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4-DIMENSIONAL c-SYMPLECTIC S^1-MANIFOLDS WITH NON-EMPTY FIXED POINT SET NEED NOT BE c-HAMILTONIAN

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The aim of this article is to answer a question posed by J. Oprea in his talk at the Workshop "Homotopy and Geometry".

In equivariant symplectic geometry it is often important to give a sufficient condition for the action to be Hamiltonian. For example McDuff [M] showed that, if the circle group acts on a compact symplectic 4-manifold with non-empty fixed point set, then the action is necessarily Hamiltonian. This phenomenon does not occur in higher dimensions.

Let S^1 act on a symplectic manifold (M, ω). If X denotes the vector field on M generated by the action, then the action is called Hamiltonian if the 1-form $i(X)\omega$ is exact. This condition is also equivalent to the existence of an equivariant cohomology class in $H^2_{S^1}(M)$ descending to $[\omega] \in H^2(M)$ [A-B]. This cohomological condition can also be expressed in several apparently different ways, e.g. [H-Y, G]. Here we utilize the following one [L-O]. Let $\varphi : S^1 \times M \to M$ denote the given action and $\varphi^* : H^*(M) \to H^*(S^1 \times M) = H^*(S^1) \otimes H^*(M)$ the induced homomorphism. Here the coefficients are taken in the reals. For $x \in H^q(M)$ let $\lambda(x) \in H^{q-1}(M)$ be defined by

$$\varphi^*(x) = 1 \otimes x + u_1^* \otimes \lambda(x)$$

where $u_1^* \in H^1(S^1)$ is the standard generator. The cohomology class $[i(X)\omega]$ coincides with $\lambda[\omega]$, and hence the action is Hamiltonian if and only if $\lambda[\omega] = 0$.

Let now (M, x) be a cohomologically symplectic manifold (c-symplectic manifold in short in [L-O]). This means M is $2n$-dimensional and x is a cohomology class in $H^2(M)$ such that $x^n \neq 0$ in $H^{2n}(M)$. Suppose the group S^1 acts on M. Oprea calls (M, x) c-Hamiltonian if $\lambda(x) = 0$. He asks whether (M, x) is c-Hamiltonian when $\dim M = 4$ and the fixed point set M^{S^1} is non-empty.

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In the sequel we shall show that there are examples in which $\dim M = 4$, $M^{S^1} \neq \emptyset$ but (M, x) is not c-Hamiltonian. First remark that, if M is an orientable closed 4-manifold, there is a basis $x_1, \ldots, x_l$ of $H^2(M)$ such that $x_j^2 \neq 0$ for each j. Hence, if the assumptions $M^{S^1} \neq \emptyset$ and $x^2 \neq 0$ implied $\lambda(x) = 0$, it would follow $\lambda = 0$ on the whole $H^2(M)$. Therefore it will suffice to give an example M in which $\dim M = 4$, $M^{S^1} \neq \emptyset$ and $\lambda \neq 0$ on $H^2(M)$, or equivalently ${}^t\lambda : H_1(M) \to H_2(M)$, the transpose of λ, is non-trivial. Note that ${}^t\lambda : H_1(M) \to H_2(M)$ is given by

$$ {}^t\lambda(\alpha) = \varphi_*(u_1 \otimes \alpha), \tag{1} $$

where $u_1 \in H_1(S^1)$ is the dual of $u_1^* \in H^1(S^1)$.

We need a lemma whose proof is an easy exercise.

LEMMA. *Let S^1 act on itself by left translations. Then the actions of S^1 on the trivial vector bundle $S^1 \times \mathbf{R}^l$ which cover the action on the base S^1 are unique up to gauge equivalences. They are equivalent to the following one*:

$$ g(z, w) = (gz, w). $$

Put $M_1 = S^4$ equipped with the standard S^1 action. It has two isolated fixed points. Put also $M_2 = S^1 \times N$, where $N = S^1 \times S^2$. We let S^1 act on M_2 by

$$ g(z, y) = (gz, y), \ \ z \in S^1, \ y \in N. $$

Take a non-trivial orbit B_1 in M_1 and let V_1 be an invariant closed tubular neighborhood of B_1. Also put $B_2 = S^1 \times y_0 \subset S^1 \times N = M_2$ where $y_0 \in N$, and let V_2 be an invariant closed tubular neighborhood. Then B_1 and B_2 are identified with S^1, and V_1 and V_2 with $S^1 \times D^3$ where $D^3 \subset \mathbf{R}^4$ is the unit disk. Moreover, the action of S^1 on V_1 and V_2 can be assumed to be of the form as in Lemma. Let W_j be defined by

$$ W_j = \overline{M_j \setminus V_j} $$

for $j = 1, 2$. The boundary ∂W_j of W_j is identified with $\partial(S^1 \times D^3) = S^1 \times S^2$. We define the manifold M by gluing W_1 and W_2 along the boundary $S^1 \times S^2$ and rounding the corner. The action of S^1 on W_1 and W_2 is also glued to define an action on M.

PROPOSITION. *The above 4-manifold M provides an example for which $M^{S^1} \neq \emptyset$ and ${}^t\lambda \neq 0$.*

Proof. Clearly $M^{S^1} \neq \emptyset$.

It is easy to see that W_1 is diffeomorphic to $D^2 \times S^2$ and W_2 is diffeomorphic to $S^1 \times U$ where $U = \overline{N \setminus D^3}$. In particular U is homotopically equivalent to $S^1 \vee S^2$ and we have

$$ H_1(U) \cong \mathbf{R}, \quad H_2(U) \cong \mathbf{R}. $$

Let v_j denote a generator of $H_j(U)$ for $j = 1, 2$. Easy calculations using the Mayer-Vietoris sequence of the triple $(W_1, W_2; M)$ yield the following table:

$$ H_1(M) \cong \mathbf{R} \quad \text{generated by the image of} \ \ v_1 \in H_1(W_2), $$
$$ H_2(M) \cong H_2(W_2) \cong \mathbf{R} \oplus \mathbf{R} \quad \text{generated by} \ \ u_1 \otimes v_1 \ \text{and} \ \ v_2, $$
$$ H_3(M) \cong H_3(W_2) \cong \mathbf{R} \quad \text{generated by} \ \ u_1 \otimes v_2. $$

The image of v_1 will be denoted by the same letter. Then we have

$$^t\lambda(v_1) = u_1 \otimes v_1$$

by virtue of (1), because

$$\varphi(S^1 \times S^1_1) = S^1_2 \times S^1_1 \subset W_2 \subset M,$$

where $S^1_1 = S^1 \times y_1 \subset N$ with $y_1 \in S^2$ such that $y_0 \notin S^1_1$ and we write M_2 as $S^1_2 \times N$. Thus we have shown the above M is a desired example. ■

REMARK. It is not difficult to see the following fact. If one takes any orientable closed 4-dimensional S^1-manifold M_1 with $M_1^{S^1} \neq \emptyset$ instead of S^4 and any orientable closed 3-manifold N with non-trivial first Betti number instead of $S^1 \times S^2$, then one gets a similar example as above.

References

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