisation, it seems reasonable to consider 'true', 'old', 'large' and 'tall' as positive generators and 'false', 'young', 'small' and 'short' as negative ones.

In this section we shall also examine RHA with exactly one positive and one negative generator. Let an RHA $AX = (X, G, LH, \leq)$ be given, where the set G of generators consists of one positive and one negative generator, $G = \{t, f\}$. For every x in X, we define a so-called contradictory element of the element x as follows:

Assume that $x = h_n \dots h_1 c$, where $c \in G$, is a representation of x with respect to c. An element y is said to be a contradictory element of x if it can be represented as $h_n \dots h_1 c'$, with $c' \in G$ and $c' \neq c$. For example, $y =$ 'very very false' is a contradictory element of $x =$ 'very very true'; $v =$ 'very little bad' is a contradictory element of $u =$ 'very little good'. It is obvious that a positive generator is a contradictory element of its negative one and vice-versa. By definition, it is also obvious that if y is a contradictory element of x then x is a contradictory element of y.

Definition 5.1. An RHA $AX = (X, G, LH, \leq)$, where G consists exactly of one positive and one negative generator, is said to be a *symmetrical RHA* provided every element x in X has a unique contradictory element in X, denoted by x^-.

We now give a characterisation of symmetrical RHAs.

Theorem 5.1. *A RHA $AX = (X, G, LH, \leq)$ is symmetrical iff AX satisfies the following assumption:*

(SYM) *For every element $x \in X$, x is a fixed point iff x^- is a fixed point.*

Proof. To prove the necessity, assume the contrary that x is a fixed point and $x^- \neq hx^-$, for some $h \in LH$. By definition, $(x^-)^- = x$ and the contradictory element of $u = hx^-$ is the element $u^- = hx = x$. This shows that u and x^- are two distinct contradictory elements of x, a contradiction to the definition of symmetrical RHAs.

Now we prove the sufficiency. Assume that AX satisfies the assumption (SYM). Consider an arbitrary element $x \in X$ and let u and v be two contradictory elements of x.

Suppose that u and v are defined by $u = h_n \dots h_1 c^-$ and $v = k_m \dots k_1 c^-$, which correspond to two representations $h_n \dots h_1 c$ and $k_m \dots k_1 c$ of x, where c, $c^- \in G$ and $c \neq c^-$. It is known that there exists an index $i \leq \min(n, m)$ such that $h_i \dots h_1 c$ is the canonical representation of x w.r.t. c. This implies that $h_j = k_j$ for all $j \leq i$. It is clear that if $m = n = i$ then $u = v$. If either $i < n$ or $i < m$ then x is a fixed point. By the assumption (SYM), $h_i \dots h_1 c^-$ is also a fixed point and, hence, again $u = v$, which concludes the proof. ■

Notice that, by Theorem 4.3, every symmetrical RHA $AX = (X, G, LH, \leq)$ is a distributive lattice. Moreover, we have the following:

Theorem 5.2. *For every symmetrical RHA $AX = (X, G, LH, \leq)$, the following statements hold:*

(i) $(hx)^- = hx^-$*, for every* $h \in LH$ *and* $x \in X$

(ii) $(x^-)^- = x$*, for every* $x \in X$.

(iii) $hx > x$ *iff* $hx^- < x^-$*, for every* $h \in LH$ *and* $x \in X$.

(iv) $hx > kx$ *iff* $hx^- < kx^-$*, for any* $h, k \in LH$ *and* $x \in X$.

(v) $x < y$ *iff* $x^- > y^-$*, for any* $x, y \in X$.

(vi) $(x \cup y)^- = x^- \cap y^-$ *and* $(x \cap y)^- = x^- \cup y^-$*, for any* $x, y \in X$*, where* $\cup$ *and* $\cap$ *stand for join and meet, respectively, in* AX.

Proof. The assertion (i) is a direct consequence of the definition of the contradictory elements in AX. Assertion (ii) follows immediately from the fact that, for every $x \in X$, x^- is uniquely defined and x is a contradictory element of x^-.

