4-DIMENSIONAL c-SYMPLECTIC $S^1$-MANIFOLDS WITH NON-EMPTY FIXED POINT SET NEED NOT BE c-HAMILTONIAN

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The aim of this article is to answer a question posed by J. Oprea in his talk at the Workshop “Homotopy and Geometry”.

In equivariant symplectic geometry it is often important to give a sufficient condition for the action to be Hamiltonian. For example McDuff [M] showed that, if the circle group acts on a compact symplectic 4-manifold with non-empty fixed point set, then the action is necessarily Hamiltonian. This phenomenon does not occur in higher dimensions.

Let $S^1$ act on a symplectic manifold $(M,\omega)$. If $X$ denotes the vector field on $M$ generated by the action, then the action is called Hamiltonian if the 1-form $i(X)\omega$ is exact. This condition is also equivalent to the existence of an equivariant cohomology class in $H^2_{S^1}(M)$ descending to $[\omega] \in H^2(M)$ [A-B]. This cohomological condition can also be expressed in several apparently different ways, e.g. [H-Y, G]. Here we utilize the following one [L-O]. Let $\varphi : S^1 \times M \to M$ denote the given action and $\varphi^* : H^*(M) \to H^*(S^1 \times M) = H^*(S^1) \otimes H^*(M)$ the induced homomorphism. Here the coefficients are taken in the reals. For $x \in H^q(M)$ let $\lambda(x) \in H^{q-1}(M)$ be defined by

$$\varphi^*(x) = 1 \otimes x + u_1^* \otimes \lambda(x)$$

where $u_1^* \in H^1(S^1)$ is the standard generator. The cohomology class $[i(X)\omega]$ coincides with $\lambda[\omega]$, and hence the action is Hamiltonian if and only if $\lambda[\omega] = 0$.

Let now $(M,x)$ be a cohomologically symplectic manifold (c-symplectic manifold in short in [L-O]). This means $M$ is $2n$-dimensional and $x$ is a cohomology class in $H^2(M)$ such that $x^n \neq 0$ in $H^{2n}(M)$. Suppose the group $S^1$ acts on $M$, Oprea calls $(M,x)$ c-Hamiltonian if $\lambda(x) = 0$. He asks whether $(M,x)$ is c-Hamiltonian when $\dim M = 4$ and the fixed point set $M^{S^1}$ is non-empty.

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[91]
In the sequel we shall show that there are examples in which dim $M = 4$, $M^{S^1} \neq \emptyset$ but $(M, x)$ is not c-Hamiltonian. First remark that, if $M$ is an orientable closed 4-manifold, there is a basis $x_1, \ldots, x_l$ of $H^2(M)$ such that $x_j^2 \neq 0$ for each $j$. Hence, if the assumptions $M^{S^1} \neq \emptyset$ and $x^2 \neq 0$ implied $\lambda(x) = 0$, it would follow $\lambda = 0$ on the whole $H^2(M)$. Therefore it will suffice to give an example $M$ in which dim $M = 4$, $M^{S^1} \neq \emptyset$ and $\lambda \neq 0$ on $H^2(M)$, or equivalently $\lambda : H_1(M) \to H_2(M)$, the transpose of $\lambda$, is non-trivial. Note that $\lambda : H_1(M) \to H_2(M)$ is given by

\[ \lambda(\alpha) = \varphi_*(u_1 \otimes \alpha), \]  

where $u_1 \in H_1(S^1)$ is the dual of $u_1^* \in H^1(S^1)$.

We need a lemma whose proof is an easy exercise.

**Lemma.** Let $S^1$ act on itself by left translations. Then the actions of $S^1$ on the trivial vector bundle $S^1 \times \mathbb{R}^1$ which cover the action on the base $S^1$ are unique up to gauge equivalences. They are equivalent to the following one:

\[ g(z, w) = (gz, w). \]

Put $M_1 = S^4$ equipped with the standard $S^1$ action. It has two isolated fixed points. Put also $M_2 = S^1 \times N$, where $N = S^1 \times S^2$. We let $S^1$ act on $M_2$ by

\[ g(z, y) = (gz, y), \quad z \in S^1, \quad y \in N. \]

Take a non-trivial orbit $B_1$ in $M_1$ and let $V_1$ be an invariant closed tubular neighborhood of $B_1$. Also put $B_2 = S^1 \times y_0 \subset S^1 \times N = M_2$ where $y_0 \in N$, and let $V_2$ be an invariant closed tubular neighborhood. Then $V_1$ and $V_2$ are identified with $S^1$, and $V_1$ and $V_2$ with $S^1 \times D^3$ where $D^3 \subset \mathbb{R}^4$ is the unit disk. Moreover, the action of $S^1$ on $V_1$ and $V_2$ can be assumed to be of the form as in Lemma. Let $W_j$ be defined by

\[ W_j = M_j \setminus V_j \]

for $j = 1, 2$. The boundary $\partial W_j$ of $W_j$ is identified with $\partial(S^1 \times D^3) = S^1 \times S^2$. We define the manifold $M$ by gluing $W_1$ and $W_2$ along the boundary $S^1 \times S^2$ and rounding the corner. The action of $S^1$ on $W_1$ and $W_2$ is also glued to define an action on $M$.

**Proposition.** The above 4-manifold $M$ provides an example for which $M^{S^1} \neq \emptyset$ and $\lambda \neq 0$.

**Proof.** Clearly $M^{S^1} \neq \emptyset$.

It is easy to see that $W_1$ is diffeomorphic to $D^2 \times S^2$ and $W_2$ is diffeomorphic to $S^1 \times U$ where $U = N \setminus D^3$. In particular $U$ is homotopically equivalent to $S^1 \lor S^2$ and we have

\[ H_1(U) \cong \mathbb{R}, \quad H_2(U) \cong \mathbb{R}. \]

Let $v_j$ denote a generator of $H_j(U)$ for $j = 1, 2$. Easy calculations using the Mayer-Vietoris sequence of the triple $(W_1, W_2; M)$ yield the following table:

- $H_1(M) \cong \mathbb{R}$ generated by the image of $v_1 \in H_1(W_2)$,
- $H_2(M) \cong H_2(W_2) \cong \mathbb{R} \oplus \mathbb{R}$ generated by $u_1 \otimes v_1$ and $v_2$,
- $H_3(M) \cong H_3(W_2) \cong \mathbb{R}$ generated by $u_1 \otimes v_2$. 

The image of $v_1$ will be denoted by the same letter. Then we have

$$
\iota \lambda(v_1) = u_1 \otimes v_1
$$

by virtue of (1), because

$$
\varphi(S^1 \times S^1_1) = S^2_1 \times S^1_1 \subset W_2 \subset M,
$$

where $S^1_1 = S^1 \times y_1 \subset N$ with $y_1 \in S^2$ such that $y_0 \not\in S^1_1$ and we write $M_2$ as $S^2_1 \times N$. Thus we have shown the above $M$ is a desired example.

**Remark.** It is not difficult to see the following fact. If one takes any orientable closed 4-dimensional $S^1$-manifold $M_1$ with $M_1^{S^1} \neq \emptyset$ instead of $S^4$ and any orientable closed 3-manifold $N$ with non-trivial first Betti number instead of $S^1 \times S^2$, then one gets a similar example as above.

**References**