Now we shall prove assertion (iii) by induction on the length of the canonical representations of x w.r.t. a generator:

Let $|x| = 1$, where $|x|$ denotes the length of the canonical representation of x w.r.t. a generator c. Clearly, $x = c \in G$. If $Vc > c$ and $hc > c$ then V and h are compatible. Thus, the inequality $hc^- < c^-$ follows from the fact that $Vc^- < c^-$. If $Vc < c$ and $hc > c$ then V and h are converse. Hence, $Vc^- > c^-$ implies $hc^- < c^-$. For the other cases, the proof is similar. Therefore, the assertion (iii) is true for $|x| = 1$. Assume that (iii) holds for all x satisfying $|x| < i$. Let $u = hx$ with $|u| = i$ and consider the case that $khx > hx$. If k is positive w.r.t. h, then $hx > x$ and, by the induction hypothesis, $hx^- < x^-$. Hence, it implies that $khx^- < hx^-$, since the equality cannot occur, by assumption (SYM). By the same argument, it can be proved that $ku^- < u^-$ implies $ku > u$.

Analogously, we can prove (iii) for the other cases.

Now, we prove (iv). If h and k are converse, then $hx > x > kx$ and by (iii) it implies that $hx^- < x^- < kx^-$. If h and k are compatible then $hx > kx > x$, which implies $h > k$ in $LH^c + I$. Since, by (iii), we have $kx^- < x^-$ and, hence, $hx^- < kx^-$.

Note that, as above, the equality $hx^- = kx^-$ does not occur, since in the contrary case, x^- is a fixed point and, hence, so is its contradictory element x, by (SYM).

The proof for the two last assertions will be more complicated. First, we prove (v). It is known that if $x \in LH(c)$ and $y \in LH(c')$, with $c \neq c'$, then $c > c'$ can follow from $x > y$. By definition, $x^- \in LH(c')$ and $y^- \in LH(c)$ and, hence, $x^- < y^-$.

Suppose that $x, y \in LH(c)$ and $x > y$, and $x = h_n \ldots h_1 w$, $y = k_m \ldots k_1 w$ are, respectively, the canonical representation of x and y w.r.t. w, where $w \in LH(c)$ and $h_1 \neq k_1$. Note that one of h_1 and k_1 may be the identity I. From $x > y$ it follows that $h_1 w > k_1 w$, by Theorem 3.3 and, by (iv), we have $h_1 w^- < k_1 w^-$.

Without loss of generality, we assume $h_1 \neq I$ and shall prove the necessity of (v) by induction on the length of the string $\sigma = h_n \ldots h_1$, denoted by $|\sigma|$.

First consider the case when $|\sigma| = 1$, i.e. $x = h_1 w > y = k_m \ldots k_1 w$. If there is no index i in SI^c such that both h_1 and k_1 belong to LH_i^c, then from the fact $h_1 w^- < k_1 w^-$ above it follows that $x = h_1 w^- < y^-$, by (1),Theorem 3.3. In the opposite case, i.e. there exists an index $i \in SI^c$ such that $h_1, k_1 \in LH_i^c$ and also by (1),Theorem 3.3, from the fact $x > y$ it follows that $h_1 w \geq k_m \ldots k_2 h_1 w$. If $x = h_1 w = k_m \ldots k_2 h_1 w$ occurs in the last inequality, then x is a fixed point and, hence, $k_1 w$ is also a fixed point, i.e. $y = k_1 w$. Then, by (SYM), $h_1 w^-$ and $k_1 w^-$ are also fixed points. Thus, $x^- = h_1 w^- < k_1 w^- = y^-$. If $h_1 w > k_m \ldots k_2 h_1 w$, then $k_2 \neq I$ and, by Theorem 3.3, we have $h_1 w > k_2 h_1 w$. Hence, it follows from (iii) that $h_1 w^- < k_2 h_1 w^-$. Again by Theorem 3.3, the last inequality implies $h_1 w^- < k_m \ldots k_2 h_1 w^-$. Moreover, by (R3), we have $k_m \ldots k_2 h_1 w^- < k_m \ldots k_2 k_1 w^-$ and so, $x^- = h_1 w^- < k_m \ldots k_2 h_1 w^- < k_m \ldots k_2 k_1 w^- = y^-$, which is what we require to prove for the case $|\sigma| = 1$.

Now let us assume the induction hypothesis, that $x = h_n \ldots h_1 w > y = k_m \ldots k_1 w$ implies that $x^- = h_n \ldots h_1 w^- > y^- = k_m \ldots k_1 w^-$ for all strings of hedges σ satisfying $|\sigma| < p$, and for any $w \in LH(c)$. To prove the induction conclusion let us consider $x = h_p \ldots h_1 w$, i.e. $|\sigma| = p$.

If there is no index i in SI^c such that both h_1 and k_1 belong to LH_i^c, then from $h_1 w^- < k_1 w^-$ it follows that $x^- < y^-$, by (1), Theorem 3.3. If there exists an index

i in SI^c such that both h_1 and k_1 belong to LH_i^c, then from $x = h_p \dots h_1 w > y = k_m \dots k_1 w$, it follows, by Theorem 3.3, that $x = h_p \dots h_1 w \geq k_m \dots k_2 h_1 w$. Putting $y_1 = k_m \dots k_2 h_1 w$, it is clear that $y_1 > y$, by (R3). Since $h_1 w^- < k_1 w^-$, as a consequence of (R3), we have $y_1^- < y^-$. Applying Theorem 3.3 to two elements x and y_1, it follows that there exists an index j such that $2 \leq j \leq \min(p, m) + 1$ and $h_{j'} = k_{j'}$, for $2 \leq j' < j$. If $x = y_1$ then we have $p = m = j$ and $h_j w_j = k_j w_j$, where $w_j = h_{j-1} \dots h_1 w$. Therefore, we have $h_j w_j^- = k_j w_j^-$, by (iv). Hence, $x^- = h_j w_j^- = k_j w_j^- = y_1^- < y^-$.

Assume that $x > y_1$, by Theorem 3.3, it follows that $h_j w_j > k_j w_j$, and, thus, $h_j w_j^- < k_j w_j^-$, by (iv). Note that the length of the string $\sigma' = h_p \dots h_{j+1}$ is less than or equal to p. Therefore, by the induction hypothesis, it follows that $x^- = h_p \dots h_{j+1} h_j w_j^- < y_1^- = k_n \dots k_{j+1} k_j w_j^-$. Consequently, we have $x^- < y^-$. On account of (ii), it is evident that the sufficiency of (v) can be deduced directly from the necessity. This concludes the proof of (v).

To prove (vi), we find first, by (v), that $x = y$ iff $x^- = y^-$ and that x and y are incomparable iff x^- and y^- are incomparable. We shall prove the validity of $(x \cup y)^- = x^- \cap y^-$. The proof for $(x \cap y)^- = x^- \cup y^-$ can be obtained by duality.

If x and y are comparable then the assertion follows directly from (v). Suppose that x and y are incomparable and $x = h_n \dots h_1 w$, $y = k_m \dots k_1 w$ are, respectively, the canonical representation of x and y w.r.t. w, where $w \in LH(c)$, for some $c \in G$, such that $h_1 \neq k_1$. We shall prove the assertion by induction on the length of the string $\sigma = h_n \dots h_1$, denoted by $|\sigma|$.

First, let us suppose that $|\sigma| = 1$, i.e. $x = h_1 w$. By Theorem 3.3, it follows that there exists an index i in SI^c such that both h_1 and k_1 belong to LH_i^c. By Theorem 4.1, we have

$$x \cup y = \begin{cases} (h_1 \vee k_1) w \cup k_m \dots k_2 (h_1 \vee k_1) w \text{ if } h_1 w > w, \\ (h_1 \wedge k_1) w \cup k_m \dots k_2 (h_1 \wedge k_1) w \text{ if } h_1 w < w. \end{cases}$$

Recall that LH_i^c is a sublattice of $LH^c + I$. Hence, if $h_1 w > w$ then $(h_1 \vee k_1) w > w$. If $h_1 w$ is a fixed point then so are $(h_1 \vee k_1) w$ and $k_1 w$, by Proposition 3.1. Hence, $y = k_1 w$ and $x \cup y = (h_1 \vee k_1) w$. Furthermore, by (SYM), it follows that $h_1 w^-$, $k_1 w^-$, $(h_1 \vee k_1) w^-$ are also fixed points, i.e. $x^- = h_1 w^-, y^- = k_1 w^-$. On the other hand, by (iii), it follows from $h_1 w > w$ that $h_1 w^- < w^-$. Thus, by Theorem 4.1, we have $x^- \cap y^- = (h_1 \vee k_1) w^- = (x \cup y)^-$.

Now assume that $h_1 w$ is not a fixed point. If k_2 is positive w.r.t. $(h_1 \vee k_1)$, then $k_2(h_1 \vee k_1) w > (h_1 \vee k_1) w$. Notice that equality cannot occur, since if $k_2(h_1 \vee k_1) w = (h_1 \vee k_1) w$, then $(h_1 \vee k_1) w$ is a fixed point and, hence, so is $h_1 w$, a contradiction. By Theorem 3.3, we have $k_m \dots k_2 (h_1 \vee k_1) w > (h_1 \vee k_1) w$, which yields $x \cup y = k_m \dots k_2 (h_1 \vee k_1) w$. By definition, we have $(x \cup y)^- = k_m \dots k_2 (h_1 \vee k_1) w^-$. On the other hand, by (iii), from $k_2(h_1 \vee k_1) w > (h_1 \vee k_1) w$ it follows that $k_2 (h_1 \vee k_1) w^- < (h_1 \vee k_1) w^-$. Consequently, $k_m \dots k_2 (h_1 \vee k_1) w^- < (h_1 \vee k_1) w^-$, by Theorem 3.3. Thus, $x^- \cap y^- = k_m \dots k_2 (h_1 \vee k_1) w^-$, which is the requirement.

Since the proof for the case $h_1 w < w$ is similar, this concludes the proof for the case $|\sigma| = 1$.

Now, let us suppose that $(x \cup y)^- = x^- \cap y^-$ holds for all x and y with the string σ of hedges satisfying $|\sigma| < p$, and with any $w \in LH(c)$. We shall prove the induction conclusion for $x = h_p \dots h_1 w$, where $|\sigma| = p$.

Since x and y are incomparable, it follows from Theorem 3.3 that there exists an index i in SI^c such that both $h_1, k_1 \in LH_i^c$. Moreover, by Theorem 4.1, we have

$$x \cup y = \begin{cases} h_p \dots h_2(h_1 \vee k_1)w \cup k_m \dots k_2(h_1 \vee k_1)w \text{ if } h_1 w > w, \\ h_p \dots h_2(h_1 \wedge k_1)w \cup k_m \dots k_2(h_1 \wedge k_1)w \text{ if } h_1 w < w. \end{cases}$$

First, assume that $h_1 w > w$. By (iii), $h_1 w^- < w^-$ and by (v), x^- and y^- are incomparable. So, on account of Theorem 4.1, we have $x^- \cap y^- = h_p \dots h_2(h_1 \vee k_1)w^- \cap k_m \dots k_2(h_1 \vee k_1)w^-$, where $x^- = h_p \dots h_1 w^-$ and $y^- = k_m \dots k_1 w^-$. Further, if $h_p \dots h_2(h_1 \vee k_1)w$ and $k_m \dots k_2(h_1 \vee k_1)w$ are comparable, then, by (v), it is obvious that

$$\begin{aligned}(x \cup y)^- &= (h_p \dots h_2(h_1 \vee k_1)w \cup k_m \dots k_2(h_1 \vee k_1)w)^- \\ &= h_p \dots h_2(h_1 \vee k_1)w^- \cap k_m \dots k_2(h_1 \vee k_1)w^- \\ &= x^- \cap y^-.\end{aligned}$$

If $x_1 = h_p \dots h_2(h_1 \vee k_1)w$ and $y_1 = k_m \dots k_2(h_1 \vee k_1)w$ are incomparable then, by Theorem 3.3, there exists an index j satisfying $2 \le j < \min(p, m) + 1$ such that $h_{j'} = k_{j'}$ for all j' satisfying $2 \le j' < j$, and there exists an index i' in SI^c such that $h_j, k_j \in LH_i^c$. Thus, it follows from Theorem 4.1 that

$$x_1 \cup y_1 = \begin{cases} h_p \dots h_{j+1}(h_j \vee k_j)w_j \cup k_m \dots k_{j+1}(h_j \vee k_j)w_j \text{ if } h_j w_j > w_j, \\ h_p \dots h_{j+1}(h_j \wedge k_j)w_j \cup k_m \dots k_{j+1}(h_j \wedge k_j)w_j \text{ if } h_j w_j < w_j, \end{cases}$$

where $w_j = h_{j-1} \dots h_2(h_1 \vee k_1)w$. Clearly, $|\sigma'| < p$, where $\sigma' = h_p \dots h_{j+1}$. As proved above, x_1^- and y_1^- must be incomparable.

If $h_j w_j > w_j$ then $h_j w_j^- < w_j^-$, by (iii). Therefore, again by Theorem 4.1,

$$x^- \cap y^- = h_p \dots h_{j+1}(h_j \vee k_j)w_j^- \cap k_m \dots k_{j+1}(h_j \vee k_j)w_j^- = x_1^- \cap y_1^-.$$

Now, combining the obtained equalities and taking into account the induction hypothesis, we obtain $(x \cup y)^- = (h_p \dots h_{j+1}(h_j \vee k_j)w_j \cup k_m \dots k_{j+1}(h_j \vee k_j)w_j)^- = (x_1 \cup y_1)^- = x_1^- \cap y_1^- = x^- \cap y^-$.

For the case $h_j w_j < w_j$, the proof is similar.

Since the proof for the case $h_1 w < w$, can be obtained by duality, the theorem is completely proved. ∎

6. RHA of linguistic truth variable as an algebraic foundation of linguistic-valued logic. It is known that linguistic variables, especially linguistic truth variable, which were interpreted in the framework of fuzzy set theory by Zadeh as quintuple $(X, T(X), U, G, M)$ (see, e.g., [26-28]), have an important role in investigation of fuzzy logic and approximate reasoning methods. In the same time, the symmetrical EHA of linguistic truth variable can be taken as a basic algebraic structure for linguistic-valued fuzzy logic and linguistic reasoning methods developed in [5-7]. In the previous section, we have examined symmetrical RHAs and proved their several important properties. Particularly, all those properties hold for each RHA of linguistic truth variables. However, as an algebraic structure modelling domains of linguistic truth variable, we shall discuss in more detail the semantics of negation and implication and show that their properties may be appropriate for a certain fuzzy logic.

Let us consider a symmetrical RHA $AT = (T, C, LH, \leq)$ of linguistic truth variable generated by two primary generators 'True' and 'False', where 'True' is the positive generator and 'False' is the negative one, i.e. $C = \{\text{True}, \text{False}\}$. For simplicity, we assume that AT is finite.

It is known that the RHA AT under consideration is a distributive lattice. Thus, the lattice operations of join and meet can model the semantics of the logical disjunction and conjunction. Now, we show that the operator "$-$" can be interpreted as a negation.

Let $AT = (T, C, LH, \leq)$ be a symmetrical RHA of linguistic truth variable, where the underlying set T is defined as follows:

First we define $LH_n[C]$, for $n \geq 0$, by the following procedure:

$$LH_0[C] = C, \quad LH_{n+1}[C] = LH[LH_n[C]].$$

Notice that, by our convention, the identity I will only stand in a prefix of an expression, for instance $I \ldots Ih \ldots h'x$, and it means that if I occurs explicitly in an expression, then every hedge operation applying to I has no effect, i.e. $hIu = Iu = u$. Therefore, it is easily seen that $C \subset LH[C] \subset LH_2[C] \subset \ldots \subset LH_n[C] \subset \ldots$. In general, this chain is infinite. However, in applications, we use only a bounded number of hedges in concatenation and, hence, we require the above chain of inclusions to be finite.

Let p be a fixed positive integer. For any $x \in LH_p[C]$ and $x \notin LH_{p-1}[C]$, we define $hx = x$, for every $h \in LH$ and, so, we have $C \subset LH[C] \subset LH_2[C] \subset \ldots \subset LH_p[C]$. Let $T = LH_p[C]$. Clearly, if AT is finite, then there exists $p \geq 0$ such that $T = LH_p[C]$. It is known that this algebra AT is a complete distributive lattice. In addition, based on the properties of the unit-operation V, it is easy to see that the elements $V^p\text{True}$ and $V^p\text{False}$ are the greatest and least elements in AT and they will be denoted by 1 and by 0, respectively.

As observed by Ho & Wechler in [14], the negation of vague concept may often be its contradictory concept, if it exists. For example, 'good' and 'true' are vague concepts and they involve an intuitively intended meaning. Refuting this meaning, one may often think of the meaning of the concepts 'bad' and 'false', which are the contradictory concepts of 'good' and 'true', respectively, and vice-versa. This interpretation was adopted in many investigations of fuzzy reasoning (see, e.g., [25-28]). Furthermore, it may still be possible to discuss how to refute statements containing vague concepts which are not primary concepts, for example, the concept 'Very little true'. It is natural to regard the negation of 'Very little true' as to be a concept of 'false' and it may most probably be the concept 'Very little false', a contradictory concept of the concept 'Very little true'.

Therefore, analogous to the paper [14] by Ho and Wechler, we now define the negation of an element x in AT to be its contradictory element, i.e. $-x = x^-$. This operation $-$ is called negation operation. The implication operation, denoted by $\Rightarrow$, in this algebra is defined in this paper in a regular way, i.e. by means of the negation operation and the join operation, as follows:

$$x \Rightarrow y = \neg x \cup y, \text{ for any } x \text{ and } y \text{ of } AT.$$

We introduce in AT a new generator W defined by $LH(\text{True}) > W > LH(\text{False})$ and $hW = W$, for all $h \in LH$. This element W can be understood as 'Unknown'.

Note that the algebra AT with the new element W also preserves all properties of the symmetrical RHA, where $W = W^-$. Therefore, without loss of generality we assume that the set of generators of the symmetrical RHA AT consists of three elements True, W and False, where W is defined as above and True $> W >$ False.

Let $AT = (T, C, LH, \leq)$, with $C = \{\text{True}, W, \text{False}\}$ and underlying set T defined as above, be a symmetrical RHA of the linguistic truth variable. As examined above, the operations $\cup$, $\cap$, $\neg$, $\Rightarrow$ can be derived in AT and, so, we can write

$$AT = (T, C, LH, \leq, \neg, \cup, \cap, \Rightarrow, 0, 1).$$

Throughout this section we always write simply AT for such an algebra.

We are now ready to discuss some elementary properties of the negation operation and the implication operation. From the definition of these operations and Theorem 5.2, it is not difficult to see that the following holds.

THEOREM 6.1. *Let AT be a symmetrical RHA of the linguistic truth variable. Then*

(i) $\neg(hx) = h\neg x$, *for every $h \in LH$ and $x \in T$.*
(ii) $\neg(\neg x) = x$, *for all $x \in T$.*
(iii) $\neg(x \cup y) = \neg x \cap \neg y$ *and* $\neg(x \cap y) = \neg x \cup \neg y$, *for all $x, y \in T$.*
(iv) $x \cap \neg x \leq y \cup \neg y$, *for all $x, y \in T$.*
(v) $x \cap \neg x \leq W \leq x \cup \neg x$, *for all $x \in T$.*
(vi) $\neg 1 = 0$, $\neg 0 = 1$ *and* $\neg W = W$.
(vii) $x > y$ *iff* $\neg x < \neg y$, *for all $x, y \in T$.*

It is worth to mention that the statements (ii)-(iv) of Theorem 6.1 show that the algebra AT is a Kleen algebra in the sense of Skala [24] and (vi) shows that this algebra includes the 3-valued Łukasiewicz algebra $\{0, W, 1\}$ as its subalgebra.

As a consequence of the definition of the implication operation and Theorem 6.1, we have the following

THEOREM 6.2. *Let AT be a symmetrical RHA of the linguistic truth variable. Then*

(i) $x \Rightarrow y = \neg y \Rightarrow \neg x$.
(ii) $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$.
(iii) $x \Rightarrow y \geq x' \Rightarrow y'$ *if* $x \leq x$ *and/or* $y \geq y'$.
(iv) $x \Rightarrow y = 1$ *iff either* $x = 0$ *or* $y = 1$.
(v) $1 \Rightarrow x = x$ *and* $x \Rightarrow 1 = 1$; $0 \Rightarrow x = 1$ *and* $x \Rightarrow 0 = \neg x$.
(vi) $x \Rightarrow y \geq W$ *iff either* $x \leq W$ *or* $y \geq W$, *and*
$x \Rightarrow y \leq W$ *iff* $x \geq W$ *and* $y \leq W$.

7. Conclusions. In this paper RHA has been introduced and investigated. We have proved that RHA with a chain of the primary generators is a distributive lattice. We would like to note that the primary generators of almost linguistic variables constitute linearly ordered sets. Furthermore, in the symmetrical RHAs of linguistic truth variable we are able to define negation operation and implication operation. Note that a method in linguistic reasoning based on linguistic-valued fuzzy logic corresponding to the symmetrical EHAs has been established in [7]. In this direction, we hope that it is possible to develop deductive reasoning methods based on RHAs. Remember that the symmetrical RHAs have a finer structure than that of the symmetrical EHAs and their operations may model the semantics of logical connectives more appropriately.

Some researchers, who are familiar with the fuzzy sets theory, might have some criticisms on the way we have defined negation and implication as above. In the authors' opinion, in an algebraic approach to fuzzy logic, the way we define negation, which satisfies (ii) and (vii) of Theorem 6.1, may be unique. Remember that there exists only a unique complement operation in a finite linear set, that satisfies these two properties.

The important thing which justifies the reasonableness of negation and implication is their properties, which have been proved in the algebras under consideration. Theorem 6.1 and 6.2 show that the symmetrical RHAs of the linguistic truth variable are logically rich enough to examine a kind of fuzzy logic, called linguistic-valued logic, and develop linguistic reasoning methods.

For comparison of our study with fuzzy set approach to fuzzy logics, we present roughly here a general idea of fuzzy logics based on fuzzy sets theory. As we have said previously, a basic notion to construct approximate reasoning methods is the concept of linguistic variable (see [28]), which is interpreted as quintuple $(X, T(X), U, G, M)$, where X is the name of the variable; $T(X)$ denotes the term-set of X, U is a universe of discourse of the base variable, G is a syntactic rule for generating linguistic terms of $T(X)$, and M is a semantic rule which is a mapping assigning to each linguistic term a fuzzy set on U, i.e. a function from U into the unit interval $[0,1]$. Let us denote by $F(U,[0,1])$ the set of all functions from U into the unit interval $[0,1]$. If τ is a linguistic value in $T(X)$ then $M(\tau) \in F(U,[0,1])$, the set of all functions from U into $[0,1]$, is a meaning of τ, and if, for example, a connective OR occurs in τ, for example $\tau =$ App.True OR Poss.True, then $M(OR)$ is an operation on $F(U,[0,1])$, e.g. $M(OR) =$ Max or $M(OR)$ is a t-norm operation. So, a non-computational structure $T(X)$, from the viewpoint of fuzzy set theory, is embedded in $F(U,[0,1])$, a computational structure.

The authors emphasise that in approximate reasoning methods, the semantic mapping M is rather subjective and, in applications, its reasonableness is justified by experiments. However, it is clear that there is an intuitive structure of the set $T(X)$ and, then, a question arises on mathematical point of view, whether M preserves this structure, or, more exactly, whether M models the intuitive structure of $T(X)$ appropriately.

Based on our study, the answer is no by the following reasons.

First, we have pointed out in the paper that $T(X)$ has a rich enough algebraic structure and M does not preserve even the ordering relation of $T(X)$.

Second, from the algebraic point of view we should use the mathematical structure of the image $M(T(X)) \subseteq F(U,[0,1])$ as an underlying structure to investigate and construct fuzzy reasoning methods. However, it can be seen that this structure is too weak and, hence, one has to use the functional structure of the whole set $F(U,[0,1])$ instead of $M(T(X))$ to develop fuzzy reasoning methods, irrespectively of whether the set $F(U,[0,1])$ models the structure of $M(T(X))$ suitably or not. Note that the set $M(T(X))$ is countably infinite and in applications it is in general finite only, and we can see that the difference between the structures of $M(T(X))$ and $F(U,[0,1])$ is too big and, so, this may be a main reason which causes certain unreasonable questions, in the authors' opinion, and large errors in application of fuzzy reasoning methods (see [3,16,17,19]). For example, the operation MAX on fuzzy sets (as functions) in $M(T(X))$, which has to be defined meaningfully only in $F(U,[0,1])$, cannot model the connective OR in natural language reasonably, especially in the case $M(T(X))$ is finite.

From this point of view, our main contribution is the following.

1. We have proved that $T(X)$ has a good enough mathematical structure, denoted by $\mathbf{A}(T(X))$ and it is also a computational structure. Particularly, the structure $\mathbf{A}(T(\text{Truth}))$ of linguistic variable of Truth can be considered as a rich enough logical foundation for approximate reasoning.

2. Now, linguistic variable can be interpreted as a quartuple $(X, T(X), G, M)$, where X, $T(X)$ and G are the same as above, but M is a mapping from $T(X)$ onto $\mathbf{A}(T(X))$ which models rather well the meaning of the terms in $T(X)$.

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